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ESTIMATES OF THE ZEROS OF SOME COUNTING POLYNOMIALS

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Abstract. In this work we study the zeros of the Eulerian and Bell polynomials and their generalizations. More concretely, lower estimates for the leftmost zeros of these polynomials will be given, complementing earlier results where upper estimations were presented.

1. Introduction

The Bell polynomials are very well-known in combinatorics; they can be defined, for example, by their generating function:

$$\sum_{n\geq 0} B_n(x)\frac{y^n}{n!} = e^{x(e^y-1)}.$$

Where $B_n(x)$ counts the number of partitions on n elements such that (if x is a positive integer) the blocks in the partitions are independently colored with one of x fixed colors. Equivalently, $B_n(x)$ can be defined as

$$B_n(x) = \sum_{k=0}^{n} \left\{ n \atop k \right\} x^k \quad (n \geq 0),$$

where $\left\{ n \atop k \right\}$ is a Stirling number of the second kind.

The Bell polynomials have a generalization: the r-Bell polynomials [8]. These have the generating function

$$\sum_{n\geq 0} B_{n,r}(x)\frac{y^n}{n!} = e^{x(e^y-1)+ry}.$$

The $B_{n,r}(x)$ polynomials count similar colored partitions on $n+r$ elements such that the first r of the elements are restricted to be in pairwise different

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blocks. The coefficients of these polynomials are the r-Stirling numbers [8]:

$$B_n(x) = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r x^k \quad (n \geq 0),$$

where the $\left\{ {n+r \atop k+r} \right\}_r$ numbers are the r-Stirling numbers of the second kind.

Both of these classes of polynomials have only real and nonpositive zeros, and these zeros are interlacing (see [4] for $B_n(x)$, and [6] for $B_{n,r}(x)$). Consequently, the leftmost zero of the nth polynomial, which will be denoted by z_n^* and $z_{n,r}^*$, is monotone increasing as n grows. It is therefore interesting to ask, how rapidly do these leftmost zeros grow. Upper estimations were given in [9]: for all[1] $n > 0$

$$|z_n^*| < \frac{1}{n}\binom{n}{2} + \frac{n-1}{n}\sqrt{\binom{n}{2}^2 - \frac{2n}{n-1}\left(\binom{n}{3} + 3\binom{n}{4}\right)} \sim \frac{1}{2}\sqrt{\frac{5}{3}}n^{\frac{3}{2}}.$$

It follows that

$$|z_n^*| = O\left(n^{\frac{3}{2}}\right).$$

A similar but somewhat more complicated upper estimation holds for $|z_{n,r}^*|$. Its consequence is that

$$|z_{n,r}^*| = O\left(n^{\frac{3}{2}}\right), \tag{1.1}$$

similar to the $r = 0$ case.

It was conjectured in [9] that $|z_n^*| \sim cn$ as n tends to infinity. By further numerical analysis it seems that $|z_n^*|/n$ converges to a constant c close to e. For example,

$$\frac{|z_{20\,000}^*|}{20\,000} \approx \frac{54\,230}{20\,000} = 2.7115,$$

while $e = 2.71828\ldots$

In this paper we first give a lower estimation for $|z_n^*|$ and $|z_{n,r}^*|$, then we do a similar study for further classes of polynomials, the Eulerian polynomials and their "r-version".

Very recently, the results in [9] were extended to the case of r-Dowling polynomials, and similar results were given for the r-Lah and r-Dowling-Lah polynomials. See the paper of G. Rácz [11].

2. Lower estimations for $|z_n^*|$ and $|z_{n,r}^*|$

In the proof of the upper estimates, a theorem of Samuelson was our main tool, see [9]. We establish the lower estimates by using Colucci's theorem [2, 10]. This theorem says that if the roots of a polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \tag{2.1}$$

[1] In [9] it was stated in Theorem 3.2 that the statement is valid for large n only. In fact, it is true for all $n > 0$.

with real or complex coefficients does not exceed M, then for the kth derivative of $p(x)$,

$$|p^{(k)}(x)| \leq k!\binom{n}{k}|a_n|(|x|+M)^{n-k} \quad (k=0,1,\dots,n). \tag{2.2}$$

Taking $k=0$ and $x=1$ we immediately get that

$$B_n \leq 0!\binom{n}{0}\left\{ n \atop n \right\}(1+|z_n^*|)^n,$$

that is (since $\left\{ n \atop n \right\}=1$),

$$|z_n^*| \geq \sqrt[n]{B_n}-1.$$

This is not as good, however, as the following estimate.

Theorem 2.1. *For all $n>0$ and $r\geq 0$,*

$$r+\frac{n-1}{2} \leq |z_{n,r}^*|.$$

In particular,

$$\frac{n-1}{2} \leq |z_n^*|.$$

Proof. We make use of the theorem of Colucci (2.2) with $k=n-1$. First note that for all $r\geq 0$

$$B_{n,r}^{(n-1)}(x)=(n-1)!\left\{ n+r \atop n-1+r \right\}_r+n!\left\{ n+r \atop n+r \right\}_r x.$$

The r-Stirling number $\left\{ n+r \atop k+r \right\}_r$ counts the partitions of a set of $n+r$ elements into $k+r$ subsets such that the first r elements (which we call distinguished) are in separate blocks. Then, clearly, $\left\{ n+r \atop n+r \right\}_r=1$, and it is not hard to see that

$$\left\{ n+r \atop n-1+r \right\}_r=rn+\binom{n}{2}. \tag{2.3}$$

Indeed, in a partition where the $n+r$ elements are put into $n-1+r$ blocks, there will be only one block with two elements, the others are singletons. The block with two elements can be of two types. 1) it contains no distinguished elements; or 2) one of the two elements is distinguished. In the first case we choose two of the nondistinguished elements in $\binom{n}{2}$ ways. In the second case we choose one distinguished element from r and one nondistinguished from n; these give us rn possibilities. We thus infer that (2.3) indeed holds.

By the reasons above, $B_{n,r}^{(n-1)}(x)$ has the simple form

$$B_{n,r}^{(n-1)}(x)=(n-1)!\left(rn+\binom{n}{2}\right)+n!x. \tag{2.4}$$

Colucci's estimate therefore gives (with $k=n-1$ and $x=1$) that

$$(n-1)!\left(rn+\binom{n}{2}\right)+n! \leq (n-1)!\binom{n}{n-1}\left\{ n+r \atop n+r \right\}_r(1+|z_{n,r}^*|).$$

Since $(n-1)!\binom{n}{n-1} = n!$ and $\left\{ {n+r \atop n+r} \right\}_r = 1$, we get the simpler form

$$\frac{1}{n}\left(rn + \binom{n}{2}\right) \le |z^*_{n,r}|,$$

from where the statement of the theorem follows. □

Our statement, and the earlier statement (1.1), plus the fact that $|z^*_{n,r}|$ is monotone increasing, yield that there exists a constant $\alpha \in [1, 3/2]$ such that

$$\lim_{n\to\infty} \frac{|z^*_{n,r}|}{n^\alpha} = c_r.$$

We argued above that $\alpha = 1$ is a possible candidate and $c_0 \approx e$. It would be interesting to describe these c_r constants for all $r \ge 0$.

3. The Eulerian polynomials, their "r-version", and their leftmost zeros

We now carry out the analysis of the leftmost zeros of the Eulerian and r-Eulerian polynomials. The Eulerian polynomials are defined as

$$E_n(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k,$$

where the coefficients $\left\langle {n \atop k} \right\rangle$ are the Eulerian numbers; $\left\langle {n \atop k} \right\rangle$ is the number of permutations on n elements with k ascents [3].

The r-Eulerian polynomials $E_{n,r}(x)$ were studied in the PhD thesis of the first author,

$$E_{n,r}(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle_r x^k,$$

where $\left\langle {n \atop k} \right\rangle_r$ counts permutations on $n+r$ elements with $k+r$ ascents such that not both of the elements in the ascent are distinguished.

Both families of polynomials have only real roots [1, 7]. (In [1] the $A_n(x)$ polynomials are used, $E_n(x)$ and $A_n(x)$ are reciprocals of each other: $A_n(x) = x^n E_n(1/x)$. But this does not influence the real-zero property.) Note that $E_n(x)$ is of degree $n-1$ while $E_{n,r}(x)$ is a degree n polynomial. The leftmost of these zeros will be denoted by x^*_n and $x^*_{n,r}$. We have the following statement.

Theorem 3.1. *For large n,*

$$|x^*_n| = O(2^n),$$

and for all n,

$$|x^*_n| \ge \frac{2^n}{n-1} - 1 - \frac{2}{n-1}.$$

Moreover, for the leftmost zero of the r-Eulerian polynomials,

$$|x^*_{n,r}| = O\left(\left(\frac{r+1}{r}\right)^n\right) \quad (n \to \infty),$$

and for all n,

$$|x_n^*| \geq (r+1)\frac{(r+1)^n - r^n}{r^n n} - 1.$$

Or, the weaker form of this:

$$|x_n^*| = O\left(\frac{1}{n}\left(\frac{r+1}{r}\right)^n\right) \quad (n \to \infty).$$

Proof. We prove the theorem for the slightly more complicated r-Eulerian polynomials; for the classical Eulerian polynomials the calculations are similar. Samuelson's estimate [12] (which is a rediscovery of Laguerre's theorem [5]) says that for a polynomial (2.1) the roots are contained in the interval $[x_-, x_+]$, where

$$x_\pm = -\frac{a_{n-1}}{na_n} \pm \frac{n-1}{na_n}\sqrt{a_{n-1}^2 - \frac{2n}{n-1}a_{n-2}a_n}. \tag{3.1}$$

It is known [7] that

$$\left\langle {n \atop n} \right\rangle_r = r!r^n. \tag{3.2}$$

The other values we need, namely $\left\langle {n \atop n-1} \right\rangle_r$ and $\left\langle {n \atop n-2} \right\rangle_r$ can be calculated by the formula

$$\left\langle {n \atop k} \right\rangle_r = \sum_{i=0}^{n}(i+r)!\left\{ {n+r \atop i+r} \right\}_r \binom{n-i}{k}(-1)^{n-i-k}.$$

We get that

$$\begin{aligned} \left\langle {n \atop n-1} \right\rangle_r &= r!\left[(r+1)((r+1)^n - r^n) - nr^n\right], \\ \left\langle {n \atop n-2} \right\rangle_r &= r!r^n\binom{n}{n-2} - (r+1)!(n-1)[(r+1)^n - r^n] \\ &\quad + (r+2)!\left[\frac{1}{2}(r+2)^n - (r+1)^n + \frac{1}{2}r^n\right]. \end{aligned} \tag{3.3}$$

These exact expressions imply

$$\left\langle {n \atop n-1} \right\rangle_r \sim (r+1)!(r+1)^n,$$
$$\left\langle {n \atop n-2} \right\rangle_r \sim \frac{1}{2}(r+2)!(r+2)^n.$$

Substituting these into (3.1), some algebra gives that

$$|x_{n,r}^*| = (r+1)\left(\frac{r+1}{r}\right)^n - 1 + o(1).$$

The consequence of this estimate is that

$$|x^*_{n,r}| = O\left(\left(\frac{r+1}{r}\right)^n\right),$$

as it is in the statement.

Now we turn to the lower estimate, and we invoke Colucci's estimate with $k = n-1$ and $x = 1$. We make similar calculations as around (2.4) but now with respect to the polynomial $E_{n,r}(x)$.

$$(n-1)!\left\langle {n \atop n-1} \right\rangle_r + n!\left\langle {n \atop n} \right\rangle_r \le n!\left\langle {n \atop n} \right\rangle_r (1 + |x^*_{n,r}|).$$

From here, by recalling (3.2) and (3.3),

$$\begin{aligned} |x^*_{n,r}| \ge \frac{1}{r!r^n}\frac{1}{n}\left\langle {n \atop n-1} \right\rangle_r &= \frac{1}{r!r^n}\frac{1}{n}r!\left[(r+1)((r+1)^n - r^n) - nr^n\right] \\ &= \frac{(r+1)[(r+1)^n - r^n]}{r^n n} - 1. \end{aligned}$$

This was our last statement to be proved. □

We give some remarks regarding the theorem. If we use (3.1) in its exact form, we get surprisingly sharp estimates for the roots of the Eulerian polynomials. For example,

$$x^*_5 \approx -23.2,$$

while (3.1) gives that

$$x_- \approx -23.28.$$

Similarly,

$$x^*_{10} \approx -963.85,$$

and the Samuelson estimate gives

$$x_- \approx -964.48.$$

These and Theorem 3.1 make us curious about constants $\beta \in [0,1]$ and $d > 0$ where

$$\lim_{n\to\infty} \frac{n^\beta |x^*_n|}{2^n} = d.$$

Similar to the case of the r-Eulerian polynomials, there might exist constants $\beta \in [0,1]$ and $d_r > 0$ such that

$$\lim_{n\to\infty} \frac{n^\beta |x^*_{n,r}|}{\left(\frac{r+1}{r}\right)^n} = d_r.$$

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