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ON PARITY AND RECURRENCES FOR CERTAIN PARTITION FUNCTIONS

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Abstract. In this paper, parity and recurrence formulas for some partition functions are given. In particular, a new recurrence for the number of partitions of a positive integer into distinct parts is derived and some identities reminiscent of Legendre's partition-theoretic interpretation of Euler's pentagonal numbers theorem are obtained.

1. Introduction

A partition of a positive integer n is a representation $(\lambda_1, \lambda_2, \lambda_3, \dots)$ where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 1$ and $\sum_{i\geq 1} \lambda_i = n$. The number of partitions of n , usually denoted by $p(n)$, has been well studied. In particular, there are several recurrence formulas for $p(n)$ (for instance, see [5]). Finding exact formulas for the parity of $p(n)$ remains a thorny issue, although its recurrences can be used to recursively deduce $p(n) \pmod 2$. There are also known parity recurrence formulas for $p(n)$ (see [4], [2]). At times, focus is on restricted partition functions. In this case, certain conditions are imposed on the parts of partitions, and the resulting enumerating function is studied. One such function is the number of partitions of n into distinct parts. Denote this number by $c(n)$. It is much easier to deduce parity of $c(n)$ by considering Euler's pentagonal numbers theorem. If $c_e(n)$ (resp. $c_o(n)$) denotes the number of $c(n)$-partitions into an even (resp. odd) number of parts, then it turns out that

$$c_e(n) - c_o(n) = \begin{cases} (-1)^j, & n = \frac{j}{2}(3j \pm 1), j \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

The above identity is the partition-theoretic interpretation of Euler's pentagonal numbers theorem due to Legendre [3]. Since $c(n) = c_e(n) + c_o(n) \equiv c_e(n) - c_o(n) \pmod 2$, we have an exact parity formula for $c(n)$.
In this note, we derive parity and recurrence formulas for certain restricted partition functions. As it will turn out, parity shall follow from identities in the spirit of (1.1). In particular, we obtain a recurrence formula for the

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number of partitions of n into distinct parts. Our main tools involve the following well-known q-identities.

$$\prod_{n=1}^{\infty}\frac{1-q^n}{1+q^n}=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}, \tag{1.2}$$

$$\prod_{n=1}^{\infty}(1-q^n)(1+zq^n)(1+z^{-1}q^{n-1})=\sum_{n=0}^{\infty}z^n q^{n(n+1)/2}, \tag{1.3}$$

$$\prod_{n=1}^{\infty}(1-q^n)^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}. \tag{1.4}$$

2. Results

Lemma 2.1. *Let $k\in\mathbb{Z}_{\geq 2}$. Then*

$$\prod_{n=1}^{\infty}(1-q^n)^3\equiv$$
$$\prod_{n=1}^{\infty}\left(1+q^{2^k n}\right)\left(1+\sum_{j=1}^{2^{k-1}-1}q^{(2n-1)j}\right)\prod_{j=1}^{2^{k-2}-1}\left(1+q^{2^{k-1}n-2j)}\right)\pmod 2.$$

Proof.

$$\begin{aligned}\prod_{n=1}^{\infty}(1-q^n)^3 &= \prod_{n=1}^{\infty}\frac{(1+q^{2^k n})}{(1+q^{2^k n})}\frac{(1-q^n)^4}{(1-q^n)}\\
&\equiv \prod_{n=1}^{\infty}\frac{(1+q^{2^k n})(1-q^{4n})}{(1+q^{2^{k-1}n})(1+q^{2^{k-1}n})(1-q^n)}\pmod 2\\
&= \prod_{n=1}^{\infty}\frac{(1+q^{2^k n})(1-q^{4n})(1-q^{2^{k-1}(2n-1)})}{(1+q^{2^{k-1}n})(1-q^n)}\\
&= \prod_{n=1}^{\infty}\frac{(1+q^{2^k n})(1-q^{4n})}{(1+q^{2^{k-1}n})(1-q^{2^{k-1}n})(1-q^{2^{k-1}n-1})}\\
&\qquad\times\frac{(1-q^{2^{k-1}(2n-1)})}{(1-q^{2^{k-1}n-2})\dots(1-q^{2^{k-1}n-2^{k-1}+1})}\\
&= \prod_{n=1}^{\infty}\frac{(1+q^{2^k n})(1-q^{2^{k-1}(2n-1)})}{(1+q^{2^{k-1}n})(1-q^{2n-1})(1-q^{2^{k-1}n})}\\
&\qquad\times\frac{(1-q^{4n})}{\prod_{j=1}^{2^{k-2}-1}(1-q^{2^{k-1}n-2j})}\\
&= \prod_{n=1}^{\infty}\frac{(1+q^{2^k n})(1-q^{2^{k-1}(2n-1)})(1-q^{4n})}{(1-q^{2^k n})(1-q^{2n-1})\prod_{j=1}^{2^{k-2}-1}(1-q^{2^{k-1}n-2j})}\end{aligned}$$

$$= \prod_{n=1}^{\infty} \frac{(1+q^{2^k n})(1-q^{2^{k-1}(2n-1)})(1-q^{2^k n})}{(1-q^{2^k n})(1-q^{2n-1})}$$
$$\times \frac{(1-q^{2^k n-4})(1-q^{2^k n-8})\ldots(1-q^{2^k n-2^k+4})}{\prod_{j=1}^{2^{k-2}-1}(1-q^{2^{k-1}n-2j})}$$
$$= \prod_{n=1}^{\infty} \frac{(1-q^{2^{k+1}n})(1-q^{2^{k-1}(2n-1)})\prod_{j=1}^{2^{k-2}-1}(1-q^{2^k n-4j})}{(1-q^{2^k n})(1-q^{2n-1})\prod_{j=1}^{2^{k-2}-1}(1-q^{2^{k-1}n-2j})}$$
$$= \prod_{n=1}^{\infty} \frac{(1-q^{2(2^k n)})}{(1-q^{2^k n})} \frac{(1-q^{2^{k-1}(2n-1)})}{(1-q^{2n-1})} \prod_{j=1}^{2^{k-2}-1} \frac{(1-q^{2(2^{k-1}n-2j)})}{(1-q^{2^{k-1}n-2j})}$$
$$= \prod_{n=1}^{\infty}(1+q^{2^k n})(1+q^{2n-1}+q^{2(2n-1)}+\ldots+q^{(2^{k-1}-1)(2n-1)})$$
$$\times \prod_{j=1}^{2^{k-2}-1}(1+q^{2^{k-1}n-2j})).$$

□

In fact, the above proposition has a direct combinatorial interpretation in terms of partitions. Define $s(n,k)$ as the number of partitions of n in which even parts are distinct and congruent to $0 \pmod{2^k}$, $-2j \pmod{2^{k-1}}$ where $j = 1, 2, \ldots, 2^{k-2}-1$, and odd parts appear at most $2^{k-1}-1$ times. Then the following theorem is an immediate consequence of Lemma 2.1 and (1.4).

Theorem 2.2. *For all integers $k \geq 2$,*

$$s(n,k) \equiv \begin{cases} 1 \pmod{2}, & \text{if } n = \frac{m(m+1)}{2}, m \geq 0; \\ 0 \pmod{2}, & \text{otherwise.} \end{cases}$$

Now set $k = 2$, we have

$$\sum_{n=0}^{\infty} s(n,2)q^n = \prod_{n=1}^{\infty}(1+q^{4n})(1+q^{2n-1}).$$

If we use the notation $s_e(n,2)$ (resp. $s_o(n,2)$) to denote the number of $s(n,2)$-partitions into an even (resp. odd) number of even parts, it is clear that $s(n,2) \equiv s_e(n,2) - s_o(n,2) \pmod{2}$ but in the spirit of (1.1), we have the following:

Theorem 2.3. *For $n \geq 0$, we have*

$$s_e(n,2) - s_o(n,2) = \begin{cases} 1, & \text{if } n = \frac{m(m+1)}{2}, m \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty}(s_e(n,2)-s_o(n,2))q^n &= \prod_{n=1}^{\infty}(1-q^{4n})(1+q^{2n-1})\\ &= \prod_{n=1}^{\infty}(1-q^{4n})(1+q^{4n-1})(1+q^{4n-3})\\ &= \sum_{n=-\infty}^{\infty} q^{2n^2+n} \text{ (by (1.3) with } q:=q^4,\ z:=q^{-1})\\ &= \sum_{n=0}^{\infty} q^{n(2n\pm1)}.\end{aligned}$$

□

The proof above where (1.3) is invoked points towards some sort of generalisation. Set $q := q^2$ in (1.3) so that

$$\sum_{n=-\infty}^{\infty}(zq)^n q^{n^2} = \prod_{m=1}^{\infty}(1-q^{2m})(1+zq^{2m})(1+z^{-1}q^{2m}).$$

Now replace zq by z in the preceding identity; we obtain

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{m=1}^{\infty}(1-q^{2m})(1+zq^{2m-1})(1+z^{-1}q^{2m-1}). \tag{2.1}$$

Let $q := q^{2r}$ and $z := q$ in (2.1) so that

$$\sum_{n=-\infty}^{\infty} q^{2rn^2+n} = \prod_{m=1}^{\infty}(1-q^{4rm})(1+q^{4rm-(2r-1)})(1+q^{4km-(2r+1)}), \tag{2.2}$$

whose partition-theoretic consequence is given in the following theorem.

Theorem 2.4. *Let $d(n,r)$ denote the number of partitions of n into distinct parts that are congruent to $0, 2r\pm1 \pmod{4r}$. Let $d_e(n,r)$ enumerate those $d(n,r)$-partitions in which the number of parts congruent to $0 \pmod{4r}$ is even. Similarly, $d_o(n,r)$ for odd number of aforesaid parts. Then*

$$d_e(n,r)-d_o(n,r) = \begin{cases}1, & \text{if } n = m(2rm\pm1), m\geq 0;\\ 0, & \text{otherwise.}\end{cases}$$

Corollary 2.5. *For all $n \geq 1$, $d(n,r) \equiv 1 \pmod 2$ if and only if $n = j(2rj\pm1)$ for some $j \in \mathbb{Z}_{\geq 0}$.*

Observe that Theorem 2.4 generalises Theorem 2.3 ($r = 1$). From (2.2), we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{2rn^2+n} &= \prod_{m=1}^{\infty} \frac{1-q^{4rm}}{1+q^{4rm}} \\ &\qquad \times \prod_{m=1}^{\infty}(1+q^{4rm})(1+q^{4rm-(2r-1)})(1+q^{4km-(2r+1)}) \\ &= \prod_{m=1}^{\infty} \frac{1-q^{4rm}}{1+q^{4rm}} \sum_{n=0}^{\infty} d(n,r)q^n \\ &= \sum_{m=-\infty}^{\infty} (-1)^m q^{4rm^2} \sum_{n=0}^{\infty} d(n,r)q^n \text{ by (1.2)} \\ &= \left(1 + 2\sum_{m=1}^{\infty}(-1)^m q^{4rm^2}\right) \sum_{n=0}^{\infty} d(n,r)q^n \\ &= \sum_{n=0}^{\infty} d(n,r)q^n + 2\left(\sum_{n=1}^{\infty}(-1)^n q^{4rn^2}\right)\sum_{n=0}^{\infty} d(n,r)q^n \\ &= \sum_{n=0}^{\infty} d(n,r)q^n + 2\sum_{n=0}^{\infty}\left(\sum_{m=1}^{\lfloor\sqrt{\frac{n}{4r}}\rfloor}(-1)^m d(n-4rm^2,r)\right) q^n \\ &= \sum_{n=0}^{\infty}\left(d(n,r) + 2\sum_{m=1}^{\lfloor\sqrt{\frac{n}{4r}}\rfloor}(-1)^m d(n-4rm^2,r)\right) q^n.\end{aligned}$$

By comparing the coefficients on the right and left-hand sides, we obtain the following recurrence.

Theorem 2.6.

$$d(n,r) = \begin{cases} 1 + 2\sum_{m=1}^{\lfloor\sqrt{\frac{n}{4r}}\rfloor}(-1)^{m+1}d(n-4rm^2,r), & \textit{if } n = j(2rj \pm 1), j \geq 0; \\ 2\sum_{m=1}^{\lfloor\sqrt{\frac{n}{4r}}\rfloor}(-1)^{m+1}d(n-4rm^2,r), & \textit{otherwise}; \end{cases}$$

where $d(0,r) := 1$.

We look at an example demonstrating the above recurrence.

Example 2.7. *Consider* $n = 10$ *and* $r = 1$.

Using the formula, we have

$$d(10,1) = 1 + 2d(6,1) = 1 + 2(1 + 2d(2)) = 1 + 2(1 + 2(0)) = 3.$$

However, the $d(10,1)$-partitions are; $(9,1),(7,3),(5,4,1)$. In either case, the answer is 3. Furthermore, from (2.2), we observed that

$$\sum_{n=-\infty}^{\infty} q^{2rn^2+n} = \prod_{m=1}^{\infty} \frac{1-q^{4rm}}{1+q^{4rm}} \times \prod_{m=1}^{\infty} (1+q^{4rm})(1+q^{4rm-(2r-1)})(1+q^{4km-(2r+1)}).$$

This means

$$\sum_{n=0}^{\infty}(d_e(n,r)-d_o(n,r))q^n = \sum_{m=-\infty}^{\infty} (-1)^m q^{4rm^2} \sum_{n=0}^{\infty} d(n,r)q^n$$

so that

$$\sum_{n=0}^{\infty}(d_e(n,r)+d_o(n,r))q^n = \sum_{m=-\infty}^{\infty} (-1)^m q^{4rm^2} \sum_{n=0}^{\infty} d(n,r)q^n + 2\sum_{n=0}^{\infty} d_o(n,r)q^n,$$

i.e.,

$$\sum_{n=0}^{\infty} d(n,r)q^n = \sum_{n=0}^{\infty}\left(d(n,r) + 2\sum_{m=1}^{\lfloor\sqrt{\frac{n}{4r}}\rfloor} (-1)^m d(n-4rm^2,r)\right) q^n + 2\sum_{n=0}^{\infty} d_o(n,r)q^n.$$

Thus the following formula follows.

Corollary 2.8. *For all $n \geq 1$, we have*

$$d_o(n,r) = \sum_{m=1}^{\lfloor\sqrt{\frac{n}{4r}}\rfloor} (-1)^{m+1} d(n-4rm^2,r).$$

Besides, one can simply use the fact that $d_e(n,r) = d(n,r) - d_o(n,r)$ to derive a similar formula as in the above corollary for $d_e(n,r)$.

In Theorem 2.4, the modulus $4r$ is even, similarly, for odd moduli, let $q := 2r+1$, $z := q$ in (2.1) so that

$$\sum_{n=-\infty}^{\infty} q^{n+(2r+1)n^2} = \prod_{m=1}^{\infty} (1-q^{(4r+2)m})(1+q^{(4r+2)m-2r})(1+q^{(4r+2)m-2r-2}).$$

Now set $q := q^{\frac{1}{2}}$ in the preceding identity to obtain

(2.3)
$$\sum_{n=-\infty}^{\infty} q^{(n+(2r+1)n^2)/2} = \prod_{m=1}^{\infty} (1-q^{(2r+1)m})(1+q^{(2r+1)m-k})(1+q^{(2r+1)m-r-1}).$$

Thus the following theorem follows.

Theorem 2.9. *Let $c(n,r)$ denote the number of partitions of n into distinct parts that are congruent to $0, \pm r \pmod{2r+1}$. Let $c_e(n,r)$ enumerate those $c(n,r)$-partitions with an even number of parts congruent to $0 \pmod{2r+1}$. Similarly, $c_o(n,r)$ for odd number of the aforesaid parts. Then*

$$c_e(n,r) - c_o(n,r) = \begin{cases} 1, & \text{if } n = \frac{m}{2}[(2r+1)m \pm 1], m \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Remark. The partition function $c(n,r)$ is well-known (see [1]) as it is used to generalise Legendre's partition theoretic interpretation of Euler's pentagonal numbers theorem. In this paper, we have considered it in a different way by effecting parity restriction only on those parts divisible by $2r+1$, while in the known case, parity affects all parts.

We now obtain a recurrence for $c(n,r)$. Equation (2.3) implies

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q^{(n+(2r+1)n^2)/2} &= \prod_{m=1}^{\infty} \frac{1-q^{(2r+1)m}}{1+q^{(2r+1)m}} \prod_{m=1}^{\infty} (1+q^{(2r+1)m})(1+q^{(2r+1)m-k}) \\ &\qquad \times \prod_{m=1}^{\infty} (1+q^{(2r+1)m-r-1}) \\ &= \prod_{m=1}^{\infty} \frac{1-q^{(2r+1)m}}{1+q^{(2r+1)m}} \sum_{n=0}^{\infty} c(n,r) q^n. \end{aligned}$$

Proceeding in the same manner as done in deriving the recurrence in Theorem 2.6, we obtain the following theorem.

Theorem 2.10.

$$c(n,r) = \begin{cases} 1 + 2 \displaystyle\sum_{m=1}^{\lfloor \sqrt{\frac{n}{2r+1}} \rfloor} (-1)^{m+1} c(n-(2r+1)m^2, r), \\ \qquad\qquad \text{if } n = \frac{j}{2}[(2r+1)j \pm 1], j \geq 0; \\ 2 \displaystyle\sum_{m=1}^{\lfloor \sqrt{\frac{n}{2r+1}} \rfloor} (-1)^{m+1} c(n-(2r+1)m^2, r), \\ \qquad\qquad\qquad\qquad \text{otherwise}; \end{cases}$$

where $c(0,r) := 1$.

Remark. Observe that $c(n,1)$ is the number of partitions of n into distinct parts. Thus we have a new recurrence for this popular partition function. Furthermore, formulas expressing $c_e(n,r)$ and $c_e(n,r)$ in terms of $c(n,r)$ can be derived via a similar approach to the one used in arriving at Corollary 2.8.

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