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ON UNIFORMLY RESOLVABLE $(C_4, K_{1,3})$-DESIGNS

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Abstract. In this paper, we consider the uniformly resolvable decompositions of the complete graph K_v minus a 1-factor ($K_v - I$) into subgraphs where each resolution class contains only blocks isomorphic to the same graph. We completely determine the spectrum for the case in which all the resolution classes consist of either 4-cycles or 3-stars.

1. Introduction

Given a collection of graphs $\mathcal{H}$, an *$\mathcal{H}$-decomposition* of a graph G is a decomposition of the edges of G into isomorphic copies of graphs in $\mathcal{H}$, called *blocks*. Such a decomposition is *resolvable* if it is possible to partition the blocks into classes called *resolution classes* such that every vertex of G appears exactly once in each resolution class.

A resolvable $\mathcal{H}$-decomposition of G is also referred to as an *$\mathcal{H}$-factorization of G* and each resolution class is called an *$\mathcal{H}$-factor of G*. The case where $\mathcal{H}$ is the singleton $\{K_2\}$ is known as a *1-factorization of G* and for $G = K_v$ it is well known to exist if and only if v is even. A resolution class of a 1-factorization, a pairing of all points, is also known as a *1-factor* or a *perfect matching*.

In many cases we wish to impose further constraints on the resolution classes of an $\mathcal{H}$-decomposition. For example, a resolution class is said to be *H-uniform* if its blocks are all isomorphic to the graph H. An H-decomposition is uniformly resolvable if each resolution class is H-uniform for a suitable $H \in \mathcal{H}$. In particular, by writing (H_1, H_2)-URD(r_1, r_2) of G we will mean a uniformly resolvable $\{H_1, H_2\}$-decomposition of G with exactly r_i resolution classes that are H_i-uniform for $i = 1, 2$. Uniformly resolvable decompositions of K_v and $K_v - I$, where I is a 1-factor of K_v have been studied widely in the literature, see for example [1]–[6], [9]–[16], [18]–[23] and [25]–[30]. Note that some of the papers in these references are

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on special cases of Hamilton-Waterloo problems that are basically (H_1, H_2)-URD(r_1, r_2) of K_v, where H_i's are cycles.

In what follows, we will denote by (a_1, a_2, a_3, a_4) the 4-cycle C_4 having vertex set $\{a_1, a_2, a_3, a_4\}$ and edge set $\{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_4, a_1\}\}$ and by $(a_1; a_2, a_3, a_4)$ the 3-*star*, $K_{1,3}$, having the vertex set $\{a_1, a_2, a_3, a_4\}$, and the edge set $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}\}$.

Given $v \equiv 0 \pmod 4$, define $I(v)$ accordingly to Table 1. It was proven

v	$I(v)$
$0 \pmod{12}$	$\{(\frac{v-2}{2} - 3x, 4x) : x = 0, 1, \ldots, \frac{v-6}{6}\}$
$4 \pmod{12}$	$\{(\frac{v-2}{2} - 3x, 4x) : x = 0, 1, \ldots, \frac{v-4}{6}\}$
$8 \pmod{12}$	$\{(\frac{v-2}{2} - 3x, 4x) : x = 0, 1, \ldots, \frac{v-2}{6}\}$

Table 1. The set $I(v)$.

in [11] that a $(C_4, K_{1,3})$-URD(r, s) of λK_v (the λ-fold complete graph) cannot exist for λ odd. So in this paper, as is customary in such situations, we remove a 1-factor I from K_v, note that this implies that v is even. Furthermore, as both $K_{1,3}$ and C_4 have 4 vertices, $v \equiv 0 \mod 4$ is a necessary condition for the existence of a $(C_4, K_{1,3})$-URD(r, s). Note that we also use the notation $(C_4, K_{1,3})$-URD$(v; r, s)$ if the order v needs to be specified. We therefore investigate the existence of a $(C_4, K_{1,3})$-URD(r, s) of $K_v - I$ and prove the following result:

Main Theorem. *There exists a* $(C_4, K_{1,3})$-URD(r, s) *of* $K_v - I$ *if and only if* $v \equiv 0 \pmod 4$ *and* $(r, s) \in I(v)$, *where* $I(v)$ *is given in Table* 1.

In this paper we will use the following notation. Let G be the complete multipartite graph in which each partite set of the set of vertices is $\{0_i, 1_i, \ldots, (v-1)_i\}$, for $i = 1, \ldots, m$. If H is a subgraph of G, we denote by $H + j$, for any $j \in \{0, 1, \ldots, v-1\}$, the subgraph isomorphic to H obtained under the natural action of $\mathbb{Z}_v$ on G, defined by $x_i \to (x+j)_i$ for any $x, j \in \{0, 1, \ldots, v-1\}$ and $i \in \{1, \ldots, m\}$.

2. Necessary conditions

In this section, we will give the necessary conditions for the existence of a uniformly resolvable decomposition of $K_v - I$ into r classes of 4-cycles and s classes of 3-stars.

Lemma 2.1. *If there exists a* $(C_4, K_{1,3})$-URD(r, s) *of* $K_v - I$, *then* $(r, s) \in I(v)$.

Proof. Let D be a $(C_4, K_{1,3})$-URD$(v; r, s)$ of $K_v - I$. Counting the edges of $K_v - I$ that appear in D we obtain

$$\frac{4rv}{4} + \frac{3sv}{4} = \frac{v(v-2)}{2},$$

and hence

$$4r + 3s = 2(v - 2). \tag{2.1}$$

The above identity gives $s \equiv 0 \pmod 4$. Now letting $s = 4x$ in Equation (2.1), we obtain $r = (v-2)/2 - 3x$; since r and s cannot be negative, and x is an integer, the value of x has to be in the range given in the definition of $I(v)$. □

3. Constructions and related structures

In this section, we will introduce some useful definitions, results and discuss constructions we will use in proving the main result. For missing terms or results that are not explicitly explained in the paper, the reader is referred to [8] and its online updates. A resolvable $\mathcal{H}$-decomposition of the complete multipartite graph with u parts, each of size g is known as a resolvable group divisible design $\mathcal{H}$-RGDD of type g^u, the parts of size g are called the groups of the design. When $\mathcal{H} = K_n$ we will call it an n-RGDD.

A $(C_4, K_{1,3})$-URGDD (r, s) of type g^u is a uniformly resolvable decomposition of the complete multipartite graph with u parts each of size g into r classes containing only copies of 4-cycles and s classes containing only copies of 3-stars.

If the blocks of an $\mathcal{H}$-RGDD of type g^u can be partitioned into partial parallel classes, each of which contain all points except those of one group, we refer to the decomposition as a *frame*. When $\mathcal{H} = K_n$ we will call it an *n-frame* and it is easy to deduce that the number of partial parallel classes missing a specified group G is $|G|/(n-1)$.

An incomplete resolvable $(C_4, K_{1,3})$-decomposition of K_{v+h} with a hole of size h is a $(C_4, K_{1,3})$-decomposition of $K_{v+h} - K_h$ in which there are two types of classes, full classes and partial classes which cover every point except those in the hole (the points of K_h are referred to as the hole). Specifically a $(C_4, K_{1,3})$-IURD$(v+h, h; [(r_1, s_1)], [(r_2, s_2)])$ is a uniformly resolvable $(C_4, K_{1,3})$-decomposition of $K_{v+h} - K_h$ with r_1 partial classes of 4-cycles, s_1 partial classes of 3-stars and one partial 1-factor which cover only the points not in the hole, r_2 full classes of 4-cycles which cover every point of K_{v+h} and s_2 full classes of 3-stars which cover every point of K_{v+h}.

Let $C_m(n)$ denote the graph with the vertex set $\{\bigcup_{i=1}^m X_i\}$ with $|X_i| = n$ for $i = 1, 2, ..., m$ and $|X_i| \cap |X_j| = \emptyset$ for $i \neq j$, and the edge set

$$\{\{u, v\} \ : u \in X_i,\ v \in X_j,\ i - j \equiv 1 \pmod m \text{ or } j - i \equiv 1 \pmod m\}$$

We also need the following definitions. Let (s_1, t_1) and (s_2, t_2) be two pairs of nonnegative integers. Define $(s_1, t_1) + (s_2, t_2) = (s_1 + s_2, t_1 + t_2)$. If X and Y are two sets of pairs of nonnegative integers, then $X + Y$ denotes the set $\{(s_1, t_1) + (s_2, t_2) : (s_1, t_1) \in X, (s_2, t_2) \in Y\}$. If X is a set of pairs of nonnegative integers and h is a positive integer, then $h * X$ denotes the set of all pairs of nonnegative integers which can be obtained by adding any h elements of X together (repetitions of elements of X are allowed).

The following result can be proven in a similar manner as in [17].

Theorem 3.1. *Let v, g, t, h and u be positive integers such that $v = gtu+h$. If there exists*

(i) a 2-frame $\mathcal{F}$ of type g^u;
(ii) a $(C_4, K_{1,3})$-URD(r_1, s_1) of $K_h - I$ with $(r_1, s_1) \in J_1$;
(iii) a $(C_4, K_{1,3})$-URGDD(r_2, s_2) of type t^2 with $(r_2, s_2) \in J_2$;
(iv) a $(C_4, K_{1,3})$-IURD$(gt+h, h; [(r_1, s_1)], [(r_3, s_3)])$ with $(r_1, s_1) \in J_1$ and $(r_3, s_3) \in J_3 \subseteq g * J_2$;

then there exists a $(C_4, K_{1,3})$-URD(r, s) for each $(r, s) \in J_1 + u * J_3$.

4. Small cases

Lemma 4.1. *There exists a $(C_4, K_{1,3})$-URD$(1, 0)$ of $K_4 - I$.*

Proof. Let $V(K_4) = \mathbb{Z}_4$ and $I = \{\{0, 2\}, \{1, 3\}\}$. The class of 4-cycle is $\{(0, 1, 2, 3)\}$. □

Lemma 4.2. *There exists a $(C_4, K_{1,3})$-URGDD(r, s) of type 2^4, with*

$$(r, s) \in \{(3, 0), (0, 4)\}.$$

Proof. The case $(3, 0)$ is given in [9, Theorem 2.2] and the case $(0, 4)$ in [18, Lemma 4.1]. □

Lemma 4.3. *There exists a $(C_4, K_{1,3})$-URD(r, s) of $K_8 - I$ with*

$$(r, s) \in \{(3, 0), (0, 4)\}.$$

Proof. The result follows from Lemma 4.2. □

Lemma 4.4. *There exists a $(C_4, K_{1,3})$-URD(r, s) of $K_{12} - I$ with*

$$(r, s) \in \{(5, 0), (2, 4)\}.$$

Proof. Let $V(K_{12}) = \mathbb{Z}_{12}$. Then the case $(5, 0)$ corresponds to a $(C_4, K_{1,3})$-URGDD$(5, 0)$ of type 2^6 which is known to exist by [9, Theorem 2.2].

For the case $(2, 4)$ take

$$I = \{\{4, 8\}, \{1, 3\}, \{7, 11\}, \{6, 10\}, \{5, 9\}, \{0, 2\}\}$$

and the classes listed below:

$$\{(0; 4, 5, 6), (7; 8, 9, 10), (11; 1, 2, 3)\},$$
$$\{(1; 5, 6, 7), (4; 9, 10, 11), (8; 0, 2, 3)\},$$
$$\{(2; 4, 6, 7), (5; 8, 10, 11), (9; 0, 1, 3)\},$$
$$\{(3; 4, 5, 7), (6; 8, 9, 11), (10; 0, 1, 2)\},$$
$$\{(0, 1, 4, 7), (2, 3, 6, 5), (8, 11, 9, 10)\},$$
$$\{(0, 11, 10, 3), (1, 2, 9, 8), (4, 6, 7, 5)\}.$$

□

Lemma 4.5. *There exists a* $(C_4, K_{1,3})$-URGDD(r,s) *of type* 12^2*, with*

$$(r,s) \in \{(6,0),(3,4),(0,8)\}.$$

Proof. Take the groups to be $\{0,1,\dots,11\}$ and $\{0',1',\dots,11'\}$. The case $(6,0)$ is given in [9, Theorem 2.2] and the case $(0,8)$ in [7, Lemma 2.3]. For the case $(3,4)$ the classes are:

$$\{(1;4',9',10'),(2;0',6',8'),(3;5',7',11'),$$
$$(1';6,7,10),(2';0,5,9),(3';4,8,11)\},$$
$$\{(4;2',7',11'),(5;10',9',1'),(6;3',0',8'),$$
$$(4';2,9,0),(5';1,8,11),(6';3,7,10)\},$$
$$\{(7;3',5',0'),(8;11',2',4'),(9;6',1',10'),$$
$$(7';1,5,11),(8';10,3,4),(9';0,2,6)\},$$
$$\{(0;6',8',1'),(10;3',5',7'),(11;2',4',9'),$$
$$(0';3,4,8),(10';2,6,7),(11';1,5,9)\},$$
$$\{(1',1,0',11),(2',10,10',3),(2,3',0,11'),$$
$$(4,6',8,9'),(5,5',9,8'),(6,4',7,7')\},$$
$$\{(2',1,8',7),(2,1',8,7'),(3,3',9,9'),$$
$$(4,5',0,10'),(5,4',10,0'),(6,6',11,11')\},$$
$$\{(3',1,6',5),(2',6,5',2),(3,1',4,4'),$$
$$(7,9',10,11'),(8,8',11,10'),(9,7',0,0')\}.$$

□

Lemma 4.6. *There exists a* $(C_4, K_{1,3})$-URGDD(r,s) *of type* 12^3*, with*

$$(r,s) \in \{(12,0),(6,8),(0,16)\}.$$

Proof. The case $(12,0)$ is given in [9, Theorem 2.2] and the case $(0,16)$ follows by [18, Lemma 4.6]. For the case $(6,8)$ take the groups $\{0_i,1_i,\dots,11_i\}$ for $i=1,2,3$. Then consider:

$$\begin{aligned} B = &\{(j_1; j_2, (j+1)_2, (j+2)_2),\ j=0,4,8\}\\ &\cup \{(j_2; j_3, (j+1)_3, (j+2)_3),\ j=3,7,11\}\\ &\cup \{(j_3; j_1, (j+1)_1, (j-1)_1),\ j=2,6,10\},\end{aligned}$$

$$\begin{aligned} C = &\{(j_1; (j+3)_2, (j+4)_2, (j+5)_2),\ j=0,4,8\}\\ &\cup \{(j_2; (j+3)_3, (j+4)_3, (j+5)_3),\ j=2,6,10\}\\ &\cup \{(j_3; (j+2)_1, (j+7)_1, (j+9)_1),\ j=0,4,8\},\end{aligned}$$

$$\begin{aligned} D = &\{(j_1, (j+8)_2, (j+1)_1, (j-1)_2),\ j=0,4,8\}\\ &\cup \{(j_1, (j+4)_3, (j+9)_1, (j+6)_3),\ j=2,6,10\}\\ &\cup \{(j_2, (j+8)_3, (j+9)_2, (j+6)_3),\ j=1,5,9\},\end{aligned}$$

$$E = \{(j_1, (j+6)_2, (j+4)_3, (j+9)_2),\ j = 0, 4, 8\}$$
$$\cup \{(j_1, (j+6)_2, (j+4)_3, (j+9)_2),\ j = 2, 6, 10\}$$
$$\cup \{(j_1, (j+2)_3, (j+6)_1, (j+8)_3),\ j = 1, 5, 9\}.$$

Then the classes are $B+i$, $C+i$, $D+i$ for $i = 0, 1, 2, 3$ and $E+i$ for $i = 0, 1$. □

Lemma 4.7. *There exists a* $(C_4, K_{1,3})$-URGDD(r, s) *of type* 12^5*, with*

$$(r, s) \in \{(24, 0), (12, 16), (0, 32)\}.$$

Proof. There exists a $(C_4, K_{1,3})$-URGDD(r, s) of $C_5(12)$ with

$$(r, s) \in \{(12, 0), (0, 16)\}.$$

For the case $(0, 16)$ see [18]. For the case $(12, 0)$ we take a 1-factorization of $C_5(6)$ with 12 factors, expand each point by two and replace each edge by a 4-cycle. Since a 5-partite graph with part size 12 can be obtained by the union of two $C_5(12)$, we obtain a $(C_4, K_{1,3})$-URGDD(r, s) of type 12^5 with $(r, s) \in \{(24, 0), (12, 16), (0, 32)\}$. □

Lemma 4.8. *There exists a* $(C_4, K_{1,3})$-URD(r, s) *of* $K_{24} - I$ *with*

$$(r, s) \in \{(11, 0), (8, 4), (5, 8), (2, 12)\} = I(24).$$

Proof. Take a $(C_4, K_{1,3})$-URGDD(r, s) of type 12^2 with

$$(r, s) \in \{(6, 0), (3, 4), (0, 8)\}$$

which exists by Lemma 4.5. Replace each group of size 12 with the same $(C_4, K_{1,3})$-URD$(12; x, y)$, where $(x, y) \in \{(5, 0), (2, 4)\}$ which exists by Lemma 4.4. By combining these resolutions for possible values of (r, s) and (x, y) we get the result. □

Lemma 4.9. *There exists a* $(C_4, K_{1,3})$-URD(r, s) *of* $K_{36} - I$ *with*

$$(r, s) \in \{(17, 0), (14, 4), (11, 8), (8, 12), (5, 16), (2, 20)\} = I(36).$$

Proof. Take a $(C_4, K_{1,3})$-URGDD(r, s) of type 12^3 with

$$(r, s) \in \{(12, 0), (6, 8), (0, 16)\}$$

which exists by Lemma 4.6. Replace each group of size 12 with the same $(C_4, K_{1,3})$-URD$(12; x, y)$, where $(x, y) \in \{(5, 0), (2, 4)\}$ which exists by Lemma 4.4. By combining these resolutions for possible values of (r, s) and (x, y) we get the result. □

Lemma 4.10. *There exists a* $(C_4, K_{1,3})$-URGDD(r, s) *of type* 4^4 *with*

$$(r, s) \in \{(6, 0), (3, 4), (0, 8)\}.$$

Proof. The case $(6, 0)$ is given in [9, Theorem 2.2], the case $(0, 8)$ in [7, Lemma 2.6]. For the case $(3, 4)$, take the groups to be

$$\{x_i, i \in \mathbb{Z}_4\}, \{y_i, i \in \mathbb{Z}_4\}, \{z_i, i \in \mathbb{Z}_4\} \text{ and } \{t_i, i \in \mathbb{Z}_4\}.$$

The 4 resolution classes of 3-stars can be obtained from the base blocks:

$$(x_0; y_2, z_1, t_1), (y_1; z_3, t_2, x_2), (z_2; t_0, x_3, y_3), (t_3; x_1, y_0, z_0).$$

The 3 resolution classes of 4-cycles can be obtained from the base blocks:

$$(x_0, y_0, z_0, t_0), (x_0, z_0, t_1, y_1), (x_0, z_2, y_1, t_3).$$

□

Lemma 4.11. *There exists a* $(C_4, K_{1,3})$-URD(r, s) *of* $K_{16} - I$ *with*

$$(r, s) \in \{(7, 0), (4, 4), (1, 8)\} = I(16).$$

Proof. Take a $(C_4, K_{1,3})$-URGDD(r, s) of type 4^4 with

$$(r, s) \in \{(6, 0), (3, 4), (0, 8)\}$$

which exists by Lemma 4.10. Replace each group of size 4 with the same $(C_4, K_{1,3})$-URD$(4; 1, 0)$, which exists by Lemma 4.1. By combining these resolutions for possible values of (r, s) we get the result. □

Lemma 4.12. *There exists a* $(C_4, K_{1,3})$-IURD$(16, 4; [(1, 0)], [(r_2, s_2)])$ *with*

$$(r_2, s_2) \in \{(6, 0), (3, 4), (0, 8)\}.$$

Proof. Start from a $(C_4, K_{1,3})$-URGDD(r, s) of type 4^4 with $(r, s) \in \{(6, 0), (3, 4), (0, 8)\}$ which exists by Lemma 4.10 and fill in the three groups with a copy of a $(C_4, K_{1,3})$-URD$(4; 1, 0)$ in order to obtain a $(C_4, K_{1,3})$-IURD$(16, 4; [(1, 0)], [(r_2, s_2)])$ with $(r_2, s_2) \in \{(6, 0), (3, 4), (0, 8)\}$ and one group as the hole. □

Lemma 4.13. *There exists a* $(C_4, K_{1,3})$-IURD$(20, 8; [(r_1, s_1)], [(r_2, s_2)])$ *with*

$$(r_1, s_1) \in \{(3, 0), (0, 4)\} \text{ and } (r_2, s_2) \in \{(6, 0), (3, 4), (0, 8)\}.$$

Proof. Let the point set of K_{20} be $\mathbb{Z}_{20}$ and the point set $\{0, 1, \ldots, 7\}$ be the hole.

(1) Let $(r_1, s_1) = (3, 0)$. These are the 3 resolution classes of 4-cycles on $K_{20} - K_8$ on the vertices from 8 to 19:

$$\{(8, 14, 9, 15), (10, 18, 11, 19), (12, 16, 13, 17)\},$$
$$\{(8, 12, 9, 13), (10, 16, 11, 17), (14, 18, 15, 19)\},$$
$$\{(8, 16, 9, 17), (10, 14, 11, 15), (12, 18, 13, 19)\}.$$

(a) Let $(r_2, s_2) = (6, 0)$. The followings are the 6 resolution classes of 4-cycles on K_{20}:

$$\{(0,8,1,9),(2,10,3,11),(4,12,5,13),$$
$$(6,14,7,15),(16,18,17,19)\},$$
$$\{(0,10,1,11),(2,12,3,13),(4,14,5,15),$$
$$(6,16,7,17),(18,8,19,9)\},$$
$$\{(0,12,1,13),(2,14,3,15),(4,16,5,17),$$
$$(6,18,7,19),(8,10,9,11)\},$$
$$\{(0,14,1,15),(2,16,3,17),(4,18,5,19),$$
$$(6,8,7,9),(10,12,11,13)\},$$
$$\{(0,16,1,17),(2,18,3,19),(4,8,5,9),$$
$$(6,10,7,11),(12,14,13,15)\},$$
$$\{(0,18,1,19),(2,8,3,9),(4,10,5,11),$$
$$(6,12,7,13),(14,16,15,17)\}.$$

The one factor is:

$$\{\{8,9\},\{10,11\},\{12,13\},\{14,15\},\{16,17\},\{18,19\}\}.$$

(b) Let $(r_2, s_2) = (3, 4)$. The followings are the 4 resolution classes of 3-stars:

$$\{(0;8,9,10),(1;11,12,13),(2;14,15,16),$$
$$(17;3,4,18),(19;5,6,7)\},$$
$$\{(3;8,9,10),(4;11,12,13),(5;14,15,17),$$
$$(16;0,1,19),(18;2,6,7)\},$$
$$\{(6;8,10,13),(9;1,18,19),(12;2,5,11),$$
$$(14;0,4,17),(15;3,7,16)\},$$
$$\{(7;9,16,17),(8;2,18,19),(10;1,4,12),$$
$$(11;0,3,6),(13;5,14,15)\}$$

and these are the 3 resolution classes of 4-cycles:

$$\{(0,17,1,18),(2,13,3,19),(4,8,5,16),$$
$$(6,14,12,15),(7,10,9,11)\},$$
$$\{(0,15,17,19),(1,8,7,14),(2,10,13,11),$$
$$(3,12,6,16),(4,9,5,18)\},$$
$$\{(0,12,7,13),(1,15,4,19),(2,9,6,17),$$
$$(3,14,16,18),(5,10,8,11)\}.$$

The one-factor is:

$$\{\{8,9\},\{10,11\},\{12,13\},\{14,15\},\{16,17\},\{18,19\}\}.$$

(c) Let $(r_2, s_2) = (0, 8)$. The followings are the 8 resolution classes of 3-stars:

$$\{(0;8,9,10),(4;12,13,14),(11;1,2,3),$$
$$(15;5,6,7),(16;17,18,19)\},$$
$$\{(1;12,13,14),(5;16,17,18),(15;0,2,3),$$
$$(19;4,6,7),(8;9,10,11)\},$$
$$\{(2;16,17,18),(6;8,9,10),(19;0,1,3),$$
$$(11;4,5,7),(12;13,14,15)\},$$
$$\{(0;11,12,13),(1;8,9,15),(3;10,16,17),$$
$$(14;2,5,6),(18;4,7,19)\},$$
$$\{(2;8,9,19),(3;12,13,18),(10;5,7,11),$$
$$(16;0,1,14),(17;4,6,15)\},$$
$$\{(4;8,15,16),(9;3,5,18),(12;2,6,10),$$
$$(13;7,11,14),(17;0,1,19)\},$$
$$\{(5;12,13,19),(6;11,16,18),(7;8,9,17),$$
$$(10;1,2,4),(14;0,3,15)\},$$
$$\{(7;12,14,16),(8;3,5,19),(9;4,10,11),$$
$$(13;2,6,15),(18;0,1,17)\}.$$

The one-factor is:

$$\{\{8,18\},\{9,19\},\{10,13\},\{11,12\},\{14,17\},\{15,16\}\}.$$

(2) Let $(r_1, s_1) = (0, 4)$. These are the 4 resolution classes of 3-stars on $K_{20} - K_8$ on the vertices from 8 to 19:

$$\{(8;9,12,13),(10;11,16,17),(14;15,18,19)\},$$
$$\{(9;12,13,14),(16;8,11,17),(18;10,15,19)\},$$
$$\{(11;14,17,18),(12;13,16,19),(15;8,9,10)\},$$
$$\{(13;14,16,18),(17;8,9,12),(19;10,11,15)\}.$$

(a) Let $(r_2, s_2) = (6, 0)$. The followings are the 6 resolution classes of 4-cycles on K_{20}:

$$\{(0,8,1,9),(2,10,3,11),(4,12,5,13),$$
$$(6,14,7,15),(16,18,17,19\},$$
$$\{(0,10,1,11),(2,8,3,9),(4,14,12,15),$$
$$(5,16,6,18),(7,17,13,19)\},$$
$$\{(0,12,1,13),(2,14,5,15),(3,17,6,19),$$
$$(4,16,7,18),(8,10,9,11)\},$$

$$\{(0,14,1,17),(2,18,9,19),(3,13,15,16),$$
$$(4,8,5,11),(6,10,7,12)\},$$
$$\{(0,15,1,16),(2,12,10,13),(3,14,8,18),$$
$$(4,17,5,19),(6,9,7,11)\},$$
$$\{(0,18,1,19),(2,16,14,17),(3,12,11,15),$$
$$(4,9,5,10),(6,8,7,13)\}.$$

The one factor is:

$$\{\{8,19\},\{9,16\},\{10,14\},\{11,13\},\{12,18\},\{15,17\}\}.$$

(b) Let $(r_2,s_2)=(3,4)$. The followings are the 4 resolution classes of 3-stars:

$$\{(0;8,9,10),(1;11,12,13),(2;14,15,16),$$
$$(17;3,4,18),(19;5,6,7)\},$$
$$\{(3;8,9,10),(4;11,12,13),(5;14,15,17),$$
$$(16;0,1,19),(18;2,6,7)\},$$
$$\{(6;8,9,10),(11;0,5,7),(12;2,15,18),$$
$$(13;3,17,19),(14;1,4,16)\},$$
$$\{(7;12,14,17),(8;11,18,19),(9;1,4,16),$$
$$(10;2,5,13),(15;0,3,6)\}$$

and these are the 3 resolution classes of 4-cycles:

$$\{(0,12,6,14),(1,15,17,19),(2,9,11,13),$$
$$(3,16,5,18),(4,8,7,10)\},$$
$$\{(0,13,6,17),(1,10,9,18),(2,11,3,19),$$
$$(4,15,7,16),(5,8,14,12)\},$$
$$\{(0,18,4,19),(1,8,2,17),(3,12,10,14),$$
$$(5,9,7,13),(6,11,15,16)\}.$$

The one factor is:

$$\{\{8,10\},\{9,19\},\{11,12\},\{13,15\},\{14,17\},\{16,18\}\}.$$

(3) Let $(r_1,s_1)=(0,4)$ and $(r_2,s_2)=(0,8)$. These are the 4 resolution classes of 3-stars on $K_{20}-K_8$ on the vertices from 8 to 19:

$$\{(8;9,12,18),(11;13,14,19),(17;10,15,16)\},$$
$$\{(8;11,13,17),(10;12,15,18),(16;9,14,19)\},$$
$$\{(9;17,18,19),(14;10,12,15),(16;8,11,13)\},$$
$$\{(12;9,11,16),(13;10,14,18),(19;8,15,17)\}.$$

The followings are the 8 resolution classes of 3-stars:

$$\{(0;8,9,10),(1;11,12,13),(2;14,15,16),$$
$$(17;3,4,5),(18;6,7,19)\},$$
$$\{(0;11,12,13),(1;8,9,10),(2;17,18,19),$$
$$(14;3,4,5),(15;6,7,16)\},$$
$$\{(3;8,9,10),(4;11,12,15),(5;16,18,19),$$
$$(13;2,6,7),(14;0,1,17)\},$$
$$\{(3;11,12,13),(4;8,9,16),(6;14,17,19),$$
$$(10;2,5,7),(15;0,1,18)\},$$
$$\{(5;8,9,11),(7;14,16,17),(10;4,6,19),$$
$$(12;2,13,15),(18;0,1,3)\},$$
$$\{(6;8,9,16),(11;2,7,10),(15;3,5,13),$$
$$(17;0,1,18),(19;4,12,14)\},$$
$$\{(7;8,12,19),(9;2,10,14),(11;6,15,18),$$
$$(13;4,5,17),(16;0,1,3)\},$$
$$\{(8;2,10,15),(9;7,11,13),(12;5,6,17),$$
$$(18;4,14,16),(19;0,1,3)\}.$$

The one factor is:

$$\{\{8,14\},\{9,15\},\{10,16\},\{11,17\},\{12,18\},\{13,19\}\}.$$

□

Lemma 4.14. *There exists a* $(C_4, K_{1,3})$-URD(r, s) *of* $K_{20} - I$ *with*

$$(r, s) \in \{(9,0),(6,4),(3,8),(0,12)\} = I(20).$$

Proof. The result follows by Lemmas 4.3 and 4.13. □

In the following lemma we will use the following notation; if $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, we denote by $[A, B]$ the 4-cycle (a_1, b_1, a_2, b_2).

Lemma 4.15. *There exists a* $(C_4, K_{1,3})$-IURD$(28, 4; [(1,0)], [(r_2, s_2)])$ *with*

$$(r_2, s_2) \in \{(12,0),(9,4),(6,8),(3,12),(0,16)\}.$$

Proof.

(1) Let $(r_2, s_2) = (12, 0)$. On the complete graph K_{12} with vertex set $\{0, 1, \ldots, 11\}$ consider the following partial 1-factors:

$$F_0 = \{\{2,10\},\{3,9\},\{4,8\},\{5,7\},\{6,11\}\},$$
$$F_1 = \{\{0,10\},\{3,11\},\{4,9\},\{5,8\},\{6,7\}\},$$
$$F_2 = \{\{0,1\},\{4,10\},\{5,11\},\{6,8\},\{7,9\}\},$$
$$F_3 = \{\{0,2\},\{1,11\},\{5,9\},\{6,10\},\{7,8\}\},$$

$$\begin{aligned}
F_4 &= \{\{0,3\},\{1,2\},\{6,9\},\{7,11\},\{8,10\}\},\\
F_5 &= \{\{0,4\},\{1,3\},\{2,11\},\{7,10\},\{8,9\}\},\\
F_6 &= \{\{0,5\},\{1,4\},\{2,3\},\{8,11\},\{9,10\}\},\\
F_7 &= \{\{0,6\},\{1,5\},\{2,4\},\{3,10\},\{9,11\}\},\\
F_8 &= \{\{0,7\},\{1,6\},\{2,5\},\{3,4\},\{10,11\}\},\\
F_9 &= \{\{0,8\},\{1,7\},\{2,6\},\{3,5\},\{4,11\}\},\\
F_{10} &= \{\{0,9\},\{1,8\},\{2,7\},\{3,6\},\{4,5\}\},\\
F_{11} &= \{\{1,9\},\{2,8\},\{3,7\},\{4,6\},\{5,10\}\}.
\end{aligned}$$

We will construct a $(C_4, K_{1,3})$-IURD$(28, 4; [1, 0], [12, 0])$ on $\mathbb{Z}_{28}$ with the point set $\{0, 1, \ldots, 27\}$ and with the hole $\{24, 25, 26, 27\}$. Let the partial factor be $I = \{\{2i, 2i + 1\} \mid i = 0, 1, \ldots, 11\}$. Let $A_i = \{2i, 2i + 1\}$ for $i = 0, 1, \ldots, 11$. Then the 12 resolution classes of 4-cycles are:

$$\begin{aligned}
C_j = \{&[A_r, A_s] \mid \{r, s\} \in F_j\}\\
&\cup \{[\{24, 25\}, A_j], [\{26, 27\}, A_{j+1}]\}
\end{aligned}$$

for $j = 0, 1, \ldots, 11$ (where we take $A_{12} = A_0$). The resolution class of 4-cycles on $K_{28} - K_4$ on the vertices from 0 to 23 is:

$$D = \{[A_i, A_{11-i}] \mid i = 0, 1, \ldots, 5\}.$$

(2) Let $(r_2, s_2) = (9, 4)$. Consider the vertex set as:

$$X = \{0_i, 1_i, 2_i, 3_i, 4_i, 5_i, 6_i \mid i = 1, 2, 3, 4\}.$$

Let the partial 1-factor be $I = \{\{i_1, i_3\}, \{i_2, i_4\} \mid i = 0, 1, 2, 3, 4, 6\}$. We will construct a $(C_4, K_{1,3})$-IURD$(28, 4; [1, 0], [9, 4])$ with the point set X and with the hole $\{5_1, 5_2, 5_3, 5_4\}$. Let $\mathcal{B}$ be the following class of 4-cycles:

$$\begin{aligned}
\mathcal{B} = \{&(0_1, 1_1, 4_3, 2_1), (0_2, 1_2, 4_4, 2_2), (0_3, 1_3, 4_2, 2_3),\\
&(0_4, 1_4, 4_1, 2_4), (3_1, 6_1, 3_2, 6_2), (3_3, 6_3, 3_4, 6_4),\\
&(5_1, 5_2, 5_3, 5_4)\},
\end{aligned}$$

then 9 classes of 4-cycles are $\mathcal{B}_i = \mathcal{B} + i$, for $i = 1, 2, 3, 4, 5, 6$ and the other 3 are:

$$\begin{aligned}
\mathcal{B}_8 &= \{(i_1, (i + 1)_2, (i + 2)_4, (i + 1)_3) \mid i = 0, 1, 2, 3, 4, 5, 6\},\\
\mathcal{B}_9 &= \{(i_1, (i + 2)_2, (i + 3)_3, (i + 1)_4) \mid i = 0, 1, 2, 3, 4, 5, 6\},\\
\mathcal{B}_{10} &= \{(i_1, (i + 4)_3, (i + 5)_2, (i + 2)_4) \mid i = 0, 1, 2, 3, 4, 5, 6\}.
\end{aligned}$$

The 4 classes of $K_{1,3}$ are:

$$\mathcal{C}_1 = \{(i_1; (i+5)_2, (i+5)_3, (i+3)_4) \mid i = 0, 1, 2, 3, 4, 5, 6\},$$
$$\mathcal{C}_2 = \{(i_2; (i-6)_1, (i+3)_3, (i+5)_4) \mid i = 0, 1, 2, 3, 4, 5, 6\},$$
$$\mathcal{C}_3 = \{(i_3; (i-6)_1, (i-2)_2, (i+2)_4) \mid i = 0, 1, 2, 3, 4, 5, 6\},$$
$$\mathcal{C}_4 = \{(i_4; (i-6)_1, (i-6)_2, (i-6)_3) \mid i = 0, 1, 2, 3, 4, 5, 6\}.$$

The resolution class of 4-cycles on $K_{28} - K_4$ on the vertices $X \setminus \{5_1, 5_2, 5_3, 5_4\}$ is $\mathcal{B} \setminus \{5_1, 5_2, 5_3, 5_4\}$.

(3) Let $(r_2, s_2) = (6, 8)$. Let the vertex set be X and the partial 1-factor I as in the previous case. We will construct a $(C_4, K_{1,3})$-IURD$(28, 4; [1, 0], [6, 8])$ with the point set X and with the hole $\{5_1, 5_2, 5_3, 5_4\}$. Let $\mathcal{B}$ as in the previous case.

The 6 classes of 4-cycles are $\mathcal{B}_i = \mathcal{B} + i$, for $i = 1, 2, 3, 4, 5, 6$. The 8 classes of $K_{1,3}$ are:

$$\mathcal{C}_1 = \{(i_1; (i+1)_2, (i+1)_3, (i+1)_4) \mid i = 0, 1, 2, 3, 4, 5, 6\},$$
$$\mathcal{C}_2 = \{(i_2; (i-2)_1, (i+1)_3, (i+1)_4) \mid i = 0, 1, 2, 3, 4, 5, 6\},$$
$$\mathcal{C}_3 = \{(i_3; (i-4)_1, (i-2)_2, (i+1)_4) \mid i = 0, 1, 2, 3, 4, 5, 6\},$$
$$\mathcal{C}_4 = \{(i_4; (i-2)_1, (i-4)_2, (i-2)_3) \mid i = 0, 1, 2, 3, 4, 5, 6\},$$
$$\mathcal{C}_5 = \{(i_1; (i+5)_2, (i+5)_3, (i+3)_4) \mid i = 0, 1, 2, 3, 4, 5, 6\},$$
$$\mathcal{C}_6 = \{(i_2; (i-6)_1, (i+3)_3, (i+5)_4) \mid i = 0, 1, 2, 3, 4, 5, 6\},$$
$$\mathcal{C}_7 = \{(i_3; (i-6)_1, (i-6)_2, (i+5)_4) \mid i = 0, 1, 2, 3, 4, 5, 6\},$$
$$\mathcal{C}_8 = \{(i_4; (i-6)_1, (i-6)_2, (i-6)_3) \mid i = 0, 1, 2, 3, 4, 5, 6\}.$$

The resolution class of 4-cycles on $K_{28} - K_4$ on the vertices $X \setminus \{5_1, 5_2, 5_3, 5_4\}$ is $\mathcal{B} \setminus \{5_1, 5_2, 5_3, 5_4\}$.

(4) Let $(r_2, s_2) = (0, 16)$. We will use the construction given in [7, Lemma 3.3]. So let the point set be $\mathbb{Z}_{24} \cup \{A, B, C, D\}$ with the hole $\{A, B, C, D\}$. Let the partial 1-factor be

$$I = \{\{2i + 1, 2i + 2\} \mid i = 0, 1, \ldots, 11\}.$$

Its 16 parallel classes of 3-stars are denoted by Q_1,…,Q_{16}, where Q_1, Q_2, Q_3 and Q_4 are listed in the following table, and $Q_{i+4j} = Q_i + 6j$ for all $i = 1, 2, 3, 4$ and $j = 1, 2, 3$ (here A, B, C, D are fixed under the action of $\mathbb{Z}_{24}$ on the set of vertices).

$$Q_1 = \{(A; 13, 18, 20), (19; B, C, D), (0; 2, 3, 4), (5; 1, 7, 8),$$
$$(6; 11, 12, 14), (9; 15, 16, 22), (17; 10, 21, 23)\},$$

$$Q_2 = \{(B; 14, 15, 18), (21; A, C, D), (1; 3, 4, 6), (2; 5, 7, 9),$$
$$(8; 10, 12, 17), (11; 0, 16, 19), (13; 20, 22, 23)\},$$

$$Q_3 = \{(C;6,14,16),(22;A,B,D),(3;5,13,17),(12;1,9,21),\\ (15;0,19,23),(18;4,8,11),(20;2,7,10)\},$$

$$Q_4 = \{(D;8,11,18),(23;A,B,C),(4;6,12,17),(7;0,1,15),\\ (10;2,13,19),(14;3,5,9),(16;20,21,22)\}.$$

The partial class of 4-cycles is

$$\{(2i,2i+12,2i+13,2i+1) \mid i=0,1,\dots,5\}.$$

(5) Let $(r_2,s_2)=(3,12)$. Consider the solution in the previous case, that is $(r_2,s_2)=(0,16)$. Replace Q_k, for $k=1,2,...,7$ by the following 3 classes of 4-cycles and 3 classes of $K_{1,3}$:

$$\{(0,2,5,3),(1,4,12,6),(7,9,19,10),(8,13,23,15),\\ (11,14,26,16),(17,21,27,22),(18,24,20,25)\},$$
$$\{(0,4,15,7),(1,3,14,5),(2,13,10,20),(6,9,12,17),\\ (8,18,11,27),(16,21,24,23),(19,25,22,26)\},$$
$$\{(0,10,2,24),(1,26,3,27),(4,6,8,17),(5,7,11,19),\\ (9,14,25,23),(12,18,15,21),(13,20,16,22)\},$$
$$\{(0;14,15,17),(1;7,21,25),(3;13,18,24),(8;5,10,12),\\ (9;11,16,22),(19;2,4,27),(26;6,20,23)\},$$
$$\{(2;8,9,16),(6;10,11,21),(12;1,20,26),(18;7,14,27),\\ (23;3,5,17),(24;13,19,22),(25;0,4,15)\},$$
$$\{(4;18,24,27),(7;2,12,20),(11;0,8,13),(14;6,16,23),\\ (15,9,19,22),(17;1,3,10),(21;5,25,26)\}.$$

□

Lemma 4.16. *There exists a* $(C_4,K_{1,3})$-URD(r,s) *of* $K_{28}-I$ *with*

$$(r,s)\in\{(13,0),(10,4),(7,8),(4,12),(1,16)\}=I(28).$$

Proof. The result follows by Lemmas 4.1 and 4.15. □

Lemma 4.17. *There exists a* $(C_4,K_{1,3})$-URD(r,s) *of* $K_{52}-I$ *with*

$$(r,s)\in\{(25,0),(22,4),(19,8),(16,12),(13,16),\\ (10,20),(7,24),(4,28),(1,32)\}.$$

Proof. For the case $(r,s)=(1,32)$ we consider a $K_{1,3}$-frame of type 12^4 from Lemma 2.8 in [7]. There are 8 partial parallel classes missing each group. We add 4 new points and take them as the hole. Then we place on each group with the hole a copy of a $(C_4,K_{1,3})$-IURD$(16,4;[1,0],[0,8])$ by Lemma 4.12.

For the case $(r,s)=(25,0)$ we consider a 4-cycle frame of type 12^4 instead of the $K_{1,3}$-frame in the previous case $(r,s)=(1,32)$. (Note that there will be 6 partial parallel classes missing each group.)

Let $(r, s) = (13, 16)$. Let the point set be $\{0_i, 1_i, \dots, 12_i : i = 1, 2, 3, 4\}$. The 13 resolution classes of 4-cycles are:

$$A_1 = \{(i_1, (i+1)_2, (i+3)_4, (i+2)_3) \mid i = 0, \dots, 12\},$$
$$A_2 = \{(i_1, (i+2)_2, (i+5)_4, (i+3)_3) \mid i = 0, \dots, 12\},$$
$$A_3 = \{(i_1, (i+3)_2, (i+8)_4, (i+4)_3) \mid i = 0, \dots, 12\},$$
$$A_4 = \{(i_1, (i+4)_2, (i+10)_4, (i+7)_3) \mid i = 0, \dots, 12\},$$
$$A_5 = \{(i_1, (i+6)_2, (i+7)_3, (i+5)_4)) \mid i = 0, \dots, 12\},$$
$$A_6 = \{(i_1, (i+7)_2, (i+9)_3, (i+6)_4) \mid i = 0, \dots, 12\},$$
$$A_7 = \{(i_1, (i+8)_2, (i+12)_3, (i+7)_4) \mid i = 0, \dots, 12\},$$
$$A_8 = \{(i_1, (i+9)_2, (i+1)_3, (i+10)_4) \mid i = 0, \dots, 12\},$$
$$A_9 = \{(i_1, (i+11)_3, i_2, (i+11)_4) \mid i = 0, \dots, 12\},$$
$$A_{10} = \{(i_1, (i+9)_3, i_2, (i+9)_4) \mid i = 0, \dots, 12\},$$
$$A_{11} = \{(i_1, (i+8)_3, i_2, (i+8)_4) \mid i = 0, \dots, 12\},$$
$$A_{12} = \{(i_1, (i+5)_3, (i+8)_2, (i+2)_4) \mid i = 0, \dots, 12\},$$
$$A_{13} = \{(i_1, (i+10)_2, (i+9)_3, (i+3)_4) \mid i = 0, \dots, 12\}.$$

The 16 resolution classes of $K_{1,3}$ are:

$$\begin{aligned}
B_1 = \{&(0_1; 1_1, 2_1, 3_1), (0_2; 1_2, 2_2, 3_2),\\
&(0_3; 1_3, 2_3, 3_3), (0_4; 1_4, 2_4, 3_4),\\
&(9_3; 9_1, 9_2, 9_4), (10_2; 4_2, 5_2, 6_2), (10_3; 4_3, 5_3, 6_3),\\
&(10_4; 4_4, 5_4, 6_4), (7_1; 12_2, 8_3, 7_4), (8_2; 8_1, 11_3, 8_4),\\
&(10_1; 4_1, 5_1, 6_1), (11_4; 12_1, 7_2, 12_3), (12_4; 11_1, 11_2, 7_3)\},
\end{aligned}$$
$$B_i = B_1 + i - 1 \text{ for } i = 2, \dots, 13,$$
$$B_{14} = \{(i_1; (i+11)_2, (i+6)_3, (i+4)_4) \mid i = 0, \dots, 12\},$$
$$B_{15} = \{(i_2; (i+1)_1, (i+6)_3, (i+10)_4) \mid i = 0, \dots, 12\},$$
$$B_{16} = \{(i_3; (i+3)_1, (i+6)_2, (i+6)_4) \mid i = 0, \dots, 12\}.$$

The one factor is

$$\{\{i_1, (i-1)_3\} \mid i = 0, \dots, 12\} \cup \{\{i_2, (i-1)_4\} \mid i = 0, \dots, 12\}.$$

From this case we can get the $(r, s) = (10, 20)$ case just by replacing the 3 classes of 4-cycles A_1, A_5 and A_9 with the following 4 classes of $K_{1,3}$:

$$B_{17} = \{(i_1; (i+1)_2, (i+2)_3, (i+5)_4) \mid i = 0, \dots, 12\},$$
$$B_{18} = \{(i_2; (i+7)_1, (i+1)_3, (i+2)_4) \mid i = 0, \dots, 12\},$$
$$B_{19} = \{(i_3; (i+2)_1, (i+2)_2, (i+1)_4) \mid i = 0, \dots, 12\},$$
$$B_{20} = \{(i_4; (i+2)_1; (i+2)_2, (i+2)_3) \mid i = 0, \dots, 12\}.$$

In a similar way, from the $(10, 20)$ case we can get the $(r, s) = (7, 24)$ case just by replacing the 3 classes of 4-cycles A_2, A_6 and A_{10} with the following

4 classes of $K_{1,3}$:

$$\begin{aligned}
B_{21} &= \{(i_1; (i+2)_2, (i+3)_3, (i+6)_4) \mid i = 0, \dots, 12\},\\
B_{22} &= \{(i_2; (i+6)_1, (i+2)_3, (i+3)_4) \mid i = 0, \dots, 12\},\\
B_{23} &= \{(i_3; (i+4)_1, (i+4)_2, (i+2)_4) \mid i = 0, \dots, 12\},\\
B_{24} &= \{(i_4; (i+4)_1, (i+4)_2, (i+3)_3) \mid i = 0, \dots, 12\}.
\end{aligned}$$

Again, from the $(7, 24)$ case we can get the $(r, s) = (4, 28)$ case just by replacing the 3 classes of 4-cycles A_3, A_7 and A_{11} with the following 4 classes of $K_{1,3}$:

$$\begin{aligned}
B_{25} &= \{(i_1; (i+3)_2, (i+4)_3, (i+7)_4) \mid i = 0, \dots, 12\},\\
B_{26} &= \{(i_2; (i+5)_1, (i+4)_3, (i+5)_4) \mid i = 0, \dots, 12\},\\
B_{27} &= \{(i_3; (i+5)_1, (i+5)_2, (i+4)_4) \mid i = 0, \dots, 12\},\\
B_{28} &= \{(i_4; (i+5)_1, (i+5)_2, (i+5)_3) \mid i = 0, \dots, 12\}.
\end{aligned}$$

Let $(r, s) = (22, 4)$. Again, let the point set be $\{0_i, 1_i, \dots, 12_i : i = 1, 2, 3, 4\}$. The 22 resolution classes of 4-cycles are:

$$\begin{aligned}
A_1 = \{&(0_1, 1_1, 5_2, 2_1), (3_1, 6_1, 5_3, 7_1),\\
&(0_3, 1_3, 8_1, 2_3), (0_2, 1_2, 5_1, 2_2),\\
&(3_2, 6_2, 8_3, 7_2), (4_2, 9_2, 8_4, 10_2), (4_1, 9_1, 5_4, 10_1),\\
&(3_3, 6_3, 8_2, 7_3), (4_3, 9_3, 11_4, 10_3), (0_4, 1_4, 11_1, 2_4),\\
&(3_4, 6_4, 11_2, 7_4), (4_4, 9_4, 11_3, 10_4), (12_1, 12_2, 12_4, 12_3)\},\\
A_i = {}&A_1 + i - 1 \text{ for } i = 2, \dots, 13,\\
A_{14} = {}&\{(i_1, (i+1)_2, (i+1)_3, (i+5)_4) \mid i = 0, \dots, 12\},\\
A_{15} = {}&\{(i_1, (i+2)_2, (i+5)_3, (i+10)_4) \mid i = 0, \dots, 12\},\\
A_{16} = {}&\{(i_1, (i+5)_2, (i+9)_3, (i+12)_4) \mid i = 0, \dots, 12\},\\
A_{17} = {}&\{(i_1, (i+6)_2, (i+7)_4, (i+1)_3) \mid i = 0, \dots, 12\},\\
A_{18} = {}&\{(i_1, (i+7)_2, (i+9)_4, (i+2)_3) \mid i = 0, \dots, 12\},\\
A_{19} = {}&\{(i_1, (i+8)_2, (i+11)_4, (i+3)_3) \mid i = 0, \dots, 12\},\\
A_{20} = {}&\{(i_1, (i+4)_3, (i+7)_2, (i+11)_4) \mid i = 0, \dots, 12\},\\
A_{21} = {}&\{(i_1, (i+5)_3, i_2, (i+6)_4) \mid i = 0, \dots, 12\},\\
A_{22} = {}&\{(i_1, (i+8)_3, (i+2)_2, (i+7)_4) \mid i = 0, \dots, 12\}.
\end{aligned}$$

The 4 resolution classes of $K_{1,3}$ are:

$$\begin{aligned}
B_1 &= \{(i_1; (i+11)_2, (i+9)_3, i_4) \mid i = 0, \dots, 12\},\\
B_2 &= \{(i_2; (i+1)_1, (i+7)_3, (i+7)_4) \mid i = 0, \dots, 12\},\\
B_3 &= \{(i_3; (i+3)_1, (i+5)_2, (i+9)_4) \mid i = 0, \dots, 12\},\\
B_4 &= \{(i_4; (i+12)_1, (i+3)_2, (i+3)_3) \mid i = 0, \dots, 12\}.
\end{aligned}$$

The one factor is

$$\{\{i_1, (i+2)_4\} \mid i = 0, \dots, 12\} \cup \{\{i_2, (i+9)_3\} \mid i = 0, \dots, 12\}.$$

From this case we can get the $(r, s) = (19, 8)$ case just by replacing the 3 classes of 4-cycles A_{14}, A_{17} and A_{20} with the following 4 classes of $K_{1,3}$:

$$\begin{aligned}
B_5 &= \{(i_1; (i+1)_2, (i+1)_3, (i+5)_4) \mid i = 0, \dots, 12\},\\
B_6 &= \{(i_2; (i+7)_1, i_3, (i+1)_4) \mid i = 0, \dots, 12\},\\
B_7 &= \{(i_3; (i+9)_1, (i+3)_2, (i+4)_4) \mid i = 0, \dots, 12\},\\
B_8 &= \{(i_4; (i+2)_1, (i+9)_2, (i+7)_3) \mid i = 0, \dots, 12\}.
\end{aligned}$$

At last, from the $(19, 8)$ case we can get the $(r, s) = (16, 12)$ just by replacing the 3 classes of 4-cycles A_{15}, A_{18} and A_{21} with the following 4 classes of $K_{1,3}$:

$$\begin{aligned}
B_9 &= \{(i_1; (i+2)_2, (i+2)_3, (i+10)_4) \mid i = 0, \dots, 12\},\\
B_{10} &= \{(i_2; (i+6)_1, (i+3)_3, (i+2)_4) \mid i = 0, \dots, 12\},\\
B_{11} &= \{(i_3; (i+8)_1, (i+8)_2, (i+5)_4) \mid i = 0, \dots, 12\},\\
B_{12} &= \{(i_4; (i+7)_1, (i+7)_2, (i+6)_3) \mid i = 0, \dots, 12\}.
\end{aligned}$$

□

Lemma 4.18. *There exists a* $(C_4, K_{1,3})$-URD(r, s) *of* $K_{60} - I$ *with*

$$(r, s) \in \{(29, 0), (26, 4), (23, 8), (20, 12), (17, 16), (14, 20), (11, 24), (8, 28), (5, 32), (2, 36)\}.$$

Proof. For the cases $(r, s) \in \{(29, 0), (26, 4), (17, 16), (14, 20), (5, 32), (2, 36)\}$ we take a $(C_4, K_{1,3})$-URGDD(r, s) of type 12^5 with $(r, s) \in \{(24, 0), (12, 16), (0, 32)\}$ which exists by Lemma 4.7 and replace each group of size 12 with the same $(C_4, K_{1,3})$-URD$(12; r, s)$, where $(r, s) \in \{(5, 0), (2, 4)\}$ which exists by Lemma 4.4.

For the cases $(r, s) \in \{(23, 8), (20, 12)\}$ we take a $(C_4, K_{1,3})$-URGDD(r, s) of type 20^3 with $(r, s) = (20, 0)$ which exists in [9] and replace each group of size 20 with the same $(C_4, K_{1,3})$-URD$(20; r, s)$, where $(r, s) \in \{(3, 8), (0, 12)\}$ which exists by Lemma 4.14.

Let $(r, s) = (11, 24)$. Let the point set be $\{0_i, 1_i, \dots, 14_i \colon i = 1, 2, 3, 4\}$. The 11 resolution classes of 4-cycles are:

$$\begin{aligned}
A_1 &= \{(i_1, i_2, (i+2)_4, (i+3)_3) \mid i = 0, \dots, 14\},\\
A_2 &= \{(i_1, (i+1)_2, (i+5)_3, (i+2)_4) \mid i = 0, \dots, 14\},\\
A_3 &= \{(i_1, (i+4)_3, (i-1)_2, (i+3)_4) \mid i = 0, \dots, 14\},\\
A_4 &= \{(i_1, (i+3)_2, (i+8)_4, (i+5)_3) \mid i = 0, \dots, 14\},\\
A_5 &= \{(i_1, (i+4)_2, (i+10)_3, (i+4)_4) \mid i = 0, \dots, 14\},\\
A_6 &= \{(i_1, (i+6)_3, (i-1)_2, (i+5)_4)) \mid i = 0, \dots, 14\},
\end{aligned}$$

$$A_7 = \{(i_1, (i+5)_2, (i-2)_3, (i+6)_4) \mid i = 0, \dots, 14\},$$
$$A_8 = \{(i_1, (i+6)_2, (i+9)_4, (i+7)_3) \mid i = 0, \dots, 14\},$$
$$A_9 = \{(i_1, (i+8)_3, (i-1)_2, (i+7)_4) \mid i = 0, \dots, 14\},$$
$$A_{10} = \{(i_1, (i+7)_2, (i-1)_4, (i+9)_3) \mid i = 0, \dots, 14\},$$
$$A_{11} = \{(i_1, (i+8)_2, (i+4)_3, (i+8)_4) \mid i = 0, \dots, 14\}.$$

The 24 resolution classes of $K_{1,3}$ are:

$$\begin{aligned} B_1 = \{&(11_1; 13_2, 11_3, 11_4), (12_2; 13_1, 12_3, 12_4), (13_3; 12_1, 11_2, 13_4),\\ &(4_2; 8_2, 9_2, 10_2), (7_2; 14_2, 6_3, 6_4), (0_3; 1_3, 2_3, 3_3),\\ &(4_3; 8_3, 9_3, 10_3), (7_3; 14_3, 5_4, 5_1), (0_4; 1_4, 2_4, 3_4),\\ &(4_4; 8_4, 9_4, 10_4), (7_4; 14_4, 6_1, 6_2), (0_1; 1_1, 2_1, 3_1),\\ &(4_1; 8_1, 9_1, 10_1), (7_1; 14_1, 5_2, 5_3), (0_2; 1_2, 2_2, 3_2)\}, \end{aligned}$$
$$B_i = B_1 + i - 1 \text{ for } i = 2, \dots, 15,$$
$$B_{16} = \{(i_1; (i+9)_2, (i+10)_3, (i+9)_4) \mid i = 0, \dots, 14\},$$
$$B_{17} = \{((i_2; (i+5)_1, (i+1)_3, (i+9)_4 \mid i = 0, \dots, 14\},$$
$$B_{18} = \{(i_3; (i+4)_1, (i+12)_2, (i+1)_4) \mid i = 0, \dots, 14\},$$
$$B_{19} = \{(i_4; (i+5)_1, (i+5)_2, (i+9)_3) \mid i = 0, \dots, 14\},$$
$$B_{20} = \{(i_1; (i+11)_2, (i+12)_3, (i+11)_4) \mid i = 0, \dots, 14\},$$
$$B_{21} = \{(i_2; (i+3)_1, (i+10)_3, (i+11)_4) \mid i = 0, \dots, 14\},$$
$$B_{22} = \{(i_3; (i+1)_1, (i+3)_2, (i+7)_4) \mid i = 0, \dots, 14\},$$
$$B_{23} = \{(i_4; (i+3)_1, (i+3)_2, (i+5)_3) \mid i = 0, \dots, 14\},$$
$$B_{24} = \{(i_4; (i+2)_1, (i+2)_2, (i+4)_3) \mid i = 0, \dots, 14\}.$$

The one factor is

$$\{\{i_1, (i-1)_4\} \mid i = 0, \dots, 14\} \cup \{\{i_2, (i-2)_3\} \mid i = 0, \dots, 14\}.$$

From this case we can get the $(r, s) = (8, 28)$ case just by replacing the 3 classes of 4-cycles A_9, A_{10} and A_{11} with the following 4 classes of $K_{1,3}$:

$$B_{25} = \{(i_1; (i+7)_2, (i+8)_3, (i+7)_4) \mid i = 0, \dots, 14\},$$
$$B_{26} = \{(i_2; (i+7)_1, (i+9)_3, (i+8)_4) \mid i = 0, \dots, 14\},$$
$$B_{27} = \{(i_3; (i+6)_1, (i+4)_2, (i+5)_4) \mid i = 0, \dots, 14\},$$
$$B_{28} = \{(i_4; (i+7)_1, (i+8)_2, (i+11)_3) \mid i = 0, \dots, 14\}.$$

□

5. Main results

Lemma 5.1. *For every* $v \equiv 8 \pmod{24}$, *there exists a* $(C_4, K_{1,3})$-URD(r, s) *of* $K_v - I$ *with* $(r, s) \in I(v)$.

Proof. Let us denote by $R_1, R_2, \dots, R_{(v-2)/6}$ be the parallel classes of a resolvable K_4-decomposition R of $K_{v/2}$. Give weight 2 to each point of R and place on each block of R the same $(C_4, K_{1,3})$-URGDD (r, s) of type 2^4 with $(r, s) \in \{(3, 0), (0, 4)\}$ (see Lemma 4.2). Since R contains $(v-2)/6$ parallel classes the result is a $(C_4, K_{1,3})$-URD$(v; r, s)$ of $K_v - I$ for each $(r, s) \in (v-2)/6 * \{(3, 0), (0, 4)\}$. Since

$$\frac{v-2}{6} * \{(3,0),(0,4)\} = \left\{ \left(\frac{v-2}{2} - 3x, 4x \right) : x = 0, \dots, \frac{v-2}{6} \right\} = I(v),$$

we obtain the result. □

Lemma 5.2. *For every* $v \equiv 0 \pmod{24}$*, there exists a* $(C_4, K_{1,3})$-URD(r, s) *of* $K_v - I$ *with* $(r, s) \in I(v)$*.*

Proof. Start with a 1-factorization G of $K_{v/12}$. Give weight 12 to each point of this 1-factorization and place on each edge of a given 1-factor the same $(C_4, K_{1,3})$-URGDD(x, y) of type 12^2, with $(x, y) \in \{(6, 0), (3, 4), (0, 8)\}$, which exists by Lemma 4.5. Fill the groups of size 12 with the same $(C_4, K_{1,3})$-URD$(12; u, v)$, with $(u, v) \in \{(5, 0), (2, 4)\}$, which exists by Lemma 4.4. Since G contains $(v-12)/12$ 1-factors the result is a $(C_4, K_{1,3})$-URD$(v; r, s)$ of K_v for each

$$(r, s) \in \{(5,0),(2,4)\} + \frac{v-12}{12} * \{(6,0),(3,4),(0,8)\}.$$

Since

$$\frac{v-12}{12} * \{(6,0),(3,4),(0,8)\} = \left\{ \left(\frac{v-12}{2} - 3x, 4x \right) : x = 0, \dots, \frac{v-12}{6} \right\},$$

it easy to see that

$$\{(5,0),(2,4)\} + \frac{v-12}{12} * \{(6,0),(3,4),(0,8)\} = I(v).$$

This completes the proof. □

Lemma 5.3. *For every* $v \equiv 12 \pmod{24}$*, there exists a* $(C_4, K_{1,3})$-URD(r, s) *of* $K_v - I$ *with* $(r, s) \in I(v)$*.*

Proof. The cases $v = 12, 36, 60$ are covered by Lemmas 4.4, 4.9 and 4.18. For $v > 60$ start from a 2-frame of type $2^{(v-12)/24}$ [24, Theorem 1.4] and apply Theorem 3.1 with $g = 2, u = (v-12)/24, t = 12$ and $h = 12$. The input designs are: a $(C_4, K_{1,3})$-URD$(12; r_1, s_1)$ with $(r_1, s_1) \in \{(5, 0), (2, 4)\}$, which exists by Lemma 4.4; a $(C_4, K_{1,3})$-URGDD(r_2, s_2) of type 12^2 with $(r_2, s_2) \in \{(6, 0), (3, 4), (0, 8)\}$ which exists by Lemma 4.5; a $(C_4, K_{1,3})$-IURD$(36, 12; [(r_1, s_1)], [(r_3, s_3)])$ with $(r_1, s_1) \in \{(5, 0), (2, 4)\}$ and $(r_3, s_3) \in \{(12, 0), (6, 8), (0, 16)\}$, which exists by Lemmas 4.4 and 4.6. Since

$$\frac{v-12}{24} * \{(12,0),(6,8),(0,16)\} = \left\{ \left(\frac{v-12}{2} - 3x, 4x \right), x = 0, \dots, \frac{v-12}{6} \right\},$$

it easy to see that

$$\{(5,0),(2,4)\} + \frac{v-12}{24} * \{(12,0),(6,8),(0,16)\} = I(v).$$

This completes the proof. □

Lemma 5.4. *For every* $v \equiv 20 \pmod{24}$*, there exists a* $(C_4, K_{1,3})$-URD(r,s) *of* $K_v - I$ *with* $(r,s) \in I(v)$.

Proof. The case $v = 20$ follows by Lemma 4.14. For $v \geq 44$ start from a 2-frame of type $1^{(v-8)/12}$ and apply Theorem 3.1 with $g = 1$, $u = (v-8)/12$, $t = 12$ and $h = 8$. The input designs are: a $(C_4, K_{1,3})$-URD$(8; r_1, s_1)$ with $(r_1, s_1) \in \{(3,0),(0,4)\}$, which exists by Lemma 4.3; a $(C_4, K_{1,3})$-URGDD(r_2, s_2) of type 12^2 with $(r_2, s_2) \in \{(6,0),(3,4),(0,8)\}$ which exists by Lemma 4.5; a $(C_4, K_{1,3})$-IURD$(20, 8; [(r_1, s_1)], [(r_3, s_3)])$ with $(r_1, s_1) \in \{(3,0),(0,4)\}$ and $(r_3, s_3) \in \{(6,0), (3,4), (0,8)\}$, which exists by Lemma 4.13. This gives a $(C_4, K_{1,3})$-URD$(v; r, s)$ of $K_v - I$, with

$$(r,s) \in \{(3,0),(0,4)\} + \frac{(v-8)}{12} * \{(6,0),(3,4),(0,8)\}.$$

Proceeding as in Lemma 5.3 we obtain the result. □

Lemma 5.5. *For every* $v \equiv 16 \pmod{24}$*, there exists a* $(C_4, K_{1,3})$-URD(r,s) *of* $K_v - I$ *with* $(r,s) \in I(v)$.

Proof. The case $v = 16$ follows by Lemma 4.11.

For $v \geq 40$ start from a 2-frame of type $1^{(v-4)/12}$ and apply Theorem 3.1 with $g = 1, u = (v-4)/12, t = 12$ and $h = 4$. The input designs are: a $(C_4, K_{1,3})$-URD$(4; 1, 0)$ which exists by Lemma 4.1; a $(C_4, K_{1,3})$-URGDD(r_2, s_2) of type 12^2 with $(r_2, s_2) \in \{(6,0),(3,4),(0,8)\}$ which exists by Lemma 4.5; a $(C_4, K_{1,3})$-IURD$(16, 4; [(1,0)], [(r_3, s_3)])$ with $(r_3, s_3) \in \{(6,0),(3,4),(0,8)\}$, which exists by Lemma 4.12. This gives a $(C_4, K_{1,3})$-URD$(v; r, s)$ of $K_v - I$, with

$$(r,s) \in \{(1,0)\} + \frac{(v-4)}{12} * \{(6,0),(3,4),(0,8)\}.$$

Proceeding as in Lemma 5.3 we obtain the result. □

Lemma 5.6. *For every* $v \equiv 4 \pmod{24}$*, there exists a* $(C_4, K_{1,3})$-URD(r,s) *of* $K_v - I$ *with* $(r,s) \in I(v)$.

Proof. The cases $v = 28, 52$ are covered by Lemmas 4.16 and 4.17. For $v > 52$ start from a 2-frame of type $2^{(v-4)/24}$ [24, Theorem 1.4] and apply Theorem 3.1 with $g = 2, u = (v-4)/24, t = 12$ and $h = 4$. The input designs are: a $(C_4, K_{1,3})$-URD$(4; 1, 0)$ which exists by Lemma 4.1; a $(C_4, K_{1,3})$-URGDD(r_2, s_2) of type 12^2 with $(r_2, s_2) \in \{(6,0), (3,4), (0,8)\}$ which exists by Lemma 4.5; a $(C_4, K_{1,3})$-IURD$(28, 4; [(1,0)], [(r_3, s_3)])$ with

$(r_3, s_3) \in \{(12,0),(9,4),(6,8),(3,12),(0,16)\}$, which exists by Lemma 4.15. This gives a $(C_4, K_{1,3})$-URD$(v; r, s)$ of $K_v - I$, with

$$(r,s) \in \{(1,0)\} + \frac{(v-4)}{24} * \{(12,0),(6,8),(0,16)\}.$$

Proceeding as in Lemma 5.3 we obtain the result. □

Combining Lemmas 2.1, 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6 we obtain the main theorem of this paper.

Theorem 5.7. *There exists a* $(C_4, K_{1,3})$-URD(r, s) *of* $K_v - I$ *if and only if* $v \equiv 0 \pmod 4$ *and* $(r, s) \in I(v)$, *where* $I(v)$ *is given in Table 1.*

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