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A SUBSPACE BASED SUBSPACE INCLUSION GRAPH ON VECTOR SPACE

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Abstract. Let $\mathcal{W}$ be a fixed k-dimensional subspace of an n-dimensional vector space $\mathcal{V}$ such that $n-k \geq 1$. In this paper, we introduce a graph structure, called the subspace based subspace inclusion graph $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$, where the vertex set $\mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ is the collection of all subspaces $\mathcal{U}$ of $\mathcal{V}$ such that $\mathcal{U}+\mathcal{W} \neq \mathcal{V}$ and $\mathcal{U} \nsubseteq \mathcal{W}$, i.e., $\mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) = \{\mathcal{U} \subseteq \mathcal{V} \mid \mathcal{U}+\mathcal{W} \neq \mathcal{V}, \mathcal{U} \nsubseteq \mathcal{W}\}$ and any two distinct vertices $\mathcal{U}_1$ and $\mathcal{U}_1$ of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ are adjacent if and only if either $\mathcal{U}_1+\mathcal{W} \subset \mathcal{U}_2+\mathcal{W}$ or $\mathcal{U}_2+\mathcal{W} \subset \mathcal{U}_1+\mathcal{W}$. The diameter, girth, clique number, and chromatic number of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ are studied. It is shown that two subspace based subspace inclusion graphs $\mathcal{I}_n^{\mathcal{W}_1}(\mathcal{V})$ and $\mathcal{I}_n^{\mathcal{W}_2}(\mathcal{V})$ are isomorphic if and only if $\mathcal{W}_1$ and $\mathcal{W}_2$ are isomorphic. Further, some properties of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ are obtained when the base field is finite.

1. Introduction

Throughout this paper, $\mathcal{V}$ denotes a finite dimensional vector space over a field $\mathbb{F}$ and for any subspace $\mathcal{W}$ of $\mathcal{V}$, $\mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) = \{\mathcal{U} \subseteq \mathcal{V} \mid \mathcal{U}+\mathcal{W} \neq \mathcal{V}, \mathcal{U} \nsubseteq \mathcal{W}\}$. Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ be a graph, where $\mathcal{V}(\mathcal{G})$ is the set of vertices and $\mathcal{E}(\mathcal{G})$ is the set of edges of $\mathcal{G}$. We say that $\mathcal{G}$ is connected if there exists a path between any two distinct vertices of $\mathcal{G}$. For vertices a and b of $\mathcal{G}$, $\mathrm{d}(a,b)$ denotes the length of a shortest path from a to b. In particular, $\mathrm{d}(a,a) = 0$ and $\mathrm{d}(a,b) = \infty$ if there is no such path. The diameter of $\mathcal{G}$ is denoted by $\mathrm{diam}(\mathcal{G}) = \sup\{\mathrm{d}(a,b) \mid a,b \in \mathcal{V}(\mathcal{G})\}$. A cycle in a graph $\mathcal{G}$ is a path that begins and ends at the same vertex. A cycle of length n is denoted by $\mathcal{C}_n$. The girth of $\mathcal{G}$, denoted by $\mathrm{gr}(\mathcal{G})$, is the length of a shortest cycle in $\mathcal{G}$ ($\mathrm{gr}(\mathcal{G}) = \infty$ if $\mathcal{G}$ contains no cycle). A complete graph $\mathcal{G}$ is a graph where all distinct vertices are adjacent. The complete graph with $|\mathcal{V}(\mathcal{G})| = n$ is denoted by $\mathcal{K}_n$. A graph $\mathcal{G}$ is said to be complete k-bipartite if there is a partition $\cup_{i=1}^{k}\mathcal{V}_i = \mathcal{V}(\mathcal{G})$, such that $u - v \in \mathcal{E}(\mathcal{G})$ if and only if u and v are in different parts of partition. If $|\mathcal{V}_i| = n_i$, then $\mathcal{G}$ is denoted by $\mathcal{K}_{n_1,n_2,\ldots,n_k}$ and in particular $\mathcal{G}$ is called complete bipartite if $k = 2$. A graph $\mathcal{H} = (\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ is said to be a subgraph of $\mathcal{G}$ if $\mathcal{V}(\mathcal{H}) \subseteq \mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{H}) \subseteq \mathcal{E}(\mathcal{G})$. Moreover, $\mathcal{H}$ is said to be induced subgraph of $\mathcal{G}$ if

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$\mathcal{V}(\mathcal{H}) \subseteq \mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{H}) = \{u - v \in \mathcal{E}(\mathcal{G}) \mid u, v \in \mathcal{V}(\mathcal{H})\}$ and is denoted by $\mathcal{G}[\mathcal{V}(\mathcal{H})]$. Also $\mathcal{G}$ is called a null graph if $\mathcal{E}(\mathcal{G}) = \varnothing$. For a graph $\mathcal{G}$, a complete subgraph of $\mathcal{G}$ is called a clique. The clique number, $\omega(\mathcal{G})$, is the greatest integer $n \geqslant 1$ such that $\mathcal{K}_n \subseteq \mathcal{G}$, and $\omega(\mathcal{G}) = \infty$ if $\mathcal{K}_n \subseteq \mathcal{G}$ for all $n \geqslant 1$. The chromatic number $\chi(\mathcal{G})$ of a graph $\mathcal{G}$ is the minimum number of colours needed to colour all the vertices of $\mathcal{G}$ such that every two adjacent vertices get different colours. A graph $\mathcal{G}$ is perfect if $\chi(\mathcal{H}) = \omega(\mathcal{H})$ for every induced subgraph $\mathcal{H}$ of $\mathcal{G}$. Graph-theoretic terms are presented as they appear in R. Diestel [10].

Beside its combinatorial motivation, graph theory can also identify various algebraic structures. The main task of studying graphs associated with algebraic structures is to characterize algebraic structures with a graph and vice versa. To date, there has been a lot of research, see [1, 2, 3], on simple graph structures for commutative rings. Recently, some algebraic graphs associated with vector spaces were studied (see [4, 5, 6, 7, 8]). Das [6] defined the subspace inclusion graph $\mathcal{I}_n(\mathcal{V})$ on a vector space $\mathcal{V}$, where the set of vertices is a collection of all nontrivial subspaces of $\mathcal{V}$ and any two distinct vertices $\mathcal{W}_1$ and $\mathcal{W}_2$ are adjacent if and only if either $\mathcal{W}_1 \subset \mathcal{W}_2$ or $\mathcal{W}_2 \subset \mathcal{W}_1$.

Motivated by the above study, we introduce the notion of a subspace based subspace inclusion graph for a vector space $\mathcal{V}$ and denote it by $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$. The graph $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is a simple (undirected) graph with vertex set $\mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ and any two distinct vertices $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ are adjacent if and only if either $\mathcal{U}_1 + \mathcal{W} \subset \mathcal{U}_2 + \mathcal{W}$ or $\mathcal{U}_2 + \mathcal{W} \subset \mathcal{U}_1 + \mathcal{W}$. Further we investigate some basic properties of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$.

2. FUNDAMENTAL PROPERTIES OF $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$

In this section, we study the fundamental properties of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$. We show that $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is connected and $\operatorname{diam}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) \leq 3$.

Definition 2.1. *Let $\mathcal{W}$ be a subspace of a vector space $\mathcal{V}$. Then the subspace based subspace inclusion graph $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is a simple (undirected) graph with vertex set $\mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ and any two distinct vertices $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ are adjacent if and only if either $\mathcal{U}_1 + \mathcal{W} \subset \mathcal{U}_2 + \mathcal{W}$ or $\mathcal{U}_2 + \mathcal{W} \subset \mathcal{U}_1 + \mathcal{W}$.*

We have the following theorems:

Theorem 2.2. *Let $\mathcal{W}$ be a k-dimensional subspace of an n-dimensional vector space $\mathcal{V}$ over a field $\mathbb{F}$. Then the following statements hold:*

(i) *If $k = 0$, then $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}) = \mathcal{I}_n(\mathcal{V})$.*
(ii) *If $\mathcal{W}_1, \mathcal{W}_2$ are two distinct vertices of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ such that $\dim(\mathcal{W}_1 + \mathcal{W}) = \dim(\mathcal{W}_2 + \mathcal{W})$, then $\mathcal{W}_1$ is not adjacent to $\mathcal{W}_2$, i.e., $\mathcal{W}_1 \nsim \mathcal{W}_2$ in $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$.*
(iii) *If $n - k = 2$, then $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is an edgeless graph.*
(iv) *If $n - k = 1$, then $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is an empty graph.*
(v) *If $n - k \geq 4$, then $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is triangulated.*

(vi) $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ *is never complete.*

Proof.

(i) Obvious.

(ii) Let $\mathcal{W}_1, \mathcal{W}_2 \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ be two distinct subspaces of $\mathcal{V}$ and $\dim(\mathcal{W}_1 + \mathcal{W}) = \dim(\mathcal{W}_2 + \mathcal{W}) = k$. If $\mathcal{W}_1 \sim \mathcal{W}_2$, then either $\mathcal{W}_1 + \mathcal{W} \subset \mathcal{W}_2 + \mathcal{W}$ or $\mathcal{W}_2 + \mathcal{W} \subset \mathcal{W}_1 + \mathcal{W}$. Since $\dim(\mathcal{W}_1 + \mathcal{W}) = \dim(\mathcal{W}_2 + \mathcal{W}) = k$, we have $\mathcal{W}_1 + \mathcal{W} = \mathcal{W}_2 + \mathcal{W}$, which is a contradiction.

(iii) Suppose that $\dim(\mathcal{V}) - \dim(\mathcal{W}) = 2$ and let $\mathcal{W}_1, \mathcal{W}_2 \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$. Then $\dim(\mathcal{W}_1 + \mathcal{W}) = \dim(\mathcal{W}_2 + \mathcal{W}) = k + 1$ and by (ii), $\mathcal{W}_1 \nsim \mathcal{W}_2$ in $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$.

(iv) Follows trivially.

(v) Let $\mathcal{W}_1 \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$. We have the following cases:
Case 1: $\dim(\mathcal{W} + \mathcal{W}_1) = k + 1$. There exist two subspaces $\mathcal{W}_2, \mathcal{W}_3$ of $\mathcal{V}$ such that $\dim(\mathcal{W} + \mathcal{W}_2) = k + 2$, $\dim(\mathcal{W} + \mathcal{W}_3) = k + 3$ and $\mathcal{W} + \mathcal{W}_1 \subset \mathcal{W} + \mathcal{W}_2 \subset \mathcal{W} + \mathcal{W}_3$.
Case 2: $\dim(\mathcal{W} + \mathcal{W}_1) = k + 2$. There exist two subspaces $\mathcal{W}_2, \mathcal{W}_3$ of $\mathcal{V}$ such that $\dim(\mathcal{W} + \mathcal{W}_2) = k + 1$, $\dim(\mathcal{W} + \mathcal{W}_3) = k + 3$ and $\mathcal{W} + \mathcal{W}_2 \subset \mathcal{W} + \mathcal{W}_1 \subset \mathcal{W} + \mathcal{W}_3$.
Case 3: $\dim(\mathcal{W} + \mathcal{W}_1) = k + 3$. There exist two subspaces $\mathcal{W}_2, \mathcal{W}_3$ of $\mathcal{V}$ such that $\dim(\mathcal{W} + \mathcal{W}_2) = k + 1, \dim(\mathcal{W} + \mathcal{W}_3) = k + 2$ and $\mathcal{W} + \mathcal{W}_2 \subset \mathcal{W} + \mathcal{W}_3 \subset \mathcal{W} + \mathcal{W}_1$.
Thus in all the cases we can form a triangle with the vertices $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$.

(vi) Since $\dim(\mathcal{V}) - \dim(\mathcal{W}) \geq 2$, there exist two linearly independent vectors $u, v \in \mathcal{V} \setminus \mathcal{W}$ such that $\mathrm{Span}\{u\} \nsim \mathrm{Span}\{v\}$ in $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$.

□

Theorem 2.3. *Let* $\mathcal{W}$ *be a subspace of a vector space* $\mathcal{V}$ *such that* $\dim(\mathcal{V}) - \dim(\mathcal{W}) \geq 3$. *Then* $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ *is connected and* $\mathrm{diam}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) \leq 3$.

Proof. Let $\dim(\mathcal{W}) = k$ and $\mathcal{W}_1, \mathcal{W}_2 \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$. If $\mathcal{W}_1 + \mathcal{W} \subset \mathcal{W}_2 + \mathcal{W}$ or $\mathcal{W}_2 + \mathcal{W} \subset \mathcal{W}_1 + \mathcal{W}$, then $\mathcal{W}_1 \sim \mathcal{W}_2$ and $\mathrm{d}(\mathcal{W}_1, \mathcal{W}_2) = 1$. If $\mathcal{W}_1 \nsim \mathcal{W}_2$, then we have the following cases:

Case 1: $\dim(\mathcal{W}_1 + \mathcal{W}) = \dim(\mathcal{W}_2 + \mathcal{W}) = k + 1$.

Subcase 1: $\mathcal{W}_1 + \mathcal{W} = \mathcal{W}_2 + \mathcal{W}$. There exist $w \in \mathcal{V} \setminus (\mathcal{W}_1 + \mathcal{W})$ and $(\mathcal{W}_1 + \mathcal{W}) \subset (\mathrm{Span}\{w\} + \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}) \supset (\mathcal{W}_1 + \mathcal{W})$ such that $\mathcal{W}_1 \sim (\mathcal{W}_1 + \mathcal{W}_2 + \mathrm{Span}\{w\}) \sim \mathcal{W}_2$ is a path in $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ and $\mathrm{d}(\mathcal{W}_1, \mathcal{W}_2) = 2$.

Subcase 2: $\mathcal{W}_1 + \mathcal{W} \neq \mathcal{W}_2 + \mathcal{W}$. Then $(\mathcal{W}_1 + \mathcal{W}) \subset (\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}) \supset (\mathcal{W}_2 + \mathcal{W})$ and $\mathcal{W}_1 \sim (\mathcal{W}_1 + \mathcal{W}_2) \sim \mathcal{W}_2$ is a path in $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ and $\mathrm{d}(\mathcal{W}_1, \mathcal{W}_2) = 2$.

Case 2: $\dim(\mathcal{W}_1 + \mathcal{W}) = k + 1$ and $\dim(\mathcal{W}_2 + \mathcal{W}) > k + 1$.

Let $u \in \mathcal{W}_2 + \mathcal{W} \setminus \mathcal{W}_1 + \mathcal{W}$ and $< u > + \mathcal{W} = \mathcal{W}_3$. Since $\dim(\mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}) = k + 2$, $\mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W} \neq \mathcal{V}$ and $\mathcal{W}_3 + \mathcal{W} \subset \mathcal{W}_2 + \mathcal{W}$, we have $\mathcal{W}_1 \sim \mathcal{W}_1 + \mathcal{W}_3 \sim \mathcal{W}_3 \sim \mathcal{W}_2$. Hence $\mathrm{d}(\mathcal{W}_1, \mathcal{W}_2) \leq 3$.

Case 3: $\dim(\mathcal{W}_1 + \mathcal{W}) > k + 1$ *and* $\dim(\mathcal{W}_2 + \mathcal{W}) > k + 1$.

Subcase 1: $\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W} \neq \mathcal{V}$ *or* $(\mathcal{W}_1 + \mathcal{W}) + (\mathcal{W}_2 + \mathcal{W}) \neq \mathcal{W}$. Then $\mathcal{W}_1 \sim \mathcal{W}_1 + \mathcal{W}_2 \sim \mathcal{W}_2$ or $\mathcal{W}_1 \sim (\mathcal{W}_1 + \mathcal{W}) \cap (\mathcal{W}_2 + \mathcal{W}) \sim \mathcal{W}_2$.

Subcase 2: $\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W} = \mathcal{V}$ *and* $(\mathcal{W}_1 + \mathcal{W}) \cap (\mathcal{W}_2 + \mathcal{W}) = \mathcal{W}$. Let $v \in \mathcal{W}_2 \setminus \mathcal{W}$. Since $\dim(\mathcal{W}_1 + \mathcal{W}) > k + 1$, $\dim(\mathcal{W}_2 + \mathcal{W}) > k + 1$ and $\mathcal{W}_1 + \mathcal{W} + \mathcal{W}_2 + \mathcal{W} = \mathcal{V}$, $(\mathcal{W}_1 + \mathcal{W}) \cap (\mathcal{W}_2 + \mathcal{W}) = \mathcal{W}$, we have $\dim(\mathcal{W}_1 + \mathcal{W}) < n - 1$, $\dim(\mathcal{W}_2 + \mathcal{W}) < n - 1$, and $\mathcal{W}_1 + \langle v \rangle + \mathcal{W} \neq \mathcal{V}$, $\mathcal{W}_1 \sim \mathcal{W}_1 + \langle v \rangle \sim \langle v \rangle \sim \mathcal{W}_2$.

Hence $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is connected and $\mathrm{d}(\mathcal{W}_1, \mathcal{W}_2) \leq 3$.

□

Theorem 2.4. *If* $\mathcal{W}$ *is a subspace of a vector space* $\mathcal{V}$ *such that* $\dim(\mathcal{V}) - \dim(\mathcal{W}) \geq 3$, *then* $\mathrm{diam}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) = 3$.

Proof. Let $\mathcal{W}$ be a k dimensional subspace of $\mathcal{V}$ and $\{w_1, w_2, \ldots, w_k\}$ be a basis of $\mathcal{W}$. This linearly independent subset can be extended to a basis for $\mathcal{V}$. Let $\{w_1, w_2, \ldots, w_k, \ldots, w_n\}$ be a basis for $\mathcal{V}$ and $\mathcal{W}_1 = \mathrm{Span}\{w_{k+1}\}$, $\mathcal{W}_2 = \mathrm{Span}\{w_{k+2}, w_{k+3}, \ldots, w_n\}$. Clearly, $\mathcal{W}_1, \mathcal{W}_2 \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$, $\mathcal{W}_1 \nsim \mathcal{W}_2$ and $\mathrm{d}(\mathcal{W}_1, \mathcal{W}_2) \neq 1$. If $\mathrm{d}(\mathcal{W}_1, \mathcal{W}_2) = 2$, then there exists $\mathcal{W}_3 \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) \setminus \{\mathcal{W}_1, \mathcal{W}_2\}$ such that $\mathcal{W}_1 \sim \mathcal{W}_3 \sim \mathcal{W}_2$ is a path in $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$. Since $\mathcal{W}_1 \sim \mathcal{W}_3$, either $\mathcal{W}_1 + \mathcal{W} \subset \mathcal{W}_3 + \mathcal{W}$ or $\mathcal{W}_1 + \mathcal{W} \supset \mathcal{W}_3 + \mathcal{W}$. If $\mathcal{W}_1 + \mathcal{W} \supset \mathcal{W}_3 + \mathcal{W}$, then $\mathcal{W}_3 \nsim \mathcal{W}_2$ as $(\mathcal{W}_1 + \mathcal{W}) \cap (\mathcal{W}_2 + \mathcal{W}) = \mathcal{W}$. Thus $\mathcal{W}_1 + \mathcal{W} \subset \mathcal{W}_3 + \mathcal{W}$. Again since $\mathcal{W}_3 \sim \mathcal{W}_2$, either $\mathcal{W}_2 + \mathcal{W} \subset \mathcal{W}_3 + \mathcal{W}$ or $\mathcal{W}_2 + \mathcal{W} \supset \mathcal{W}_3 + \mathcal{W}$. If $\mathcal{W}_2 + \mathcal{W} \supset \mathcal{W}_3 + \mathcal{W}$, then $\mathcal{W}_3 \nsim \mathcal{W}_1$ as $(\mathcal{W}_1 + \mathcal{W}) \cap (\mathcal{W}_2 + \mathcal{W}) = \mathcal{W}$. Thus $\mathcal{W}_2 + \mathcal{W} \subset \mathcal{W}_3 + \mathcal{W}$. Therefore we find that $\mathcal{W}_3 + \mathcal{W}$ is a subspace of $\mathcal{V}$ which contains $\mathcal{W}_1 + \mathcal{W}$ as well as $\mathcal{W}_2 + \mathcal{W}$ i.e., $\mathcal{W}_3 + \mathcal{W} = \mathcal{V}$, a contradiction. Thus $\mathrm{d}(\mathcal{W}_1, \mathcal{W}_2) \geq 3$ and by Theorem 2.3, we get $\mathrm{d}(\mathcal{W}_1, \mathcal{W}_2) \leq 3$. Thus $\mathrm{diam}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) = 3$. □

The following lemmas are essential to prove our next theorem.

Lemma 2.5. *If* $\mathcal{W}$ *is a subspace of a vector space* $\mathcal{V}$ *such that* $\dim(\mathcal{V}) - \dim(\mathcal{W}) = 3$, *then* $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ *does not contain any cycle of odd length.*

Proof. Suppose that $\mathcal{W}_1 \sim \mathcal{W}_2 \sim \cdots \sim \mathcal{W}_k \sim \mathcal{W}_1$ is a cycle of odd length in $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$. Since $\dim(\mathcal{V}) - \dim(\mathcal{W}) = 3$, the dimension of each $\mathcal{W}_i + \mathcal{W}$ is either $\dim(\mathcal{W})+1$ or $\dim(\mathcal{W})+2$ since any two distinct vertices $\mathcal{W}_1, \mathcal{W}_2 \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ such that $\mathcal{W}_1 + \mathcal{W} = \mathcal{W}_2 + \mathcal{W}$ are not adjacent in $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$. Without loss of generality we may assume that $\dim(\mathcal{W}_1 + \mathcal{W}) = \dim(\mathcal{W}) + 1$ and we get $\dim(\mathcal{W}_k + \mathcal{W}) = \dim(\mathcal{W}) + 1$ and $\mathcal{W}_1 \nsim \mathcal{W}_k$, which is a contradiction. Hence $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ does not contain any cycle of odd length. □

Lemma 2.6. *Let* $\mathcal{N}$ *be a clique in* $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$. *Then* $\{\mathcal{U} + \mathcal{W} \mid \mathcal{U} \in \mathcal{N}\}$ *is a chain of subspaces of* $\mathcal{V}$.

Proof. The proof is trivial. □

Theorem 2.7. *Let* $\mathcal{W}$ *be a subspace of a finite dimensional vector space* $\mathcal{V}$. *Then* $\dim(\mathcal{V}) - (\dim(\mathcal{W}) + 1) = m$ *if and only if* $\omega(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) = m$, *where* $m = \dim(\mathcal{V}) - (\dim(\mathcal{W}) + 1)$.

Proof. Let $\mathcal{W}$ be a k-dimensional subspace of n-dimensional vector space $\mathcal{V}$ and $\{v_1, v_2, \ldots, v_k\}$, $\{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_{n-1}\}$ be the bases of $\mathcal{W}$ and $\mathcal{V}$, respectively. Let $\mathcal{W}_j = \langle v_1, v_2, \ldots, v_j\rangle$ for $j = k+1, k+2, \ldots, n$. Clearly, $\mathcal{N} = \{\mathcal{W}_{k+1}, \mathcal{W}_{k+2}, \ldots, \mathcal{W}_{n-1}\}$ is a clique. If possible, let $\mathcal{N}\cup\{\mathcal{W}'\}$ be a clique where $\mathcal{W}' \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) \setminus \mathcal{N}$. Thus by Lemma 2.6, there exists $i \in \{k+1, k+2, \ldots, n-2\}$ such that $\mathcal{W}_i \subset \mathcal{W}' + \mathcal{W} \subset \mathcal{W}_{i+1}$. Since the inclusion is proper and $\mathcal{V}$ is finite dimensional, we have $\dim(\mathcal{W}_i) < \dim(\mathcal{W}'+\mathcal{W}) < \dim(\mathcal{W}_{i+1})$, i.e., $i < \dim(\mathcal{W}' + \mathcal{W}) < i + 1$, a contradiction. Thus $\mathcal{N}$ is a clique of size $n-(k+1)$. If possible, let $\mathcal{N}' = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{n-k}\}$ be a clique of size $n-k$ and $\mathcal{U}_1+\mathcal{W} \subset \mathcal{U}_2+ \mathcal{W} \subset \cdots \subset \mathcal{U}_{n-k}+\mathcal{W}$. Again as $\mathcal{V}$ is finite dimensional and each inclusion is proper, we have $\dim(\mathcal{W}) < \dim(\mathcal{U}_1+\mathcal{W}) < \dim(\mathcal{U}_2+\mathcal{W}) < \cdots < \dim(\mathcal{U}_{n-k}+\mathcal{W})$. Since $\dim(\mathcal{U}_i+\mathcal{W})$ are distinct integers between $k+1$ and $n-1$, we have $n-k$ integers in $[k+1, n-1]$, a contradiction. Thus, $\omega(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) = n-(k+1)$.

Conversely, suppose that $\omega(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) = m$. Let $\dim(\mathcal{V}) - (\dim(\mathcal{W}) + 1) = p \neq m$. Then by the directed part, $\omega(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) = p$ and hence $p = m$. This completes the proof. □

Theorem 2.8. *If $\mathcal{W}$ is a k-dimensional subspace of an n-dimensional vector space $\mathcal{V}$, then $\chi(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) = n-k-1$.*

Proof. By Theorem 2.7, $\omega(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) = n-k-1$, and therefore $\chi(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) \geqslant n-k-1$. To show the equality, we demonstrate a $(n-k-1)$ colouring of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$. For any $\mathcal{U} \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$, if $\dim(\mathcal{U} + \mathcal{W}) = k + j$, then color $\mathcal{U}$ with the jth color. This coloring is proper since by Lemma 2.6, any two $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ such that $\dim(\mathcal{U}_2+\mathcal{W}) = \dim(\mathcal{U}_1+\mathcal{W}) = k+j$ are never adjacent and hence the theorem follows. □

Theorem 2.9. *If $\mathcal{W}$ is a k-dimensional subspace of an n-dimensional vector space $\mathcal{V}$, then $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ contains a graph $\mathcal{G}'$ such that $\mathcal{G}' \cong \mathcal{I}_{n-k}(\mathcal{V}/\mathcal{W})$.*

Proof. We know that proper subspaces of $\mathcal{V}$ containing $\mathcal{W}$ are in one-to-one correspondence with the nontrivial subspaces of $\mathcal{V}/\mathcal{W}$, i.e., $\mathfrak{A} = \{\mathcal{U} \subset \mathcal{V} \mid \mathcal{W} < \mathcal{U} < \mathcal{V}\} \longleftrightarrow \mathfrak{B} = \{\mathcal{U}' \subset \mathcal{V}/\mathcal{W} \mid (0) < \mathcal{U}' < \mathcal{V}/\mathcal{W}\}$. Clearly, $\mathfrak{A} \subseteq \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ and $\mathfrak{B} = \mathcal{V}(\mathcal{I}_n(\mathcal{V}/\mathcal{W}))$. Now if we define $\mathcal{G}'$ on $\mathfrak{A}$ by $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})[\mathfrak{A}]$, then $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})[\mathfrak{A}] \cong \mathcal{I}_{n-k}(\mathcal{V}/\mathcal{W})$ and hence the theorem follows. □

Theorem 2.10. *If $\mathcal{W}$ is a k-dimensional subspace of an n-dimensional vector space $\mathcal{V}$ such that $n-k \geq 3$, then $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is not planar.*

Proof. We know that by Theorem 2.9, $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ contains a graph $\mathcal{G}'$ such that $\mathcal{G}' \cong \mathcal{I}_{n-k}(\mathcal{V}/\mathcal{W})$, by Theorem 5.2 of [7], $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ contains a graph which is not planar, and by Kuratowski's theorem, $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is not planar. □

Theorem 2.11. *Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two subspaces of a finite dimensional vector space $\mathcal{V}$. Then $\mathcal{I}_n(\mathcal{W}_1) \simeq \mathcal{I}_n(\mathcal{W}_2)$ if and only if $\dim(\mathcal{W}_1) = \dim(\mathcal{W}_2)$.*

Proof. Suppose that $\mathcal{W}_1$ and $\mathcal{W}_2$ are two k-dimensional subspaces of an n-dimensional vector space $\mathcal{V}$ and let $\{u_1, u_2, \ldots, u_k\}$, $\{v_1, v_2, \ldots, v_k\}$ be the bases for $\mathcal{W}_1, \mathcal{W}_2$, respectively and $\mathfrak{A} = \{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n\}$, $\mathfrak{B} = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ be the extended bases for $\mathcal{V}$. Define a map $\mathfrak{f} : \mathfrak{A} \longrightarrow \mathfrak{B}$ by $\mathfrak{f}(u_i) = v_i$ for $i = 1, 2, \ldots, n$. Clearly, the map $\mathfrak{g} : \mathcal{V}(\mathcal{I}_n(\mathcal{W}_1)) \longrightarrow \mathcal{V}(\mathcal{I}_n(\mathcal{W}_2))$ defined by $\mathfrak{g}(\mathcal{U}) = \mathfrak{f}(\mathcal{U})$ for $\mathcal{U} \in \mathcal{V}(\mathcal{I}_n(\mathcal{W}_1))$ is bijective and adjacency preserving and hence $\mathcal{I}_n(\mathcal{W}_1) \simeq \mathcal{I}_n(\mathcal{W}_2)$.

Conversely, assume that $\mathcal{I}_n(\mathcal{W}_1) \simeq \mathcal{I}_n(\mathcal{W}_2)$ and $\dim(\mathcal{W}_1) = k_1, \dim(\mathcal{W}_2) = k_2$. Then by Theorem 2.7, $\omega(\mathcal{I}_n^{\mathcal{W}_1}(\mathcal{V}))$ and $\omega(\mathcal{I}_n^{\mathcal{W}_2}(\mathcal{V}))$ are $n - k_1 - 1$ and $n-k_2-1$, respectively. Since $\mathcal{I}_n(\mathcal{W}_1) \simeq \mathcal{I}_n(\mathcal{W}_2)$, we have $n-k_1-1 = n-k_2-1$ and hence $k_1 = k_2$. □

3. When the base field $\mathbb{F}$ is finite

In this section, we study some basic properties of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{W})$ if the base field $\mathbb{F}$ is finite, i.e., $|\mathbb{F}| = q$ and $q = p^r$ for some prime p.

Theorem 3.1. *Let $\mathcal{W}$ be a k-dimensional subspace of an n-dimensional vector space $\mathcal{V}$ over a finite field $\mathbb{F}$ with q elements. Then the set containing those subspaces $\mathcal{U}$ of $\mathcal{V}$ such that $\mathcal{U} + \mathcal{W} = \mathcal{V}$ i.e., $\{\mathcal{U} \subseteq \mathcal{V} \mid \mathcal{U} + \mathcal{W} = \mathcal{V}\}$ has $(\sum_{r=0}^{k-1} n_r + 1)$ elements, where*

$$n_r = \frac{(q^k-1)(q^k-q)\cdots(q^k-q^{r-1})(q^n-q^k)(q^n-q^{k+1})\cdots(q^n-q^{n-1})}{(q^{n-k+r}-1)(q^{n-k+r}-q)\cdots(q^{n-k+r}-q^{n-k+r-1})}.$$

Proof. Since $\dim(\mathcal{W}) = k < n$ for any subspace $\mathcal{W}' \in \{\mathcal{U} \subset \mathcal{V} \mid \mathcal{U} + \mathcal{W} = \mathcal{V}\}$ of $\mathcal{V}$ has dimension at least $n - k$, i.e., if $\mathcal{W}' \in \{\mathcal{U} \subset \mathcal{V} \mid \mathcal{U} + \mathcal{W} = \mathcal{V}\}$, then $\dim(\mathcal{W}') = n-k+r$ and $\dim(\mathcal{W}' \cap \mathcal{W}) = r$ where $0 \leq r \leq k-1$. To find such subspaces $\mathcal{W}'$, we choose r linearly independent vectors from $\mathcal{W}$ and $n - k$ linearly independent vectors from $\mathcal{V} \backslash \mathcal{W}$, and generate $\mathcal{W}'$ with these $n-k+r$ linearly independent vectors. Since the number of ways we can choose r linearly independent vectors from $\mathcal{W}$ is $(q^k - 1)(q^k - q)\cdots(q^k - q^{r-1})$, the number of ways we can choose $n-k$ linearly independent vectors from $\mathcal{V} \backslash \mathcal{W}$ is $(q^n - q^k)(q^n - q^{k+1})\cdots(q^n - q^{n-1})$. The number of bases of an $(n-k+r)$-dimensional subspace is $(q^{n-k+r} - 1)(q^{n-k+r} - q)\cdots(q^{n-k+r} - q^{n-k+r-1})$, the number of subspaces $\mathcal{W}'$ with $\dim(\mathcal{W}') = n-k+r$ and $\dim(\mathcal{W} \cap \mathcal{W}') = r$ is

$$n_r = \frac{(q^k-1)(q^k-q)\cdots(q^k-q^{r-1})(q^n-q^k)(q^n-q^{k+1})\cdots(q^n-q^{n-1})}{(q^{n-k+r}-1)(q^{n-k+r}-q)\cdots(q^{n-k+r}-q^{n-k+r-1})}.$$

If $r = k$, then $\mathcal{V}$ is the only subspace which satisfies the given condition. Since $0 \leq r \leq k-1$,

$$|\{\mathcal{U} \subseteq \mathcal{V} \mid \mathcal{U} + \mathcal{W} = \mathcal{V}\}| = \sum_{r=0}^{k-1} n_r + 1.$$

□

Theorem 3.2. *Let $\mathcal{W}$ be a k-dimensional subspace of an n-dimensional vector space $\mathcal{V}$ over a finite field $\mathbb{F}$ of order q. Then $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is a graph of* *order* $\mathcal{G}(n,q) - (\mathcal{G}(k,q) + \sum_{r=0}^{k-1} n_r + 1)$*, where*

$$n_r = \frac{(q^k-1)(q^k-q)\cdots(q^k-q^{r-1})(q^n-q^k)(q^n-q^{k+1})\cdots(q^n-q^{n-1})}{(q^{n-k+r}-1)(q^{n-k+r}-q)\cdots(q^{n-k+r}-q^{n-k+r-1})}$$

and $\mathcal{G}(n,q)$ *is the Galois number. In particular, when* $\mathcal{W}=(0)$*, the order of* $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ *is* $\mathcal{G}(n,q)-2$*.*

Proof. By the definition of the graph $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$, $\mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})) = \{\mathcal{U} \subset \mathcal{V}\} \setminus (\{\mathcal{U}' \subset \mathcal{W}\} \cup \{\mathcal{U} \subset \mathcal{V} \mid \mathcal{U}+\mathcal{W}=\mathcal{V}\})$. Since the number of r-dimensional subspaces of a n-dimensional vector space over a finite field of order q is the binomial coefficient (see [7])

$$\left[{n \atop r}\right]_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)},$$

the total number of subspaces of $\mathcal{V}$ is given by

$$\sum_{r=0}^{n} \left[{n \atop r}\right]_q = \mathcal{G}(n,q)-2.$$

Similarly, the total number of subspaces of $\mathcal{W}$ is given by

$$\sum_{r=0}^{k} \left[{k \atop r}\right]_q = \mathcal{G}(k,q)-2.$$

By Theorem 3.1, $\{\mathcal{U} \subseteq \mathcal{V} \mid \mathcal{U}+\mathcal{W}=\mathcal{V}\}$ has $\sum_{r=0}^{k-1} n_r + 1$ elements, where

$$n_r = \frac{(q^k-1)(q^k-q)\cdots(q^k-q^{r-1})(q^n-q^k)(q^n-q^{k+1})\cdots(q^n-q^{n-1})}{(q^{n-k+r}-1)(q^{n-k+r}-q)\cdots(q^{n-k+r}-q^{n-k+r-1})}.$$

Since $\{\mathcal{U}' \subset \mathcal{W}\} \cap \{\mathcal{U} \subseteq \mathcal{V} \mid \mathcal{U}+\mathcal{W}=\mathcal{V}\} = \varnothing$, the order of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is $\mathcal{G}(n,q) - (\mathcal{G}(k,q) + \sum_{r=0}^{k-1} n_r + 1)$, where

$$n_r = \frac{(q^k-1)(q^k-q)\cdots(q^k-q^{r-1})(q^n-q^k)(q^n-q^{k+1})\cdots(q^n-q^{n-1})}{(q^{n-k+r}-1)(q^{n-k+r}-q)\cdots(q^{n-k+r}-q^{n-k+r-1})}$$

and $\mathcal{G}(n,q)$ is the Galois number. Trivially, when $\mathcal{W}=(0)$, the order of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is $\mathcal{G}(n,q)-2$.

□

Theorem 3.3. *Let* $\mathcal{W}$ *be a* k*-dimensional subspace of a* n*-dimensional vector space of* $\mathcal{V}$ *over a finite field* $\mathbb{F}$ *of order* q *and* $\mathcal{U} \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ *such that* $\dim(\mathcal{U}+\mathcal{W}) = l$*. Then*

$$\deg(\mathcal{U}) = \sum_{r=1}^{l-k-1} \left[{l-k \atop r}\right]_q (\sum_{i=0}^{k-1} n_i + 1) + \sum_{s=1}^{n-l-1} \left[{n-l \atop s}\right]_q (\sum_{i=0}^{k-1} p_i + 1),$$

where

$$n_i = \frac{(q^k-1)(q^k-q)\cdots(q^k-q^{i-1})}{(q^{r+i}-1)(q^{r+i}-q)} \times \frac{(q^{k+r}-q^k)(q^{k+r}-q^{k+1})\cdots(q^{k+r}-q^{k+r-1})}{(q^{r+i}-q^2)\cdots(q^{r+i}-q^{r+i-1})}$$

and

$$p_i = \frac{(q^k-1)(q^k-q)\cdots(q^k-q^{i-1})}{(q^{l+s-k+i}-1)(q^{l+s-k+i}-q)}. \times \frac{(q^{l+s}-q^k)(q^{l+s}-q^{k+1})\cdots(q^{l+s}-q^{l+s-1})}{(q^{l+s-k+i}-q^2)\cdots(q^{l+s-k+i}-q^{l+s-k+i-1})}.$$

Proof. First we find the subspaces of $\mathcal{V}$ which properly contains $\mathcal{W}$ as a subspace and properly contained in $\mathcal{U}+\mathcal{W}$. We know that there is a one-to-one correspondence between the $(k+r)$-dimensional subspaces of $\mathcal{U}+\mathcal{W}$ containing $\mathcal{W}$ and the r-dimensional subspaces of $(\mathcal{U}+\mathcal{W})/\mathcal{W}$, i.e., $\mathfrak{A} = \{\mathcal{A} \mid \mathcal{W} < \mathcal{A} < \mathcal{U}+\mathcal{W}\} \longleftrightarrow \mathfrak{B} = \{\mathcal{B} \mid (0) < \mathcal{B} < (\mathcal{U}+\mathcal{W})/\mathcal{W}\}$. It may be noted that the number of r-dimensional subspaces of $(l-k)$-dimensional vector space $(\mathcal{U}+\mathcal{W})/\mathcal{W}$ over a finite field of order q is the binomial coefficient

$$\left[{l-k \atop r}\right]_q = \frac{(q^{l-k}-1)(q^{l-k-1}-1)\cdots(q^{l-k-r+1}-1)}{(q^r-1)(q^{r-1}-1)\cdots(q-1)}.$$

Corresponding to each r-dimensional subspace in $\mathfrak{B}$, there is a $(k+r)$-dimensional subspace in $\mathfrak{A}$ and therefore the number of $(k+r)$-dimensional subspaces in $\mathfrak{A}$ is given by

$$\left[{l-k \atop r}\right]_q = \frac{(q^{l-k}-1)(q^{l-k-1}-1)\cdots(q^{l-k-r+1}-1)}{(q^r-1)(q^{r-1}-1)\cdots(q-1)}.$$

Let $\mathcal{W}' \in \mathfrak{A}$ be a $(k+r)$-dimensional subspace of $\mathcal{U}+\mathcal{W}$. If $\mathcal{W}_i \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ such that $\mathcal{W}_i + \mathcal{W} = \mathcal{W}'$, then $\mathcal{W}_i \subseteq \mathcal{W}'$. Therefore by Theorem 3.1, the number of $\mathcal{W}_i \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ such that $\mathcal{W}_i + \mathcal{W} = \mathcal{W}'$ is given by $\sum_{i=0}^{k-1} n_i$, where

$$n_i = \frac{(q^k-1)(q^k-q)\cdots(q^k-q^{i-1})}{(q^{r+i}-1)(q^{r+i}-q)} \times \frac{(q^{k+r}-q^k)(q^{k+r}-q^{k+1})\cdots(q^{k+r}-q^{k+r-1})}{(q^{r+i}-q^2)\cdots(q^{r+i}-q^{r+i-1})}.$$

Therefore, we have $\left[{l-k \atop r}\right]_q - (k+r)$-dimensional subspaces, where $r = 1, 2, \ldots, l-k-1$. Thus the number of subspaces $\mathcal{U}' \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ such that $\mathcal{U}'+\mathcal{W} \subset \mathcal{U}+\mathcal{W}$ is $\sum_{r=1}^{l-k-1}\left[{l-k \atop r}\right]_q(\sum_{i=0}^{k-1} n_i + 1)$. Now we find the subspaces of $\mathcal{V}$ which properly contains $\mathcal{U}+\mathcal{W}$ as a subspace and is properly contained in $\mathcal{V}$. There is a a one-to-one correspondence between the $(l+s)$-dimensional subspace of $\mathcal{V}$ containing $\mathcal{U}+\mathcal{W}$ and the s-dimensional subspace of $\mathcal{V}/(\mathcal{U}+\mathcal{W})$, i.e., $\mathfrak{C} = \{\mathcal{A}' \mid \mathcal{U}+\mathcal{W} < \mathcal{A}' < \mathcal{V}\} \longleftrightarrow \mathfrak{D} = \{\mathcal{B}' \mid (\mathcal{U}+\mathcal{W}) < \mathcal{B} < \mathcal{V}/(\mathcal{U}+\mathcal{W})\}$.

Note that the number of s-dimensional subspaces of the $(n-l)$-dimensional vector space $\mathcal{V}/(\mathcal{U}+\mathcal{W})$ over a finite field of order q is the binomial coefficient

$$\left[{n-l \atop s}\right]_q = \frac{(q^{n-l}-1)(q^{n-l-1}-1)\cdots(q^{n-l-s+1}-1)}{(q^s-1)(q^{s-1}-1)\cdots(q-1)}.$$

Corresponding to each s-dimensional subspace in $\mathfrak{D}$, there is a $(l+s)$-dimensional subspace in $\mathfrak{C}$. Therefore the number of $(l+s)$-dimensional subspaces in $\mathfrak{C}$ is given by

$$\left[{n-l \atop s}\right]_q = \frac{(q^{n-l}-1)(q^{n-l-1}-1)\cdots(q^{n-l-s+1}-1)}{(q^s-1)(q^{s-1}-1)\cdots(q-1)}.$$

Let $\mathcal{W}' \in \mathfrak{C}$ be a $(l+s)$-dimensional subspaces of $\mathcal{V}$. If $\mathcal{W}_i \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ such that $\mathcal{W}_i + \mathcal{W} = \mathcal{W}'$, then $\mathcal{W}_i \subseteq \mathcal{W}'$. Therefore by Theorem 3.1, the number of $\mathcal{W}_i \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ such that $\mathcal{W}_i + \mathcal{W} = \mathcal{W}'$ is given by $\sum_{i=0}^{k-1} p_i + 1$, where

$$p_i = \frac{(q^k-1)(q^k-q)\cdots(q^k-q^{i-1})}{(q^{l+s-k+i}-1)(q^{l+s-k+i}-q)} \times \frac{(q^{l+s}-q^k)(q^{l+s}-q^{k+1})\cdots(q^{l+s}-q^{l+s-1})}{(q^{l+s-k+i}-q^2)\cdots(q^{l+s-k+i}-q^{l+s-k+i-1})}.$$

Therefore we have $\left[{n-l \atop s}\right]_q - (l+s)$-dimensional subspaces, where $s = 1, 2, \ldots, n-l-1$. Thus the number of subspaces $\mathcal{U}' \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ such that $\mathcal{U}+\mathcal{W} \subset \mathcal{U}'+\mathcal{W}$ is $\sum_{s=1}^{n-l-1}\left[{n-l \atop s}\right]_q(\sum_{i=0}^{k-1} p_i + 1)$. Hence

$$\deg(\mathcal{U}) = \sum_{r=1}^{l-k-1}\left[{l-k \atop r}\right]_q(\sum_{i=0}^{k-1} n_i + 1) + \sum_{s=1}^{n-l-1}\left[{n-l \atop s}\right]_q(\sum_{i=0}^{k-1} p_i + 1).$$

□

Theorem 3.4. *Let $\mathcal{W}$ be a k-dimensional subspace of an n-dimensional vector space $\mathcal{V}$ over a finite field with q elements. Then the following statements hold.*

(i) *If q is odd, then $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is Eulerian.*
(ii) *If q is even, then $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is Eulerian if and only if $n-k$ even.*

Proof. (i) It can be easily seen that from [11, Proposition 7.1, p. 25]: $G(n+1,q) = 2G(n,q) + (q^n-1)G(n-1,q)$ with $G(0,q) = 1$ and $G(1,q) = 2$. Thus if q is odd, then all Galois numbers are even. Let $\mathcal{W} \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ such that $\dim(\mathcal{W}_1 + \mathcal{W}) = \ell$. Thus by Theorem 3.3, $\deg(\mathcal{U})$ in $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is

$$(G(\ell-k,q)-2)(\sum_{i=0}^{k-1} n_i + 1) + ((G(n-\ell,q)-2))(\sum_{i=0}^{k-1} p_i + 1),$$

an even number. Thus the degree of each vertex of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is even and hence $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is Eulerien.

(ii) If q is even, then by [11, Proposition 7.1, p. 25], $G(2m,q)$ is odd and $G(2m+1,q)$ is even for $m \in \mathbb{N} \cup \{0\}$. Now, if $\mathcal{U} \in \mathcal{V}(\mathcal{I}_n^{\mathcal{W}}(\mathcal{V}))$ such that

$\dim(\mathcal{U}+\mathcal{W}_1)=\ell$, then $\deg(\mathcal{U})$ is $(G(\ell-k,q)-2)(\sum_{i=0}^{k-1} n_i+1)+((G(n-\ell,q)-2))(\sum_{i=0}^{k-1} p_i+1)$.

If $n-k$ is even, then $G(n-\ell,q)$ and $G(\ell-k,q)$ are both either even or odd and hence the degree of $\mathcal{U}$ is even.

If $n-k$ is odd, then we have the following cases.

Case 1: n is even, k is odd, and ℓ is even.

Then $G(n-\ell,q)$ is odd and $G(\ell-k,q)$ is even, and the degree of $\mathcal{U}$ is odd.

Case 2: n is even, k is odd, and ℓ is odd.

Then $G(n-\ell,q)$ is even and $G(\ell-k,q)$ is odd and the degree of $\mathcal{U}$ is odd.

Case 3: n is odd, k is even and ℓ is even.

Then $G(n-\ell,q)$ is even and $G(\ell-k,q)$ is odd and the degree of $\mathcal{U}$ is odd.

Case 4: n is odd, k is even and ℓ is odd.

Then $G(n-\ell,q)$ is odd and $G(\ell-k,q)$ is even and the degree of $\mathcal{U}$ is odd.

Thus in all the cases degree of $\mathcal{U}$ is odd and hence $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ is not Eulerian. □

4. Conclusion

In this paper, we have introduced a subspace based subspace inclusion graph on the vector space $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ of a finite dimensional vector space $\mathbb{V}$ and investigated various interrelationships between $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ (as a graph) and $\mathbb{V}$ (as a vector space). The diameter, girth, clique number, and chromatic number of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ have been studied. It is shown that two subspace based subspace inclusion graphs $\mathcal{I}_n^{\mathcal{W}_1}(\mathcal{V})$ and $\mathcal{I}_n^{\mathcal{W}_2}(\mathcal{V})$ are isomorphic if and only if $\mathcal{W}_1$ and $\mathcal{W}_2$ are isomorphic. Further, some properties of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ have also been obtained when the base field is finite. As an area of further research, one can look into the structure of the automorphism group of $\mathcal{I}_n^{\mathcal{W}}(\mathcal{V})$ in case of a finite field.

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