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NOTE ON GROUP DISTANCE MAGICNESS OF PRODUCT GRAPHS

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ABSTRACT. If l is a bijection from the vertex set $V(G)$ of a graph G to an additive abelian group Γ of $|V(G)|$ elements such that for any vertex u of G, the weight $\sum_{v \in N_G(u)} l(v)$ is μ, where $\mu \in \Gamma$, then l is a Γ-distance magic labeling of G. A graph G that admits such an l is Γ-distance magic and if G is Γ-distance magic for every such Γ, then G is a group distance magic graph. In this paper, we provide some results on the group distance magicness of the lexicographic and direct product of two graphs. By proving a few necessary conditions, we characterize the group distance magicness of a tree. In addition, we find three techniques to construct group distance magic graphs recursively from the existing ones and with respect to any abelian group with one involution, we determine infinitely many nongroup distance magic graphs.

1. Introduction

In this paper, we consider only simple and finite graphs. We use $V(G)$ for the vertex set, and $E(G)$ for the edge set of a graph G. The neighborhood $N_G(v)$, or shortly $N(v)$ of a vertex v of G is the set of all vertices adjacent to v, and the degree $\deg_G(v)$, or shortly $\deg(v)$ of v is the number of vertices in $N_G(v)$. The distance $d_G(u,v)$ between two vertices u and v of G is the length of the shortest path connecting u and v. For more standard graph-theoretic notation and terminology, we refer the reader to Bondy and Murty [4] and Hammack et al. [14].

Let G and H be two graphs. Then the lexicographic product $G \circ H$ and the direct product $G \times H$ are graphs with the vertex set $V(G) \times V(H)$. Two vertices (g,h) and (g',h') are adjacent in:

(i) $G \circ H$ if and only if g is adjacent to g' in G, or $g = g'$ and h is adjacent to h' in H.
(ii) $G \times H$ if and only if g is adjacent to g' in G and h is adjacent to h' in H.

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A distance magic labeling of a graph G is a bijection $l : V(G) \to \{1, \dots, |V(G)|\}$ such that for any u of G, the weight of u, $w_G(u) = \sum_{v \in N_G(u)} l(v)$ is a constant μ. A graph G that admits such labeling, is called a distance magic graph [18].

The concept of distance magic labeling was studied by Vilfred [23] as sigma labeling. Later, Miller et al. [18] called it a 1-vertex magic vertex labeling and Sugeng et al. [22] referred to the same concept as distance magic labeling. The study of distance magic labeling has been motivated by the construction of magic squares. It is worth mentioning the motivation given by Froncek et al. [13] through an equalized incomplete tournament. An equalized incomplete tournament of n teams with r rounds, $\mathrm{EIT}(n, r)$ is a tournament which satisfies the following conditions:

(i) Every team plays against exactly r distinct opponents.
(ii) The total strength of the opponents, against which each team plays is a constant.

Therefore, finding a solution to an $\mathrm{EIT}(n, r)$ is equivalent to obtaining a distance magic labeling of an r-regular graph with n vertices.

The important results and problems, which are relevant and helpful in proving our results, are listed below.

Theorem 1.1 ([16, 18, 19, 23])**.** *No r-regular graph with r-odd can be a distance magic graph.*

Lemma 1.2 ([18])**.** *If G contains two vertices u and v such that $|N(u) \cap N(v)| = \deg(v) - 1 = \deg(u) - 1$, then G is not distance magic.*

Theorem 1.3 ([21])**.** *Let $r \geq 1$ and $n \geq 3$. If G is an r-regular graph and C_n the cycle of length n, then $G \circ C_n$ admits a labeling if and only if $n = 4$.*

In 2009, Shafiq et al. [21] posted a problem of the existence of distance magic labeling of the lexicographic product of a nonregular graph G with C_4.

Problem 1.4 ([21])**.** *If G is a nonregular graph, determine if there is a distance magic labeling of $G \circ C_4$.*

In 2018, Cichacz and Görlich [10] raised a similar question in the case of the direct product of G with C_4.

Problem 1.5 ([10])**.** *If G is a nonregular graph, determine if there is a distance magic labeling of $G \times C_4$.*

Anholcer et al. [1] defined a distance magic graph G with an even number of vertices, *balanced* if there exists a bijection $l : V(G) \to \{1, \dots, |V(G)|\}$ such that for any vertex u of G, the following holds: if $v \in N(u)$ with $l(v) = i$, then there exists $v' \in N(u), v' \neq v$, with $l(v') = |V(G)| + 1 - i$. Further, we call v', the twin vertex of v and vice versa.

From [1], it is clear that G is a balanced distance magic graph or shortly, balanced-*dmg* if and only if G is regular and the vertex set of G can be expressed as $\{v_i, v_i' : 1 \le i \le |V(G)|/2\}$ such that for any i, $N(v_i) = N(v_i')$, where v_i is the twin vertex of v_i'. Therefore, a balanced-*dmg* G on n vertices is always a $2r$-regular graph with magic constant $r(n+1)$.

The kth power of a graph G is a graph G^k with the same set of vertices as G, and any two vertices u and v are connected if and only if $d_G(u,v) \le k$. A graph nG is the disjoint union of n copies of the graph G.

Also, if $\Omega \subseteq V(G)$ of a graph G, then $G[\Omega]$ is the subgraph of G with the vertex set Ω and the edge set $\{uv \in E(G) : u, v \in \Omega\}$.

In 2016, Arumugam et al. [3] proved the following result as a characterization of distance magic graphs G with $\Delta(G) = |V(G)| - 1$. Note that this class of graphs is derived from the class of balanced-*dmg*s.

Theorem 1.6 ([3]). *Let G be any graph of order n with $\Delta(G) = n-1$. Then G is a distance magic graph if and only if n is odd and $G \cong (K_{n-1} - M) + K_1$, where M is a perfect matching in K_{n-1}.*

In 2004, Rao et al. [20] proved the following result.

Theorem 1.7 ([20]). *The graph $C_k \Box C_m$ is distance magic if and only if $k = m$ and $k, m \equiv 2 \mod 4$.*

Now a natural question arises for all graphs which are not distance magic. Can one introduce the concept by replacing the set of all labels $\{1, \dots, |V(G)|\}$ by an abelian group with $|V(G)|$ elements in such a way that these graphs can admit such labeling? Motivated by this idea, in 2013, Froncek [12] introduced the notion of group distance magic labeling of graphs. He proved that $C_k \Box C_m$ is $\mathbb{Z}_{km}$-distance magic if and only if km is even, where $\mathbb{Z}_{km}$ is the cyclic group of order km.

Throughout this paper, $\Gamma = (\Gamma, +)$ denotes a finite additive abelian group or shortly, an abelian group, where $+$ is a binary operation on Γ. The order and identity element of Γ are denoted by $|\Gamma|$ and $\mathbf{0}$, respectively. An element $g \ne \mathbf{0}$ is an involution of Γ if $g = -g$, where $-g$ is the additive inverse of g and Γ has an involution if and only if $|\Gamma|$ is even. Further, the sum of all elements of Γ, $\mathrm{sum}(\Gamma) = \sum_{g \in \Gamma} g$ is equal to the sum of all involutions of Γ and Γ is an elementary abelian p-group if $|\Gamma| = p^n$, and for every element $g \ne \mathbf{0}$, $g + \dots + g$ (p times) $= \mathbf{0}$. The exponent of an abelian group Γ, $\exp(\Gamma)$ is the least positive integer m such that for every element $g \ne \mathbf{0}, g + \dots + g$ (m times) is $\mathbf{0}$. Recall that if Γ_1 and Γ_2 are two abelian groups, then $\exp(\Gamma_1 \times \Gamma_2) = \mathrm{lcm}(\exp(\Gamma_1), \exp(\Gamma_2))$.

The fundamental theorem on finite abelian groups states that any finite abelian group is a direct product of cyclic groups of prime power order, where the product is unique up to the order of subgroups.

The following lemma will be useful in our proofs.

Lemma 1.8 ([11]). *Let Γ be an abelian group.*

(i) *If Γ has exactly one involution g', then* $\mathrm{sum}(\Gamma) = g'$.
(ii) *If Γ has no involutions or more than one involution, then* $\mathrm{sum}(\Gamma) = \mathbf{0}$.

For more group theory related terminology and notation, refer to Lang [17] and Herstein [15].

Definition 1.9 ([12]). *If Γ is an abelian group and G is a graph such that $|V(G)| = |\Gamma|$, then a bijection $l : V(G) \to \Gamma$ is said to be a Γ-distance magic labeling of G if for any u of G, the weight of u, $w_G(u) = \sum_{v \in N_G(u)} l(v)$ is equal to the same element μ_o of Γ. A graph G that admits such a labeling is called a Γ-distance magic graph and μ_o is called the magic constant associated with l of G. Also, if l is the Γ-distance magic labeling G, then the weight of G, $w(G) = \sum_{v \in V(G)} w_G(v) = \sum_{v \in V(G)} \deg(v) l(v) = \mu_0 |V(G)|$.*

If l is any distance magic labeling of a graph G on n vertices with the magic constant μ, then $l^*(v) = l(v) \mod n$, for all v of G, is a $\mathbb{Z}_n$-distance magic labeling of G with magic constant $\mu \mod n$. However, the converse is not true, see [12].

In 2014, Cichacz [5, 7] proved the following results.

Theorem 1.10 ([7]). *Let G be a graph of order n and Γ be an arbitrary abelian group of order $4n$ such that $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathcal{B}$ for some abelian group $\mathcal{B}$ on n vertices. Then there exists a Γ-distance magic labeling for the graph $G \circ C_4$.*

Theorem 1.11 ([7]). *Let G be a graph of order n and Γ be an abelian group of order $4n$. If $n = 2^p(2k+1)$ for some natural numbers p, k, and $\deg(v) \equiv c \mod 2^{p+1}$ for some constant c for any $v \in V(G)$, then there exists a Γ-distance magic labeling for the graph $G \circ C_4$.*

Theorem 1.12 ([5]). *Let G be a graph of order n. If $n = 2^p(2k+1)$ for some natural numbers p, k, and $\deg(v) \equiv c \mod 2^{p+2}$ for some constant c for any $v \in V(G)$, then there exists a Γ-distance magic labeling for the graph $G \times C_4$.*

Cichacz in [6] gave a characterization of group distance magicness of the complete bipartite graphs.

Theorem 1.13 ([6]). *The complete bipartite graph $K_{m,n}$ is a group distance magic graph if and only if $m + n \not\equiv 2 \mod 4$.*

Recently, Anholcer et al. [2] discussed the group distance magicness of the direct product of two graphs and obtained the following result.

Theorem 1.14 ([2]). *If G is a balanced distance magic graph and H an r-regular graph for $r \geq 1$, then $G \times H$ is a group distance magic graph.*

The results from [9, 8] characterize nongroup distance magic graphs.

Theorem 1.15 ([9]). *Let G be an r-regular graph on n vertices, where r is odd. There does not exist an abelian group Γ of order n with exactly one involution g' such that G is Γ-distance magic.*

Theorem 1.16 ([8])**.** *Let G have order $n \equiv 2 \mod 4$ with all vertices having odd degree. There does not exist an abelian group Γ of order n such that G is a Γ-distance magic graph.*

Theorem 1.17 ([8])**.** *Let $G = K_{n_1,n_2,\ldots,n_t}$ be a complete t-partite graph and $n = n_1 + n_2 + \cdots + n_t$. If $n_1 \leq n_2 \leq \cdots \leq n_t$ and $n_2 = 1$, then there does not exist an abelian group Γ of order n such that G is a Γ-distance magic graph.*

In Section 2, we provide some necessary conditions for a graph to be group distance magic and characterize the group distance magic labeling of a tree T. Further, we discuss the group distance magic labeling of (r_1, r_2)-regular graphs G with $r_1 = |V(G)| - 1$ and $r_2 = |V(G)| - 2$ and also exhibit infinite families of graphs, which are not group distance magic with respect to an abelian group with one involution. In Section 3, we give three new recursive techniques to construct new group distance magic graphs from existing ones.

Motivated by the results from [2, 5, 7], in the last section, we discuss the group distance magic labeling of $G \circ H$ and $G \times H$, where G is a nonregular graph and H is a balanced-*dmg*.

2. Group distance magicness of $(K_{n-1} - M) + K_1$ and trees

First, we prove the following lemma to show that there are infinitely many graphs, which are not group distance magic.

Lemma 2.1. *Let c be an odd integer less than 2^k, and G be a graph on $2^k t$ vertices, where $k, t \in \mathbb{N}\backslash\{1\}$ and t is odd. If for any vertex v of G, $\deg(v) \equiv c \mod 2^k$, then there exists no abelian group Γ with one involution for which G is Γ-distance magic.*

Proof. Let Γ be an abelian group with $|\Gamma| = 2^k t$, where t is odd. If Γ has exactly one involution g', then $\Gamma \cong \mathbb{Z}_{2^k} \times \mathcal{A}$, where $\mathcal{A}$ is an abelian group with $|\mathcal{A}| = t$. If g' is an involution of $\mathbb{Z}_{2^k} \times \mathcal{A}$, then $g' = (2^{k-1}, 0)$, where 0 is the identity element in $\mathcal{A}$. Therefore, $\Gamma = \{g_i, -g_i : 1 \leq i \leq 2^{k-1}t - 1\} \cup \{\mathbf{0}, g'\}$, where $\mathbf{0}$ is the identity of Γ. Let $v_1, \ldots, v_{2^k t}$ be the vertices of G. For the sake of contradiction, suppose that G is Γ-distance magic with magic constant μ_0.

Without loss of generality, consider the function l given by

$$l(v_i) = \begin{cases} g_i & \text{for } i \in \{1, \ldots, 2^{k-1}t - 1\} \\ g' & \text{for } i = 2^{k-1}t \\ -g_j & \text{for } i = 2^{k-1}t + j \text{ and } j \in \{1, \ldots, 2^{k-1}t - 1\} \\ \mathbf{0} & \text{for } i = 2^k t. \end{cases}$$

If $\deg(v) \equiv c \mod 2^k$ for every v of G, then $\deg(v_i) - \deg(v_{2^{k-1}t+i}) = 2^k s_i$, for some integer s_i, where $i \in \{1, 2, \ldots, 2^{k-1}t - 1\}$. Since $w_G(v_i) = \mu_0$ for any i, we have $w(G) = \sum_{i=1}^{2^k t} w_G(v_i) = 2^k t \mu_0 = |\Gamma| \mu_0 = \mathbf{0}$. On the other hand,

$$\begin{aligned}
w(G) &= \sum_{i=1}^{2^k t} \deg(v_i) l(v_i) \\
&= \sum_{i=1}^{2^{k-1}t-1} \deg(v_i) g_i + \deg(v_{2^{k-1}t}) g' + \sum_{i=2^{k-1}t+1}^{2^k t-1} \deg(v_i) g_i + \mathbf{0} \\
&= \sum_{i=1}^{2^{k-1}t-1} \big(\deg(v_i) - \deg(v_{2^{k-1}t+i})\big) g_i + g' = 2^k \sum_{i=1}^{2^{k-1}t-1} s_i g_i + g' \\
&= 2^k g + g', \text{ where } g = \sum_{i=1}^{2^{k-1}t-1} s_i g_i \\
&= 2^k g - g'.
\end{aligned}$$

But there exists no element g in Γ for which $2^k g = g'$, a contradiction. □

Now, let us take $c = 3, k = 2$, and $t = 3$ in Lemma 2.1 and consider all $(3,7)$-graphs of order 12. With the help of a brute-force search algorithm,

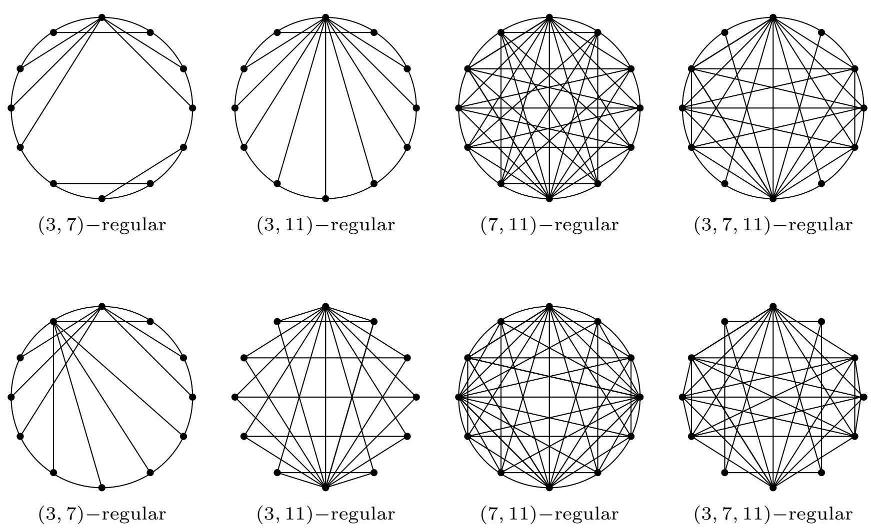

FIGURE 1. Graphs which are not $\mathbb{Z}_4 \times \mathcal{A}$-distance magic.

one can obtain 21041 graphs (up to isomorphism), which are not group distance magic with respect to any abelian group Γ, where $\Gamma \cong \mathbb{Z}_4 \times \mathcal{A}$ and $\mathcal{A}$ an abelian group with $|\mathcal{A}| = 3$. Figure 1 gives a few such nongroup distance magic graphs on 12 vertices.

Observation 2.2. *Let Γ be any abelian group and G be a graph with $|\Gamma|$ vertices. If G has at least two distinct vertices of degree $|\Gamma| - 1$, then G is not Γ-distance magic.*

Proof. Suppose that l is a Γ-distance magic labeling of the graph G with magic constant μ_0. If u and v be two distinct vertices of G with $\deg_G(u) = |\Gamma| - 1 = \deg_G(v)$, then

$$\text{sum}(\Gamma) = \mu_0 + l(u) = \mu_0 + l(v).$$

Now, the labels of u and v are equal, a contradiction. □

Lemma 2.3. *If Γ is any abelian group and if G is a graph with $|\Gamma|$ vertices such that G has two distinct vertices u and v with $|N_G(u) \cap N_G(v)| = \deg_G(u) - 1 = \deg_G(v) - 1$, then G is not Γ-distance magic.*

Proof. Suppose that l is a Γ-distance magic labeling of the graph G with magic constant μ_0. Choose two vertices u' and v' of G in such that $u' \in N_G(u)$ and $v' \in N_G(v)$ but $u' \notin N_G(v)$ and $v' \notin N_G(u)$.

By comparing the weights of u and v, we have

$$\mu_0 = w_G(u) = g + l(u') = g + l(v') = w_G(v),$$

for some g in Γ. Then the labels of u' and v' are equal, a contradiction. □

The following result characterizes the group distance magicness of a tree.

Theorem 2.4. *A nontrivial tree T is Γ-distance magic for an abelian group Γ if and only if $T \cong K_{1,n}$, with $n \not\equiv 1 \mod 4$.*

Proof. If $\text{diam}(T) = 2$, the result is straightforward by Theorem 1.13. On the other hand if $\text{diam}(T) > 2$, then T has two vertices u and v such that $d_G(u, v) = \text{diam}(T)$. Since u and v are leaves, the result follows from Lemma 2.3. □

Theorem 2.5. *Let Γ be any abelian group having at least one element $g^\dagger$ such that $g^\dagger$ is not an involution. If l is a Γ-distance magic labeling of G with magic constant μ_0, then there exists a Γ-distance magic labeling l^{-1} of G with magic constant $-\mu_0$.*

Proof. Let l be a Γ-distance magic labeling of G with magic constant μ_0. Then, define the labeling $l^{-1}(u) = -l(u)$. Clearly, l is not identically equal to l^{-1} because if $l(v) = l^{-1}(v)$ for all v, in particular, if $l(v^\dagger) = l^{-1}(v^\dagger) = g^\dagger$, then $l(v^\dagger) = -l(v^\dagger)$, which implies $2g^\dagger = \mathbf{0}$, a contradiction.

Then the weight of u of G with respect to l^{-1} is

$$\sum_{v \in N(u)} l^{-1}(v) = \sum_{v \in N(u)} -l(v) = -\sum_{v \in N(u)} l(v) = -\mu_0.$$

□

Theorem 2.6. *Let $n > 1$ be an odd integer. Let G be a graph isomorphic to $(K_{n-1} - M) + K_1$, where M is any perfect matching in K_{n-1} and v_0 be the vertex of G with degree $n - 1$. If Γ is any abelian group with $|\Gamma| = n$, then G admits a Γ-distance magic labeling l if and only if $l(v_0) = \mathbf{0}$.*

Proof. Let Γ be an abelian group with $|\Gamma| = n$. Define the vertex set of G as $\{v_0\} \cup \{v_i, v_i' : 1 \leq i \leq (n-1)/2\}$, where v_i and v_i' are twin vertices of $K_{n-1} - M$ and v_0 is the vertex, which induces K_1 of G. Let l be a Γ-distance magic labeling of G with magic constant μ_0. We know that $ng = \mathbf{0}$ for any element g of Γ, in particular, $n\mu_0 = \mathbf{0}$. By using Lemma 1.8 and by comparing the total weights, we get

$$n\mu_0 = (n-1)l(v_0) + (n-2)\big(\mathrm{sum}(\Gamma) - l(v_0)\big),$$

which implies that $l(v_0)$ is $\mathbf{0}$.

When n is odd, for every element g of Γ, there exists a unique $-g$ different from g in Γ such that $g + (-g) = \mathbf{0}$. Consider a function l from $V(G)$ to Γ as, $l(v_0) = \mathbf{0}$ and if $l(v_i) = g$, then $l(v_i') = -g$, for all $i \neq 0$. Now, one can verify that the weight of each vertex of G is $\mathbf{0}$. Thus, l is a Γ-distance magic labeling of G with magic constant $\mu_0 = \mathbf{0}$. □

Theorem 2.7. *Let Γ_1 and Γ_2 be two abelian groups. If G is an $\exp(\Gamma_2)$-regular Γ_1-distance magic graph and H is an $\exp(\Gamma_1)$-regular Γ_2-distance magic graph, then $G \circ H$ is a $(\Gamma_1 \times \Gamma_2)$-distance magic graph.*

Proof. Let l_1, l_2 be a Γ_1-distance magic labeling of G and a Γ_2-distance magic labeling of H respectively. Let μ_1, μ_2 be their respective magic constants.

For every $(u, v) \in G \circ H$, define $l_{G\circ H}$ as,

$$l_{G\circ H}\left((u,v)\right) = \big(l_1(u), l_2(v)\big).$$

Clearly $l_{G\circ H}$ is a bijection and for every (u, v) in $G \circ H$, the weight,

$$\begin{aligned}
&w_{G\circ H}\left((u,v)\right)\\
&= \sum_{(x,y)\in N_{G\circ H}((u,v))} l_{G\circ H}\left((x,y)\right)\\
&= \Bigg(l_1(u)\exp(\Gamma_1) + |V(H)| \sum_{x\in N_G(u)} l_1(x), \mathrm{sum}(\Gamma_2)\exp(\Gamma_2) + \sum_{y\in N_H(v)} l_2(y)\Bigg)\\
&= \Big(l_1(u)\exp(\Gamma_1) + |V(H)|\mu_1, \mathrm{sum}(\Gamma_2)\exp(\Gamma_2) + \mu_2\Big)\\
&= (|V(H)|\mu_1, \mu_2).
\end{aligned}$$

a constant. □

3. Three techniques to construct a group distance magic graph from the existing one.

The following theorems provide certain techniques for constructions of larger group distance magic graphs recursively from smaller group distance magic graphs using graph products.

Theorem 3.1. *Let Γ_1 be an abelian group. If G is Γ_1-distance magic graph with magic constant μ_1 such that for any v of G, exp(Γ_2) divides* $\deg_G(v)$, *then* $|\Gamma_2|G$ *is a* $(\Gamma_1\times\Gamma_2)$*-distance magic graph with a magic constant* $(\mu_1,\mathbf{0})$*, where* Γ_2 *is an abelian group with identity* $\mathbf{0}$*.*

Proof. Let G be a graph with the vertices $v_0,\ldots,v_{n-1}$ and l_1 be a Γ_1-distance magic labeling of G with magic constant μ_1. Let the vertex set of $|\Gamma_2|G$ be $\{v_i^j : 0\le i\le n-1, 1\le j\le |\Gamma_2|\}$. Consider a labeling $l_{|\Gamma_2|G}$ on $|\Gamma_2|G$ as,

$$l_{|\Gamma_2|G}(v_i^j)=(l_1(v_i),a_j),$$

where $a_j\in\Gamma_2$ for all j.

Since $\deg_G(v_i)=\deg_{|\Gamma_2|G}(v_i^j)$ for all i and j, we have $\deg_{|\Gamma_2|G}(v_i^j)=k_i\exp(\Gamma_2)$, where $1\le k_i\le(|V(G)|-1)/\exp(\Gamma_2)$, for all i. Thus the weight,

$$\begin{aligned}w_{|\Gamma_2|G}(v_i^j)&=\sum_{v^*\in N_{|\Gamma_2|G}(v_i^j)} l_{|\Gamma_2|G}(v^*)\\&=\left(\sum_{x\in N_G(v_i)} l_1(x), k_i\exp(\Gamma_2)\operatorname{sum}(\Gamma_2)a_j\right)=(\mu_1,\mathbf{0}),\end{aligned}$$

for all i and j. □

From the above theorem, we can obtain the following corollary.

Corollary 3.2. *Let* Γ_1 *be an abelian group. If* G *is an Eulerian and* Γ_1*-distance magic graph with magic constant* μ_1*, then* $|\Gamma_2|G$ *is a* $(\Gamma_1\times\Gamma_2)$*-distance magic graph with a magic constant* (μ_1,e)*, where* Γ_2 *is an elementary abelian 2-group with identity* e*.* □

Theorem 3.3. *Let* Γ_1 *be an abelian group with exponent* n*. If* G *is a* Γ_1*-distance magic graph, then* $G\circ\bar{K}_n$ *is a* $(\Gamma_1\times\Gamma_2)$*-distance magic graph, where* Γ_2 *is any abelian group of order* n *with more than one involution or no involution.*

Proof. Let l be a Γ_1-distance magic labeling of a graph G with magic constant μ. Let the vertices of G be $v_0,\ldots,v_{p-1}$ and for each $i\in\{0,\ldots,p-1\}$, let $v_i^0,\ldots,v_i^{n-1}$ be the vertices of $G\circ\overline{K}_n$ that replace the vertex v_i of G. Also, let $\Gamma_2=\{g_i : 0\le i\le n-1\}$ be an abelian group with the identity element g_0.

For all i, label the vertices of $G\circ\overline{K}_n$ as,

$$l_{G\circ\overline{K}_n}(v_i^j)=\big(l(v_i),g_j\big),$$

where $g_j\in\Gamma_2$, $j\in\{0,\ldots,n-1\}$.

Now, if for every v_i of G, $\deg_G(v_i)=k_i$, then the weight,

$$\begin{aligned}w_{G\circ\bar{K}_n}(v_i^j)&=\sum_{v^*\in N_{G\circ\bar{K}_n}(v_i^j)} l_{G\circ\overline{K}_n}(v^*)\\&=\left(\exp(\Gamma_1)\sum_{x\in N_G(v_i)} l(x), k_i\operatorname{sum}(\Gamma_2)\right)=(\mathbf{0},g_0),\end{aligned}$$

where $\mathbf{0}$ is the identity element of Γ_1. □

Corollary 3.4. *Let Γ_1 be an abelian group with the exponent n. If G is an Eulerian and Γ_1-distance magic graph, then $G \circ \bar{K_n}$ is a $(\Gamma_1 \times \Gamma_2)$-distance magic graph, where Γ_2 is any abelian group of order n having exactly one involution.*

Observation 3.5. *Theorem 3.3 and Corollary 3.4 can be recursively used to obtain an infinite number of group distance magic graphs, since $\exp(\Gamma_1 \times \Gamma_2) = \mathrm{lcm}(\exp(\Gamma_1), \exp(\Gamma_2))$ for any two abelian groups Γ_1 and Γ_2.*

Theorem 3.6. *If Γ is an abelian group with identity $\mathbf{0}$ and G is a Γ-distance magic graph such that for any v of G, $\exp(\Gamma)$ divides $\deg_G(v)$, then $G \circ G$ is a $(\Gamma \times \Gamma)$-distance magic graph.*

Proof. Let l be a Γ-distance magic labeling of the graph G with magic constant μ_0. Let the vertices of G be $v_0, \dots, v_{p-1}$ and for $i \in \{0, \dots, p-1\}$, denote $v_i^0, \dots, v_i^{p-1}$ to be the vertices of $G \circ G$ that replace the vertex v_i of G.

Consider the function,

$$l_{G\circ G}(v_i^j) = \big(l(v_i), l(v_j)\big),$$

for all i and j.

Since for any i, $\deg_G(v_i) = k_i \exp(\Gamma)$, where $1 \le k_i \le (|V(G)| - 1)/\exp(\Gamma)$, then the weights,

$$\begin{aligned} & w_{G\circ G}(v_i^j) \\ &= \sum_{v^* \in N_{G\circ G}(v_i^j)} l_{G\circ G}(v^*) \\ &= \Bigg(\Big(|\Gamma| \sum_{x\in N_G(v_i)} l(x) + k_i \exp(\Gamma) l(v_i)\Big), \sum_{x\in N_G(v_i)} l(x) + k_i \exp(\Gamma)\,\mathrm{sum}(\Gamma) \Bigg) \\ &= (\mathbf{0}, \mu_0). \end{aligned}$$

□

Observation 3.7. *If G is a p-regular graph with prime p, in Theorem 3.6, then Γ must be the elementary abelian p-group. Also, in Theorem 3.6, if G is a biregular graph with regularities p_1 and p_2, where p_1 and p_2 are distinct primes, then no abelian group Γ exists.*

4. Group distance magicness of lexicographic product and direct product of two graphs

When G is a regular graph and H is a balanced-*dmg*, the distance magicness and the group distance magicness of $G \circ H$ (only when $H \cong C_4$) and $G \times H$ are characterized by Theorem 1.3 and 1.14, respectively. In the case of a nonregular graph G, analogous to Problem 1.4 and 1.5, natural questions arise on the existence of group distance magic labeling of $G \circ H$ and $G \times H$. This section provides partial solutions to these problems.

Throughout this section, we assume that H is a balanced-*dmg* on either 2^k or $4k+2$ vertices and $\mathcal{A} = \{a_0, a_1, \ldots, a_{|\mathcal{A}|-1}\}$ is a finite abelian group with a_0 as its identity element. Observe that $C_4 \cong K_4 - M$ is balanced-*dmg* of order 4 and $C_{4k+2}^{2k} \cong K_{4k+2} - M'$ is a balanced-*dmg* of order $4k+2$, where M and M' are perfect matching in K_4 and K_{4k+2}, respectively.

Theorem 4.1. *Let G be a graph on n vertices and Γ be an abelian group with $|\Gamma| = (4k+2)n$ such that $\Gamma \cong \mathbb{Z}_{4k+2} \times \mathcal{A}$, where $k \geq 1$ and $\mathcal{A}$ an abelian group with $|\mathcal{A}| = n$.*

(i) *If the degrees of the vertices of G are either all even or all odd, then $G \circ C_{4k+2}^{2k}$ is Γ-distance magic.*
(ii) *If there exists a constant $m \in \mathbb{N}$ such that $\deg_G(u) \equiv m \pmod{4k+2}$, for all $u \in V(G)$, then $G \times C_{4k+2}^{2k}$ is Γ-distance magic.*

Proof. Let G be a graph with the vertices $u_0, \ldots, u_{n-1}$ and $H \cong C_{4k+2}^{2k}$ be a balanced-*dmg* with the vertices $x^0, x^{0'}, \ldots, x^{2k}, x^{(2k)'}$. For any $i \in \{0, \ldots, n-1\}$, let $H_i = \{x_i^0, x_i^{0'}, \ldots, x_i^{2k}, x_i^{(2k)'}\}$ be the vertices of $G \circ H$ and $G \times H$ that replace u_i of G.

Using the isomorphism $\phi : \Gamma \to \mathbb{Z}_{4k+2} \times \mathcal{A}$, we identify $g \in \Gamma$ with its image $\phi(g) = (z, a_i)$, where $z \in \mathbb{Z}_{4k+2}$ and $a_i \in \mathcal{A}$, i varies from 0 to $n-1$.

For all i and for $j \in \{0, \ldots, 2k\}$, define a function l as,

$$\begin{aligned} l(x_i^j) &= (j, a_i) \\ l(x_i^{j'}) &= (4k+1, a_0) - l(x_i^j). \end{aligned}$$

Note that the label sum of all the vertices of H_i is $(2k+1, a_0)$, which is independent of i.

For all i, if the degree of vertex u_i is $2t_i$ for some $t_i \geq 1$, then for every vertex $v \in H_i$,

$$\begin{aligned} w_{G\circ H}(v) &= \sum_{\substack{v^* \in N_{G\circ H}(v),\\ v^* \notin N_{G\circ H[H_i]}(v)}} l(v^*) + \sum_{v^{**} \in N_{G\circ H[H_i]}(v)} l(v^{**}) \\ &= 2t_i(2k+1, a_0) + 2k(4k+1, a_0) = (2k+2, a_0), \end{aligned}$$

and, if the degree of vertex u_i is $2t_i + 1$, for some $t_i \geq 0$, then for every vertex $v \in H_i$,

$$\begin{aligned} w_{G\circ H}(v) &= \sum_{\substack{v^* \in N_{G\circ H}(v),\\ v^* \notin N_{G\circ H[H_i]}(v)}} l(v^*) + \sum_{v^{**} \in N_{G\circ H[H_i]}(v)} l(v^{**}) \\ &= (2t_i+1)(2k+1, a_0) + 2k(4k+1, a_0) = (1, a_0). \end{aligned}$$

Further, the degree of each vertex u of G is congruent to m modulo $(4k+2)$. Then, for every vertex v of $G \times H$,

$$w_{G\times H}(v) = \sum_{v^* \in N_{G\times H}(v)} l(v^*)$$
$$= 2k((4k+2)k' + m)(4k+1, a_0) = (-2mk, a_0).$$

□

In the following results, we assume G is a graph with vertices $u_0, \dots, u_{n-1}$ and H is a balanced-*dmg* with the vertices $x^0, x^{0'}, \dots, x^{2^{k-1}-1}, x^{(2^{k-1}-1)'}$, in which x^j and $x^{j'}$ are the twin vertices, for all $j \in \{0, \dots, 2^{k-1}-1\}$. Moreover, we fix $\bigcup_{i=0}^{n-1} H_i$ to be the vertex set of $G \circ H$ and $G \times H$, where for any $i \in \{0, \dots, n-1\}$, $H_i = \{x_i^0, x_i^{0'}, \dots, x_i^{2^{k-1}-1}, x_i^{(2^{k-1}-1)'}\}$ is the set of vertices that replaces the vertex u_i of G. Note that x_i^j and $x_i^{j'}$ are the twin vertices in $G \circ H$ and $G \times H$.

Lemma 4.2. *Let G be a graph on n vertices and Γ be an abelian group with $|\Gamma| = 2^k n$, where $k \geq 2$ such that $\Gamma \cong \mathbb{Z}_{2^s} \times \mathcal{A}$ for $1 \leq s \leq k-1$, $\mathcal{A}$ an abelian group with $|\mathcal{A}| = 2^{k-s}n$. Let H be a balanced-dmg on 2^k vertices. Then,*

(i) *$G \circ H$ is Γ-distance magic.*
(ii) *If there exists a constant $m \in \mathbb{N}$ such that $\deg_G(u) \equiv m \mod 2^s$ for all $u \in V(G)$, then $G \times H$ is Γ-distance magic.*

Proof. Using the isomorphism $\phi : \Gamma \to \mathbb{Z}_{2^s} \times \mathcal{A}$, we identify $g \in \Gamma$ with its image $\phi(g) = (z, a_i)$, where $z \in \mathbb{Z}_{2^s}$ and $a_i \in \mathcal{A}$, i varies from 0 to $2^{k-s}n - 1$.
For $i \in \{0, \dots, n-1\}$ and $\alpha \in \{0, \dots, 2^{s-1}-1\}$, define a function l as

$$l(x_i^j) = \left(\alpha, a_{(j \bmod 2^{k-s}) + 2^{k-s} i}\right), \text{ where } \alpha 2^{k-s} \leq j \leq (\alpha+1)2^{k-s} - 1,$$
$$l(x_i^{j'}) = (2^s - 1, a_0) - l(x_i^j).$$

Now for each $i = 0, \dots, n-1$, the label sum of all the vertices of H_i is,

$$2^{k-1}\left(2^s - 1, a_0\right) = \left(-2^{k-1}, a_0\right) = (0, a_0),$$

which is the identity element of $\mathbb{Z}_{2^s} \times \mathcal{A}$ and label sum is independent of i.
Note that the degree of each vertex of H is $2r$. For all $i = 0, \dots, n-1$, the vertex $v \in H_i$ has weight,

$$w_{G\circ H}(v) = \sum_{\substack{v^* \in N_{G\circ H}(v),\\ v^* \notin N_{G\circ H[H_i]}(v)}} l(v^*) + \sum_{v^{**} \in N_{G\circ H[H_i]}(v)} l(v^{**})$$
$$= \deg_G(u_i)(0, a_0) + r(2^s - 1, a_0) = (-r, a_0).$$

Moreover, if the degree of each vertex u of G is congruent to m modulo 2^s then, for every v of $G \times H$,

$$w_{G\times H}(v) = \sum_{v^* \in N_{G\times H}(v)} l(v^*) = r(2^s k' + m)(2^s - 1, a_0) = (-mr, a_0).$$

□

Lemma 4.3. *Let G be a graph on n vertices and Γ be an abelian group with $|\Gamma| = 2^k n$, such that $\Gamma \cong \mathbb{Z}_{2^s} \times \mathcal{A}$, where $2 \le k \le s$ and $\mathcal{A}$ is an abelian group with $|\mathcal{A}| = 2^{k-s}n$. Let H be a balanced-dmg on 2^k vertices.*

(i) *If there exists a constant $m \in \mathbb{N}$ such that $\deg_G(u) \equiv m \mod 2^{s-1}$ for all $u \in V(G)$, then $G \circ H$ is Γ-distance magic.*

(ii) *If there exists a constant $m \in \mathbb{N}$ such that $\deg_G(u) \equiv m \mod 2^s$ for all $u \in V(G)$, then $G \times H$ is Γ-distance magic.*

Proof. Using the isomorphism $\phi : \Gamma \to \mathbb{Z}_{2^s} \times \mathcal{A}$, we identify $g \in \Gamma$ with its image $\phi(g) = (z, a_i)$, where $z \in \mathbb{Z}_{2^s}$ and $a_i \in \mathcal{A}$, i varies from 0 to $2^{k-s}n - 1$.

Consider the function l,

$$l(x_i^j) = \big((2^{k-1}i + j) \mod 2^{s-1}, a_{\lfloor 2^{k-s}i \rfloor}\big),$$
$$l(x_i^{j'}) = \big(2^s - 1, a_0\big) - l(x_i^j),$$

where $i \in \{0, \ldots, n-1\}$ and $j \in \{0, \ldots, 2^{k-1} - 1\}$. For each $i = 0, \ldots, n-1$, the label sum of all the vertices of H_i is $(-2^{k-1}, a_0)$, which is independent of i. Recall that the degree of any vertex of H is $2r$. Since the degree of any vertex u of G is congruent to m modulo 2^{s-1}, for all $i = 0, \ldots, n-1$, the vertex $v \in H_i$ has weight,

$$\begin{aligned} w_{G\circ H}(v) &= \sum_{\substack{v^* \in N_{G\circ H}(v),\\ v^* \notin N_{G\circ H[H_i]}(v)}} l(v^*) + \sum_{v^{**} \in N_{G\circ H[H_i]}(v)} l(v^{**}) \\ &= (2^{s-1}k' + m)(-2^{k-1}, a_0) + r(2^s - 1, a_0) = (-r - 2^{k-1}m, a_0). \end{aligned}$$

On the other hand, if the degree of any vertex u of G is congruent to m modulo 2^s then for every v of $G \times H$,

$$w_{G\times H}(v) = \sum_{v^* \in N_{G\times H}(v)} l(v^*) = r(2^s k' + m)(2^s - 1, a_0) = (-mr, a_0).$$

□

Theorem 4.4. *Let G be a graph on n vertices and Γ be an abelian group with $|\Gamma| = 2^k n$, where $k \ge 2$, $n = 2^s(2t + 1)$, for some nonnegative integers s and t. Let H be a balanced-dmg on 2^k vertices.*

(i) *If there exists a constant $m \in \mathbb{N}$ such that $\deg_G(u) \equiv m \mod 2^{k+s-1}$ for all $u \in V(G)$, then $G \circ H$ is Γ-distance magic.*

(ii) *If there exists a constant $m \in \mathbb{N}$ such that $\deg_G(u) \equiv m \mod 2^{k+s}$ for all $u \in V(G)$, then $G \times H$ is Γ-distance magic.*

Proof. By the fundamental theorem on finite abelian groups, $\Gamma \cong \mathbb{Z}_{2^{n_0}} \times \mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_r^{n_r}}$ where $2^k n = 2^{n_0} \prod_{i=1}^{r} p_i^{n_i}$, p_i's not necessarily distinct primes and $n_0 > 0$.

Note that if any vertex u of G is such that $\deg_G(u) \equiv m \mod 2^{k+s}$, then there exist unique integers m_i such that $\deg_G(u) \equiv m_i \mod 2^{n_0}$, where $n_0 \in \{1, \ldots, k+s-1\}$.

For each Γ isomorphic to $\mathbb{Z}_{2^{n_0}} \times \mathcal{A}$ with $1 \leq n_0 \leq k-1$, the result follows from Lemma 4.2 and for each Γ isomorphic to $\mathbb{Z}_{2^{n_0}} \times \mathcal{A}$ with $k \leq n_0 \leq k+s$, the result follows from Lemma 4.3, where $\mathcal{A}$ is an abelian group with $|\mathcal{A}| = n/2^{n_0-k}$. □

Now one can verify that Theorem 1.11 and Theorem 1.12 can be obtained by substituting $k = 2$ in Theorem 4.4.

Theorem 4.5. *Let G be a graph on n-odd vertices and Γ be an abelian group with $|\Gamma| = 2^k n$, where $k \geq 2$. If all the vertices of G are of even degree and H is a balanced-dmg on 2^k vertices, then $G \circ H$ is a Γ-distance magic graph.*

Proof. If Γ is isomorphic to $\mathbb{Z}_{2^p} \times \mathcal{A}$ for $p \in \{1, \ldots, k-1\}$, then the result follows from Lemma 4.2. Now suppose that Γ is isomorphic to $\mathbb{Z}_{2^k} \times \mathcal{A}$, where $|\mathcal{A}| = n$.

Using the isomorphism $\phi : \Gamma \to \mathbb{Z}_{2^k} \times \mathcal{A}$, we identify $g \in \Gamma$ with its image $\phi(g) = (z, a_i)$, where $z \in \mathbb{Z}_{2^k}$ and $a_i \in \mathcal{A}$, i varies from 0 to $n-1$.

For all $i \in \{0, \ldots, n-1\}$ and $j \in \{0, \ldots, 2^{k-1}-1\}$, define l on $G \circ H$ as,

$$l(x_i^j) = (2j, a_i) \text{ and}$$
$$l(x_i^{j'}) = (2^k - 1, a_0) - l(x_i^j).$$

Note that the label sum of all the vertices of H_i is,

$$(2^{k-1}(2^k - 1) \mod 2^k, a_0) = (-2^{k-1}, a_0),$$

which is independent of i.

Since for any u_i of G, $\deg_G(u_i) = 2t_i$ with $t_i \geq 1$ and for any x of H, $\deg_H(x) = 2r$, the degree of v in $G \circ H$ is $2(2^k t_i + r)$. Then the weight of any vertex $v \in H_i$ is,

$$\begin{aligned} w_{G\circ H}(v) &= \sum_{\substack{v^* \in N_{G\circ H}(v) \\ v^* \notin N_{G\circ H[H_i]}(v)}} l(v^*) + \sum_{v^{**} \in N_{G\circ H[H_i]}(v)} l(v^{**}) \\ &= 2t_i(-2^{k-1}, a_0) + r(2^k - 1, a_0) = (-r, a_0). \end{aligned}$$

□

Corollary 4.6. *Let t be an odd integer. Let $G \cong K_{m_1, m_2, \ldots, m_t}$ be a complete t-partite graph with, $m = \sum_{i=1}^{t} m_i$ and all m_i's are odd. If Γ is an abelian group with $|\Gamma| = 2^k m$, then $G \circ H$ is a Γ-distance magic graph, where H is a balanced-dmg on 2^k vertices.* □

For an abelian group Γ, the following result discusses the Γ-distance magic labeling of $K_{m,n} \circ H$, where m and n are of different parity and H is a balanced-*dmg* on 2^k vertices.

Theorem 4.7. *Let $K_{m,n}$ be a complete bipartite graph with m is even and n is odd. If Γ is an abelian group with $2^k(m+n)$ elements, where $k \geq 2$ and H is a $2r$-regular balanced-dmg on 2^k vertices with r-odd, then $K_{m,n} \circ H$ is Γ-distance magic.*

Proof. If Γ is isomorphic to $\mathbb{Z}_{2^p} \times \mathcal{A}$ with $p \in \{1, \dots, k-1\}$, then the assertion follows from Lemma 4.2. Suppose that Γ is isomorphic to $\mathbb{Z}_{2^k} \times \mathcal{A}$, where $|\mathcal{A}| = m + n$.

Let $G \cong K_{m,n}$ have the partition sets $X = \{u_0, \dots, u_{m-1}\}$ and $Y = \{v_0, \dots, v_{n-1}\}$. Then for each $i \in \{0, \dots, m-1\}$, let $X_i = \{x_i^0, x_i^{0'}, \dots, x_i^{2^{k-1}-1}, x_i^{(2^{k-1}-1)'}\}$ be the vertex set of $G \circ H$, that replace the vertex u_i of G. Similarly, for all $j \in \{0, \dots, n-1\}$, let $Y_j = \{y_j^0, y_j^{0'}, \dots, y_j^{2^{k-1}-1}, y_j^{(2^{k-1}-1)'}\}$ be the vertex set of $G \circ H$, that replace the vertex v_j of G.

Let the vertex set of $G \circ H$ be $X' \cup Y'$, where $X' = \bigcup_{i=0}^{m-1} X_i$ and $Y' = \bigcup_{j=0}^{n-1} Y_j$.

Using the isomorphism $\phi : \Gamma \to \mathbb{Z}_{2^k} \times \mathcal{A}$, we identify $g \in \Gamma$ with its image $\phi(g) = (z, a_i)$, where $z \in \mathbb{Z}_{2^k}$ and $a_i \in \mathcal{A}$, i varies from 0 to $m + n - 1$.

For each $q \in \{0, \dots, 2^{k-1} - 1\}$, define l on X' as,

$$l(x_i^q) = ((2^{k-1} + 1)q, a_i), \text{ and}$$
$$l(x_i^{q'}) = (2^{k-1} - 1, a_0) - l(x_i^q), \text{ for all } i = 0, \dots, m-1.$$

Again for each $q \in \{0, \dots, 2^{k-1} - 1\}$, define l on Y' as,

$$l(y_j^q) = (2q, a_{m+j}), \text{ and}$$
$$l(y_j^{q'}) = (2^k - 1, a_0) - l(y_j^q), \text{ for all } j = 0, \dots, n-1.$$

Then for all $i \in \{0, \dots, m-1\}$, the label sum of all vertices of X_i is,

$$(2^{k-1}(2^{k-1} - 1), a_0) = (2^{k-1}, a_0).$$

Similarly for all $j \in \{0, \dots, n-1\}$, the label sum of all vertices of Y_j is,

$$(2^{k-1}(2^k - 1), a_0) = (2^{k-1}, a_0).$$

Since H is $2(2t+1)$-regular graph, for every x of X_i,

$$\begin{aligned} w_{G \circ H}(x) &= \sum_{\substack{x^* \in N_{G \circ H}(x), \\ x^* \notin N_{G \circ H[X_i]}(x)}} l(x^*) + \sum_{x^{**} \in N_{G \circ H[X_i]}(x)} l(x^{**}) \\ &= n(2^{k-1}, a_0) + (2t+1)(2^{k-1} - 1, a_0) = \big(-(2t+1), a_0\big), \end{aligned}$$

and for every y of Y_j,

$$\begin{aligned} w_{G \circ H}(y) &= \sum_{\substack{y^* \in N_{G \circ H}(y), \\ y^* \notin N_{G \circ H[Y_j]}(y)}} l(y^*) + \sum_{y^{**} \in N_{G \circ H[Y_j]}(y)} l(y^{**}) \\ &= m(2^{k-1}, a_0) + (2t+1)(2^k - 1, a_0) = \big(-(2t+1), a_0\big), \end{aligned}$$

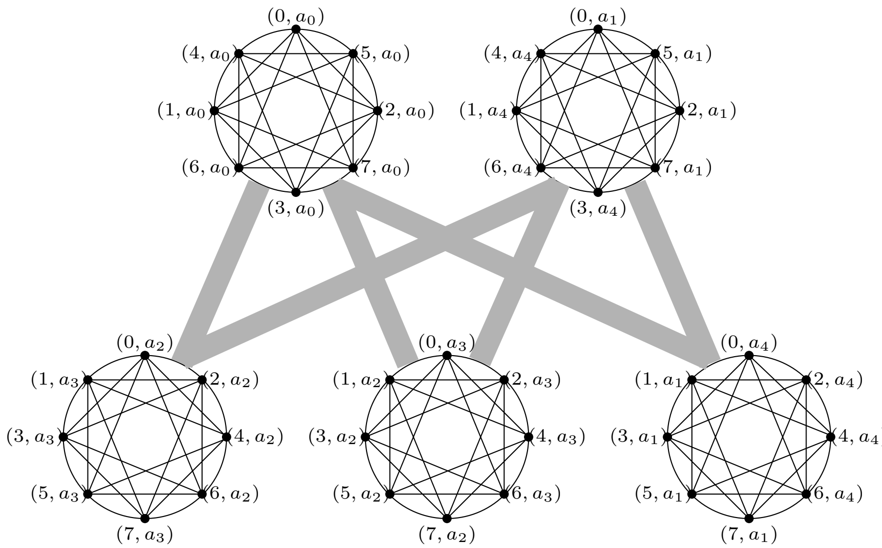

FIGURE 2. $K_{2,3} \circ (K_8 - M)$ and its $(\mathbb{Z}_8 \times \mathcal{A})$-distance magic labeling.

which completes the proof. □

Note that Figure 2 gives the $\mathbb{Z}_8 \times \mathcal{A}$-distance magic labeling of $K_{2,3} \circ (K_8 - M)$, where $\mathcal{A} = \{a_0, a_1, a_2, a_3, a_4 : a_0$ is identity and $a_1 + a_4 = a_0 = a_2 + a_3\}$ and M is any perfect matching in K_8.

5. Conclusion

In this paper, we obtain necessary conditions for a graph to be group distance magic and characterize the group distance magic labeling of a tree, a subclass of bi-regular graphs, and the lexicographic and direct product of a nonregular graph with a balanced distance magic graph. In addition, we present three techniques to build recursively larger group distance magic graphs from the existing ones. Further, we identify infinitely many graphs, which are not group distance magic with respect to any abelian group with one involution.

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