

THE FRACTIONAL LOCAL METRIC DIMENSION OF
GRAPHS

IMRAN JAVAID*, HIRA BENISH, AND MUHAMMAD MURTAZA

ABSTRACT. The fractional versions of graph-theoretic invariants multiply the range of applications in scheduling, assignment and operational research problems. For this interesting aspect of fractional graph theory, we introduce the fractional version of local metric dimension of graphs. The local resolving neighborhood $L(xy)$ of an edge xy of a graph G is the set of those vertices in G which resolve the vertices x and y . A function $f : V(G) \rightarrow [0, 1]$ is a local resolving function of G if $f(L(xy)) \geq 1$ for all edges xy in G . The minimum value of $f(V(G))$ among all local resolving functions f of G is the fractional local metric dimension of G . We study the properties and bounds of fractional local metric dimension of graphs and give some characterization results. We determine the fractional local metric dimension of strong and Cartesian product of graphs.

1. INTRODUCTION AND TERMINOLOGY

Resolving sets and the metric dimension of a graph were introduced by Slater [18] and Harary and Melter [12] independently. Currie *et al.* [7] initiated the concept of fractional metric dimension and defined it as the optimal solution of the Linear Programming relaxation of the integer programming problem of the metric dimension of graphs. The fractional metric dimension problem was further studied by Arumugam and Mathew [1] in 2012. The authors provided a sufficient condition for a connected graph G whose fractional metric dimension is $\frac{|V(G)|}{2}$. The fractional metric dimension of graphs and graph products has also been studied in [1, 9, 10, 11, 14, 19].

Okamoto *et al.* [16] initiated the study of distinguishing adjacent vertices in a graph G rather than all the vertices of G by distance. This motivated the study of local resolving sets and local metric dimension in graphs. In this paper, we introduce the fractional version of the local metric dimension of a graph. We study the local fractional metric dimension of some graphs and establish some bounds on the fractional local metric dimension of graphs. We also determine the fractional local metric dimension of strong and Cartesian products of graphs.

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Let $G = (V(G), E(G))$ be a finite, simple and connected graph. The edge between two vertices u and v is denoted by uv . If two vertices u and v are joined by an edge then they are called adjacent vertices, denoted by $u \sim v$. $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ and $N_G[u] = N(u) \cup \{u\}$ are called the *open neighborhoods* and the *closed neighborhoods* of a vertex u , respectively. For a subset U of $V(G)$, $N_G(U) = \{v \in V(G) : uv \in E(G); u \in U\}$ is the open neighborhood of U in G . The *distance* between any two vertices u and v of G is the length of a shortest $u - v$ path in G is denoted by $d(u, v)$. Two distinct vertices u, v are *adjacent twins* if $N[u] = N[v]$ and *non-adjacent twins* if $N(u) = N(v)$. Adjacent twins are called *true twins* and non-adjacent twins are called *false twins*. For two distinct vertices u and v in G , $R(u, v) = \{x \in V(G) : d(x, u) \neq d(x, v)\}$. A vertex set $W \subseteq V(G)$ is called a *resolving set* of G if $W \cap R(u, v) \neq \emptyset$ for any two distinct vertices $u, v \in V(G)$. The minimum cardinality of a resolving set of G is called the *metric dimension* of G . The function $f : V(G) \rightarrow [0, 1]$ is called a *resolving function* of G if $f(R(u, v)) \geq 1$ for any two distinct vertices u and v in G . The minimum value of $f(V(G))$ among all resolving functions f of G is called the *fractional metric dimension* of G , denoted by $\dim_f(G)$.

A vertex set $W \subset V(G)$ is called a *local resolving set* of G if $W \cap R(u, v) \neq \emptyset$ for any two adjacent vertices $u, v \in V(G)$. The minimum cardinality of a local resolving set is called the *local metric dimension* of G and it is denoted by $ldim(G)$. A local resolving set of order $ldim(G)$ is called a *local metric basis* of G . For $uv \in E(G)$, we define the *local resolving neighborhood* as $L(uv) = \{x \in V(G); d(u, x) \neq d(v, x)\}$. $L(uv) = V(G)$, for all $uv \in E(G)$, if and only if $ldim(G) = 1$. In [16], it was shown that $ldim(G) = 1$ if and only if G is a bipartite graph. Hence, $L(uv) = V(G)$ for all $uv \in E(G)$ if and only if G is a bipartite graph. Now, we define the fractional local metric dimension of a graph as follows;

Definition 1.1. A function $f : V \rightarrow [0, 1]$ is a *local resolving function LRF* of G if $f(L(uv)) \geq 1$ for all $uv \in E(G)$, where

$$f(L(uv)) = \sum_{x \in L(uv)} f(x).$$

The *weight of local resolving function f* is defined as

$$|f| = \sum_{v \in V(G)} f(v).$$

The minimum weight of a local resolving function of G is called the *fractional local metric dimension* of G and is denoted by $ldim_f(G)$.

The strong product of two graphs G and H , denoted by $G \boxtimes H$, is a graph with the vertex set $V(G \boxtimes H) = \{(u, v) : u \in V(G) \text{ and } v \in V(H)\}$ and two vertices (u_1, v_1) and (u_2, v_2) in $G \boxtimes H$ are adjacent if and only if

- $u_1u_2 \in E(G)$ and $v_1 = v_2$ or
- $u_1 = u_2$ and $v_1v_2 \in E(H)$ or

- $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

For a vertex $u \in V(G)$, the set of vertices $\{(u, v) : v \in V(H)\}$ is called an H -layer and is denoted by H^u . Similarly, for a vertex $v \in V(H)$, the set of vertices $\{(u, v) : u \in V(G)\}$ is called a G -layer and is denoted by G^v . Let $d_{G \boxtimes H}((u_1, v_1), (u_2, v_2))$ denote the distance between (u_1, v_1) and (u_2, v_2) . For $(u_1, v_1)(u_2, v_2) \in E(G \boxtimes H)$, the local resolving neighborhood of edge $(u_1, v_1)(u_2, v_2)$ is denoted by $L_{G \boxtimes H}((u_1, v_1)(u_2, v_2))$ and $L_G(u_1u_2)$ denotes the local resolving neighborhood of $u_1u_2 \in E(G)$. The following result gives the relationship between the distance of vertices in $G \boxtimes H$ and the distance of vertices in graphs G or H .

Remark: [13] Let G and H be two connected graphs. Then

$$d_{G \boxtimes H}((u_1, v_1), (u_2, v_2)) = \max\{d_G(u_1, u_2), d_H(v_1, v_2)\}.$$

This paper is organized as follows: in Section 2, we characterize the graphs G with the fractional local metric dimension $\frac{|V(G)|}{2}$ and give bounds on the fractional local metric dimension of graphs. We study the fractional local metric dimension of some families of graphs and also discuss the difference between the fractional metric dimension and the fractional local metric dimension of some families of graphs. In Section 3, we study the fractional local metric dimension of strong and Cartesian products of graphs. We establish bounds on the fractional local metric dimension of these graph products.

2. CHARACTERIZATION RESULTS AND BOUNDS ON $ldim_f(G)$

In a connected graph G , since every resolving function is also a local resolving function, it follows that

$$ldim_f(G) \leq dim_f(G)$$

Since, the characteristic function of a minimal local resolving set is an *LRF* of G , it follows that

$$1 \leq ldim_f(G) \leq ldim(G) \leq n - 1.$$

Thus, if a graph G has $ldim(G) = 1$, then $ldim_f(G) = 1$. We have the following result:

Observation 2.1. *Let G be a graph of order $n \geq 2$, Since $ldim_f(G) = 1$ if and only if $L(uv) = V(G)$ for all $uv \in E(G)$, it follows that $ldim_f(G) = 1$ if and only if G is bipartite.*

Although there is a striking difference between the fractional metric dimension and the fractional local metric dimension of graphs, the same results hold for the local metric dimension of a graph when the graph has true twin vertices. Let G be a graph and $uv \in E(G)$, then $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$ if and only if u and v are true twins. We have the following result about the local resolving neighborhood of true twin vertices:

Observation 2.2. *If u and v are two adjacent vertices of a graph G , then $L(uv) = \{u, v\}$ if and only if u and v are true twins.*

Given a graph H and a family of graphs $\mathcal{I} = \{I_v\}_{v \in V(H)}$, indexed by $V(H)$, their *generalized lexicographic product*, denoted by $H[\mathcal{I}]$, is defined as the graph with the vertex set

$$V(H[\mathcal{I}]) = \{(v, w) : v \in V(H) \text{ and } w \in V(I_v)\},$$

and the edge set

$$E(H[\mathcal{I}]) = \{(v_1, w_1)(v_2, w_2) : v_1v_2 \in E(H), \text{ or } v_1 = v_2 \text{ and } w_1w_2 \in E(I_{v_1})\}.$$

Theorem 2.3. *Let G be a connected graph of order $n \geq 2$. Then the following statements are equivalent.*

- (i) $ldim_f(G) = \frac{n}{2}$.
- (ii) *Each vertex in G has a true twin.*
- (iii) *There exist a graph H and a family of graphs $\mathcal{I} = \{I_v\}_{v \in V(H)}$, where I_v is a non-trivial complete graph, such that G is isomorphic to $H[\mathcal{I}]$.*

Proof. (i) \Rightarrow (ii) Suppose (i) holds. If there exists a vertex u in G such that u does not have a true twin, then the function $f : V(G) \rightarrow [0, 1]$,

$$f(x) = \begin{cases} 0, & \text{if } x = u, \\ \frac{1}{2}, & \text{if } x \neq u, \end{cases}$$

is a local resolving function of G by Lemma 2.2, which implies that

$$ldim_f(G) \leq \frac{n-1}{2},$$

a contradiction.

(ii) \Rightarrow (iii) Suppose (ii) holds. For $x, y \in V(G)$, define $x \equiv y$ if and only if $x = y$ or x, y are true twins. It is clear that \equiv is an equivalence relation. Let O_1, O_2, \dots, O_m be the equivalence classes. Then the induced subgraph $G[O_i]$, is either a null graph or a complete graph. Let H be the graph with the vertex set $\{O_1, \dots, O_m\}$, where two distinct vertices O_i and O_j are adjacent if there exist $x \in O_i$ and $y \in O_j$ such that x and y are adjacent in G . It is routine to verify that G is isomorphic to $H[\mathcal{I}]$, where $\mathcal{I} = \{I_{O_i} : i = 1, \dots, m\}$.

(iii) \Rightarrow (i) Suppose (iii) holds. For $v \in V(H)$, let

$$V(I_v) = \{w_v^1, \dots, w_v^{s(v)}\}.$$

where $|I_v| = s(v)$. Then $s(v) \geq 2$, and (v, w_v^i) and (v, w_v^j) are true twins in $H[\mathcal{I}]$, where $1 \leq i < j \leq s(v)$. Let h be a local resolving function of $H[\mathcal{I}]$ with $|h| = ldim_f(H[\mathcal{I}])$. By Observation 2.2, we get

$$h((v, w_v^i)) + h((v, w_v^j)) \geq 1 \quad \text{for } 1 \leq i < j \leq s(v),$$

which implies that $\sum_{k=1}^{s(v)} h(v, w_v^k) \geq \frac{s(v)}{2}$, and so

$$\begin{aligned} ldim_f(G) &= ldim_f(H[\mathcal{I}]) \\ &= |h| \\ &= \sum_{v \in V(H)} \sum_{k=1}^{s(v)} h((v, w_v^k)) \\ &\geq \sum_{v \in V(H)} \frac{s(v)}{2} \\ &= \frac{|V(H[\mathcal{I}])|}{2} \\ &= \frac{n}{2}. \end{aligned}$$

□

Let $G = G_1 + G_2$ which is the graph obtained from G_1 and G_2 by joining every vertex of G_1 with every vertex of G_2 . If each vertex in G_i has a true twin for $i = 1, 2$ then each vertex in $G_1 + G_2$ has a true twin. Hence, we have the following result.

Corollary 2.4. *Let Θ denotes the collection of all connected graphs G with $ldim_f(G) = \frac{|V(G)|}{2}$. If $G_1, G_2 \in \Theta$, then $G_1 + G_2 \in \Theta$.*

The next result is a generalization of Theorem 2.3. The clique of a graph G is a complete subgraph in G .

Theorem 2.5. *Let G be a connected graph of order n and W_1, W_2, \dots, W_k be independent cliques in G with $|W_i| \geq 3$ for all i , ($1 \leq i \leq k$). Then*

$$ldim_f(G) = \sum_{i=1}^k \frac{|V(W_i)|}{2}$$

if and only if for all $uv \in E(G) \setminus E(W_i)$, $L(xy) \subseteq L(uv)$ for some $xy \in E(W_i)$ for some i , ($1 \leq i \leq k$).

Proof. Let G be a graph with

$$ldim_f(G) = \sum_{i=1}^k \frac{|V(W_i)|}{2},$$

then there is a local resolving function f such that $f(L(uv)) \geq 1$ for all $uv \in E(G) \setminus E(W_i)$, for all $1 \leq i \leq k$. This is possible only when $L(xy) \subseteq L(uv)$ for some $xy \in E(W_i)$, for some i and f assigns 0 to the vertices of $V(G) \setminus V(W_i)$ for all i .

Conversely, suppose that for all $uv \in E(G) \setminus E(W_i)$, $L(xy) \subseteq L(uv)$ for some $xy \in E(W_i)$, for some i , ($1 \leq i \leq k$). Let $f : V(G) \rightarrow [0, 1]$ be the

function defined as:

$$f(v) = \begin{cases} 1/2 & \text{if } v \in V(W_i), 1 \leq i \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $f(L(uv)) \geq 1$ for all $uv \in E(G)$, since $L(xy) \subseteq L(uv)$. Hence f is a local resolving function of G and

$$ldim_f(G) \leq \sum_{i=1}^k \frac{|V(W_i)|}{2}.$$

To show that

$$\sum_{i=1}^k \frac{|V(W_i)|}{2} \leq ldim_f(G),$$

suppose that f is local resolving function of W_i and not a local resolving function of G . Then there exist $uv \in E(G)$ such that $f(L(uv)) < 1$. This leads to a contradiction to our supposition that $L(xy) \subseteq L(uv)$. Hence,

$$ldim_f(G) = \sum_{i=1}^k \frac{|V(W_i)|}{2}.$$

□

A lollipop graph $L_{m,n}$ is a graph obtained by joining a complete graph K_m to a pendent vertex of P_n with an edge.

Corollary 2.6. *Let $L_{m,n}$ be a lollipop graph with $m \geq 3$ and $n \geq 2$. Then $ldim_f(L_{m,n}) = \frac{m}{2}$.*

Proof. Since for all $uv \in E(P_n)$, $L(xy) \subseteq L(uv)$ for some $xy \in E(K_m)$, by Theorem 2.5 and Theorem 2.3, $ldim_f(L_{m,n}) = \frac{m}{2}$. □

Observation 2.7. *Let $l(G) = \min\{|L(uv)| : uv \in E(G)\}$. Then $f : V(G) \rightarrow [0, 1]$ defined by $f(v) = \frac{1}{l(G)}$ for all $v \in V(G)$ is trivially a local resolving function of G . Hence $ldim_f(G) \leq \frac{|V(G)|}{l(G)}$. Since $\{u, v\} \subseteq L(u, v)$ for all $uv \in E(G)$, it follows that $l(G) \geq 2$. Hence $ldim_f(G) \leq \frac{n}{2}$ for all graphs G of order n .*

Lemma 2.8. *Let G be a graph and U be a subset of $V(G)$ with cardinality $|V(G)| - ldim(G) + 1$, there exists an edge $xy \in E(G)$ such that $L(xy) \subseteq U$.*

Proof. Suppose there exists a subset U with cardinality $|V(G)| - ldim(G) + 1$ such that $L(xy) \not\subseteq U$, for all $xy \in E(G)$. Then $L(xy) \cap \{V(G) \setminus U\} \neq \emptyset$. So $V(G) \setminus U$ is a local resolving set of G . Therefore, $ldim(G) - 1 = |V(G) \setminus U| < ldim(G)$, a contradiction. □

Theorem 2.9. *Let G be a graph. Then $l(G) = |V(G)| - 1$ if and only if G is isomorphic to an odd cycle.*

Proof. It is easy to verify that $l(G) = |V(G)| - 1$ when G is an odd cycle. Conversely, let G be a graph of order $n \geq 4$ and $l(G) = |V(G)| - 1$. We further suppose that G is not a bipartite graph, since $l(G) = n$ for a bipartite graph of order n . Thus G contains an odd cycle. Let $C_p : x_1, x_2, \dots, x_p$ be an induced odd cycle, where $p \leq n$ is odd. Let $\Delta(G)$ be the maximum degree of G . We claim that $\Delta(G) = 2$. Suppose to the contrary that $\Delta \geq 3$, then odd cycle C_p must be a proper subgraph of G . Since G is connected, therefore there exists a vertex $y \in V(G) \setminus V(C_p)$ such that y is adjacent to any vertex, say x_p of C_p . Since C_p is an odd cycle, therefore $d(x_p, x_{\frac{p-1}{2}}) = d(x_p, x_{\frac{p+1}{2}})$. Thus $x_p, y \notin L(x_{\frac{p-1}{2}}x_{\frac{p+1}{2}})$. Hence $|L(x_{\frac{p-1}{2}}x_{\frac{p+1}{2}})| \leq n - 2$ which is a contradiction. Hence $\Delta(G) = 2$ and G is isomorphic to an odd cycle. \square

Using Lemma 2.8, we have the following result:

Theorem 2.10. *Let G be a graph of order n . Then*

$$ldim_f(G) \geq \frac{n}{n - ldim(G) + 1}.$$

Proof. Write $s = n - ldim(G) + 1$. Suppose f is a local resolving function of G with $|f| = ldim_f(G)$. Let $\tau = \{T : T \subset V(G), |T| = n - ldim(G) + 1\}$ and $|\tau| = \binom{|V(G)|}{s}$. For each $U \in \tau$, $f(U) \geq 1$ by Lemma 2.8. Hence,

$$\sum_{U \in \tau} f(U) \geq \binom{n}{s}.$$

Since

$$\sum_{U \in \tau} f(U) = \binom{n-1}{s-1} |f|,$$

so we accomplish our result. \square

Let G be the complete k -partite graph K_{a_1, a_2, \dots, a_k} , for $k > 2$, of order

$$n = \sum_{i=1}^k a_i.$$

Let $V(G)$ be partitioned into k -partite sets V_1, V_2, \dots, V_k , where $|V_i| = a_i$ for $1 \leq i \leq k$. Okamoto et al. proved that $ldim(K_{a_1, a_2, \dots, a_k}) = k - 1$ [16].

Lemma 2.11. *Let G be the complete k -partite graph K_{a_1, a_2, \dots, a_k} , for $k > 2$, of order $n = \sum_{i=1}^k a_i$. Then $ldim_f(K_{a_1, a_2, \dots, a_k}) = k - 1$.*

Proof. Firstly, we show that $ldim_f(G) \leq k - 1$. It is clear that all $xy \in E(K_{a_1, a_2, \dots, a_k})$ if and only if $x \in V_i$ and $y \in V_j$, $i \neq j$ and $i, j \in \{1, 2, \dots, k\}$. Note that for all $xy \in E(K_{a_1, a_2, \dots, a_k})$, $L(xy) = V_i \cup V_j$. One of the possible choices of local resolving function f of G is that f is defined as: f assigns 1 to only one vertex of $V_i \cup V_j$ and 0 to all other vertices of $V_i \cup V_j$. This implies $f(L(xy)) \geq 1$ for all $xy \in E(G)$ and $|f| = k - 1$. Thus $ldim_f(G) \leq k - 1$.

To prove $k - 1 \leq ldim_f(G)$, we suppose to the contrary that $k - 1 > ldim_f(G)$. Suppose that f is a local resolving function which obtains a minimum weight over all the local resolving functions of G , and this weight is not $k - 1$. This is only possible when f assigns 0 to all vertices of $V_r \cup V_s$, for some $r, s \in \{1, 2, \dots, k\}$. This implies $f(L(xy)) < 1$ for $xy \in E(G)$ where $x \in V_r$ and $y \in V_s$, which is a contradiction. Hence $ldim_f(G) = k - 1$. \square

In the following result, we give the fractional local metric dimension of a vertex-transitive graph G in terms of the parameter $l(G)$.

Theorem 2.12. *Let G be a vertex-transitive graph. Then $ldim_f(G) = \frac{|V(G)|}{l(G)}$.*

Proof. Let $l(G) = p$, then there exists an edge $uv \in E(G)$ such that $|L(uv)| = p$. This implies that $ldim_f(G) \geq \frac{|V(G)|}{p}$. By Observation 2.7, we have the required result. \square

Observation 2.13. *Let G be a graph and $v \in V(G)$ be the cut-vertex of G , then $ldim_f(G) - 1 \leq ldim_f(G - v)$.*

The fan graph $F_{1,n}$ of order $n + 1$ is defined as the join graph $K_1 + P_n$. Let $V(K_1) = \{u\}$ and $V(P_n) = \{u_1, u_2, \dots, u_n\}$.

Lemma 2.14. *Let $F_{1,n}$ be a fan graph with $n \geq 3$, then*

$$ldim_f(F_{1,n}) = \begin{cases} 2, & \text{if } n = 3, \\ \frac{n}{3}, & \text{if } n \geq 4. \end{cases}$$

Proof. Since $l(F_{1,3}) = 2$, therefore $ldim_f(F_{1,3}) \leq 2$ by Proposition 2.7. Now, we show that $2 \leq ldim_f(F_{1,3})$. Since $l(F_{1,3}) = 2$ and $|L(xy)| \neq 4$ for any $xy \in E(F_{1,3})$. Thus a function $f : V(F_{1,3}) \rightarrow [0, 1]$ is a local resolving function for $F_{1,3}$ if it assign $1/2$ to each vertex of $F_{1,3}$. Otherwise there exists an edge $xy \in E(F_{1,3})$ such that $L(xy) < 1$. Hence $ldim_f(F_{1,3}) = 2$.

Let $F_{1,n}$ be a fan graph with $n \geq 4$. Note that $\{u\} = V(K_1)$ does not locally resolve any $xy \in E(F_{1,n})$ for $x, y \neq u$. Let $f : V(F_{1,n}) \rightarrow [0, 1]$ be a local resolving function defined as:

$$f(v) = \begin{cases} 1/3, & \text{if } v \neq u, \\ 0, & \text{if } v = u. \end{cases}$$

$f(L(xy)) \geq 1$ for all $xy \in E(F_{1,n})$. Thus $|f| = \frac{n}{3}$. Hence $ldim_f(F_{1,n}) \leq \frac{n}{3}$.

Now we show that $\frac{n}{3} \leq ldim_f(F_{1,n})$. Note that $l(F_{1,n}) = 3$ for $n \geq 4$. f is a local resolving function as defined above. If f assigns 0 to any vertex from $V(P_n)$, then there exists an edge $xy \in E(F_{1,n})$ such that $f(L(xy)) < 1$. Hence $ldim_f(F_{1,n}) = \frac{n}{3}$ for $n \geq 4$. \square

3. THE FRACTIONAL LOCAL METRIC DIMENSION OF STRONG AND CARTESIAN PRODUCT OF GRAPHS

In this section, we study the fractional local metric dimension of strong and Cartesian product of graphs.

Lemma 3.1. *Let G and H be two graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively. Then*

$$L_{G \boxtimes H}((u_i, v_j)(u_k, v_l)) \subseteq \begin{cases} V(G) \times L_H(v_j v_l), & i = k, \\ L_G(u_i u_k) \times V(H), & j = l, \\ \{V(G) \times L_H(v_j v_l)\} \cup \{L_G(u_i u_k) \times V(H)\} & \text{otherwise.} \end{cases}$$

Proof. Let $(u_i, v_j)(u_k, v_l) \in E(G \boxtimes H)$. If $i = k$, then $v_j v_l \in E(H)$. Let $(u_i, b) \in L_{G \boxtimes H}((u_i, v_j)(u_i, v_l))$, then

$$d_{G \boxtimes H}((u_i, b), (u_i, v_j)) \neq d_{G \boxtimes H}((u_i, b), (u_i, v_l)).$$

By Remark 1.2, we have $d_H(b, v_j) \neq d_H(b, v_l)$, therefore $b \in L_H(v_j v_l)$. Thus $(u_i, b) \in \{V(G) \times L_H(v_j v_l)\}$. Analogously, if $j = l$, then $u_i u_k \in E(G)$. Let $(a, v_j) \in L_{G \boxtimes H}((u_i, v_j)(u_k, v_j))$, then $d_{G \boxtimes H}((a, v_j), (u_i, v_j)) \neq d_{G \boxtimes H}((a, v_j), (u_k, v_l))$. By Remark 1.2, we have $d_G(a, u_i) \neq d_G(a, u_k)$, therefore $a \in L_G(u_i u_k)$. Thus $(a, v_j) \in \{L_G(u_i u_k) \times V(H)\}$. Finally, if $u_i u_k \in E(G)$ and $v_j v_l \in E(H)$, then two vertices (u_i, v_j) and (u_k, v_l) are locally resolved by either (a, v_j) or (u_i, b) or both. Let $(a, v_j) \in L_{G \boxtimes H}((u_i, v_j)(u_k, v_l))$, we have

$$\begin{aligned} d_{G \boxtimes H}((u_i, v_j), (a, v_j)) &= d_G(u_i, a) \\ &\neq d_G(u_k, a) \\ &= \max\{d_G(u_k, a), 1\} \\ &= d_{G \boxtimes H}((a, v_j), (u_k, v_l)). \end{aligned}$$

Thus, $(a, v_j) \in \{L_G(u_i u_k) \times V(H)\}$. Similar arguments hold for $(u_i, b) \in L_{G \boxtimes H}((u_i, v_j)(u_k, v_l))$. Hence,

$$(a, v_j), (u_i, b) \in \{V(G) \times L_H(v_j v_l)\} \cup \{L_G(u_i u_k) \times V(H)\}$$

and we have the desired result. \square

Now, we discuss some results involving the diameter or the radius of G . For any two vertices x and y in a connected graph G , the collection of all vertices which lie on an $x - y$ path of the shortest length is known as the interval $I[x, y]$ between x and y . Given a non-negative integer k , we say that G is adjacency k -resolved if for every two adjacent vertices $x, y \in V(G)$, there exists $w \in V(G)$ such that $d_G(y, w) \geq k$ and $x \in I[y, w]$, or $d_G(x, w) \geq k$ and $y \in I[x, w]$. For example, path graphs and cyclic graphs of order $n \geq 2$ are adjacency $\lceil \frac{n}{2} \rceil$ -resolved.

Lemma 3.2. *Let G be a non-trivial graph of diameter $\text{diam}(G) < k$ and let H be an adjacency k -resolved graph of order n_2 and let $(u_i, v_j)(u_r, v_l) \in E(G \boxtimes H)$. Then*

$$L_{G \boxtimes H}((u_i, v_j)(u_r, v_l)) \subseteq \{L_G(u_i u_r) \times V(H)\}.$$

Proof. Let $L_{G \boxtimes H}((u_i, v_j)(u_r, v_l))$ be the local resolving neighborhood of

$$(u_i, v_j)(u_r, v_l) \in E(G \boxtimes H).$$

We differentiate the following two cases.

Case 1: If $j = l$, then $u_i u_r \in E(G)$. Let $(u, v_j) \in L_{G \boxtimes H}((u_i, v_j)(u_r, v_j))$ then $d_{G \boxtimes H}((u_i, v_j), (u, v_j)) \neq d_{G \boxtimes H}((u_r, v_j), (u, v_j))$. By Remark 1.2, we have $d_G(u_i, u) \neq d_G(u_r, u)$, thus $u \in L_G(u_i u_r)$.

Case 2: If $v_j v_l \in E(H)$. Since H is adjacency k -resolved, there exists $v \in V(H)$ such that $(d_H(v, v_l) \geq k$ and $v_j \in I[v, v_l])$ or $(d_H(v, v_j) \geq k$ and $v_l \in I[v, v_j])$. Say $d_H(v, v_l) \geq k$ and $v_j \in I[v, v_l]$. In such a case, as $\text{diam}(G) < k$, for every $u \in L_G(u_i u_r)$ we have

$$\begin{aligned} d_{G \boxtimes H}((u_i, v_j), (u, v)) &= \max\{d_G(u_i, u), d_H(v_j, v)\} \\ &< d_H(v, v_l) \\ &= \max\{d_G(u, u_r), d_H(v, v_l)\} \\ &= d_{G \boxtimes H}((u_r, v_l), (u, v)). \end{aligned}$$

Hence, $L_{G \boxtimes H}((u_i, v_j)(u_r, v_l)) \subseteq \{L_G(u_i u_r) \times V(H)\}$. \square

Theorem 3.3. *Let G be a non-trivial graph of diameter $\text{diam}(G) < k$ and let H be an adjacency k -resolved graph of order n_2 . Then*

$$\text{ldim}_f(G \boxtimes H) \leq n_2 \cdot \text{ldim}_f(G)$$

Proof. Let $(x, y) \in E(G \boxtimes H)$. Let $g : V(G) \rightarrow [0, 1]$ be a local resolving function of G with $|g| = \text{ldim}_f(G)$. We define a function $h : V(G \boxtimes H) \rightarrow [0, 1]$,

$$(x, y) \mapsto \begin{cases} g(x), & \text{if } (x, y) \in G^y, \\ 0, & \text{otherwise.} \end{cases}$$

Note that h is a local resolving function of $G \boxtimes H$. Since G has n_2 copies in $G \boxtimes H$, therefore $|h| \leq n_2 \cdot \text{ldim}_f(G)$. Hence, $\text{ldim}_f(G \boxtimes H) \leq n_2 \cdot \text{ldim}_f(G)$. \square

Theorem 3.4. *Let G and H be two graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively. Then*

$$2 \leq \text{ldim}_f(G \boxtimes H) \leq n_1 \cdot \text{ldim}_f(H) + n_2 \cdot \text{ldim}_f(G) - 2\text{ldim}_f(G) \cdot \text{ldim}_f(H).$$

Proof. Since $P_2 \boxtimes P_2 = K_4$ and $\text{ldim}_f(P_2 \boxtimes P_2) = 2$. So, the lower bound follows. Let $(u, v) \in V(G \boxtimes H)$. Let $g_1 : V(G) \rightarrow [0, 1]$ be a local resolving function of G with $|g_1| = \text{ldim}_f(G)$ and $g_2 : V(H) \rightarrow [0, 1]$ be a local resolving function of H with $|g_2| = \text{ldim}_f(H)$. We define a function $h : V(G \boxtimes H) \rightarrow [0, 1]$, with $h(u, v) = g_1(u) + g_2(v)$. Note that h is a local resolving function of $G \boxtimes H$. Since G has n_2 and H has n_1 copies in $G \boxtimes H$,

therefore $|h| = n_1 \cdot ldim_f(H) + n_2 \cdot ldim_f(G)$. Hence, $ldim_f(G \boxtimes H) \leq n_1 \cdot ldim_f(H) + n_2 \cdot ldim_f(G) - 2ldim_f(G) \cdot ldim_f(H)$. \square

For the sharpness of upper bound in Theorem 3.4, let $G = K_n$ and $H = K_m$. Since $K_n \boxtimes K_m \cong K_{nm}$, therefore

$$\begin{aligned} & ldim_f(K_n \boxtimes K_m) \\ &= \frac{nm}{2} \\ &= n \cdot ldim_f(K_m) + m \cdot ldim_f(K_n) - 2ldim_f(K_n) \cdot ldim_f(K_m). \end{aligned}$$

Now, we discuss general bounds for the fractional local metric dimension of Cartesian product of graphs. The Cartesian product of two graphs G and H , denoted by $G \square H$, is a graph with the vertex set $V(G \square H) = \{(u, v) : u \in V(G) \text{ and } v \in V(H)\}$ and two vertices (u_1, v_1) and (u_2, v_2) in $G \square H$ are adjacent if and only if

- $u_1 u_2 \in E(G)$ and $v_1 = v_2$ in H or
- $u_1 = u_2$ in G and $v_1 v_2 \in E(H)$.

Remark: [13] Let G and H be two connected graphs. Then

$$d_{G \square H}((u_1, v_1), (u_2, v_2)) = d_G(u_1, u_2) + d_H(v_1, v_2).$$

Lemma 3.6. *Let G and H be two graphs, then*

$$L_{G \square H}((u_i, v_j)(u_k, v_l)) = \begin{cases} \bigcup_{v \in L_H(v_j v_l)} \bigcup_{u \in V(G)} \{uv\}, & \text{if } i = k, \\ \bigcup_{u \in L_G(u_i u_k)} \bigcup_{v \in V(H)} \{uv\}, & \text{if } j = l. \end{cases}$$

Proof. For $(u_i, v_j)(u_k, v_l) \in E(G \square H)$ if $i = k$, then $v_j v_l \in E(H)$. Let $(u_i, v) \in L_{G \square H}((u_i, v_j)(u_i, v_l))$, then

$$d_{G \square H}((u_i, v), (u_i, v_j)) \neq d_{G \square H}((u_i, v), (u_i, v_l)).$$

By Remark 3.5, we have $d_H(v, v_j) \neq d_H(v, v_l)$, therefore $v \in L_H(v_j v_l)$. Thus

$$(u_i, v) \in \bigcup_{v \in L_H(v_j v_l)} \bigcup_{u \in V(G)} \{uv\}.$$

Now let

$$(u_i, v) \in \bigcup_{v \in L_H(v_j v_l)} \bigcup_{u \in V(G)} \{uv\},$$

then $d_H(v, v_j) \neq d_H(v, v_l)$. By Remark 3.5, we have $d_{G \square H}((u_i, v), (u_i, v_j)) \neq d_{G \square H}((u_i, v), (u_i, v_l))$. Thus $(u_i, v) \in L_{G \square H}((u_i, v_j)(u_i, v_l))$. Similar arguments hold for $j = l$. Hence, we have the desired result. \square

Theorem 3.7. *Let G and H be two graphs. Then*

$$ldim_f(G \square H) \geq ldim_f(G).$$

Proof. Let f be a local resolving function of $G \square H$ with $|f| = \text{ldim}_f(G \square H)$. We define a function $f_G : V(G) \rightarrow [0, 1]$ such that

$$f_G(u) = \min \left\{ 1, \sum_{v \in V(H)} f(u, v) \right\}.$$

For $u_1 u_2 \in E(G)$, we show that $f_G(L_G(u_1 u_2)) \geq 1$. If there exists an $x \in L_G(u_1 u_2)$ with $f_G(x) = 1$, then $f_G(L_G(u_1 u_2)) \geq 1$. Now, let for any $u \in V(G)$, $f_G(u) = \sum_{v \in V(H)} f(u, v)$. Then

$$f_G(L_G(u_1 u_2)) = \sum_{u \in L_G(u_1 u_2)} \sum_{v \in V(H)} f(u, v)$$

By Lemma 3.6, the above is equal

$$f(L((u_1, v_0)(u_2, v_0))) \geq 1.$$

Thus f_G is a local resolving function of G . Since

$$|f_G| \leq \sum_{u \in V(G)} \sum_{v \in V(H)} f(u, v) = |f|,$$

hence $\text{ldim}_f(G \square H) \geq \text{ldim}_f(G)$. \square

Since grid graph $P_n \square P_t$ is a bipartite graph and by Observation 2.1, we deduce $\text{ldim}_f(P_n \square P_t) = 1$.

Lemma 3.8. *Let G be a graph of order n , then $\text{ldim}_f(K_2 \square G) \leq \text{ldim}_f(G)$.*

Proof. Let $V(K_2) = \{x, y\}$, $V(G) = \{u_1, u_2, \dots, u_n\}$ and $H = K_2 \square G$. Then $V(H) = \{(x, u_i), (y, u_i) : i = 1, 2, \dots, n\}$. Let f be a local resolving function of G with $|f| = \text{ldim}_f(G)$. Now we define $g : V(H) \rightarrow [0, 1]$ by

$$g((x, u_i)) = g((y, u_i)) = \frac{f(u_i)}{2}, i = 1, 2, \dots, n.$$

We claim that g is a local resolving function for H . Let $uv \in E(H)$, if $u = (x, u_i)$ and $v = (x, u_j)$, then $\{x\} \times L_G(u_i u_j) \subseteq L_H(uv)$ and hence $g(L_H(uv)) \geq f(L_G(u_i u_j)) \geq 1$. If $u = (x, u_i)$ and $v = (y, u_i)$, then $L_H(uv) = V(H)$ and hence $g(L_H(uv)) \geq 1$. Thus, g is a local resolving function of H with $|g| = |f|$. Hence, $\text{ldim}_f(H) \leq |f| = \text{ldim}_f(G)$. \square

Remark: When G is a bipartite graph and an odd cyclic graph, the bound given in Lemma 3.8 is sharp. If G is bipartite graph, then $\text{ldim}_f(K_2 \square G) = 1 = \text{ldim}_f(G)$. If n is an odd integer with $n \geq 3$, then $\text{ldim}_f(K_2 \square C_n) = \frac{n}{n-1}$.

Let G and H be graphs with $V(H) = n$, Arumugam *et al.* proved that the fractional metric dimension of $G \square H \geq \frac{n}{2}$ if $\text{dim}_f(H) = \frac{n}{2}$ [2]. Similar result holds for the fractional local metric dimension with an alternative proof as follows:

Theorem 3.10. *Let G and H be two connected graphs with order m , n respectively and $\text{ldim}_f(H) = \frac{n}{2}$. Then $\text{ldim}_f(G \square H) \geq \frac{n}{2}$.*

Proof. Since $ldim_f(H) = \frac{n}{2}$, by Theorem 2.3, every vertex of H has a true twin. Let v has a true twin w in H then $L_H(vw) = \{v, w\}$. By Lemma 3.6, it follows that $L_{G \square H}((u, v)(u, w)) = \{(x, v) : x \in V(G)\} \cup \{(x, w) : x \in V(G)\}$.

Now, let f be a local resolving function of $G \square H$. Then

$$f(L_{G \square H}((u, v)(u, w))) \geq 1$$

for all $(u, v)(u, w) \in E(G \square H)$. Hence

$$\sum_{x \in V(G)} f((x, v)) + \sum_{x \in V(G)} f((x, w)) \geq 1$$

for all $vw \in E(H)$. Adding these n inequalities, we get

$$\sum_{x \in V(H)} \sum_{x \in V(G)} f((x, v)) + \sum_{x \in V(G)} f((x, w)) \geq n.$$

This implies $2|f| \geq n$. Hence $ldim_f(G \square H) \geq \frac{n}{2}$. \square

Corollary 3.11. *Let G and H be two connected graphs with order m , n respectively and $ldim_f(G) = \frac{m}{2}$ and $ldim_f(H) = \frac{n}{2}$. Then $ldim_f(G \square H) \geq \max\{ldim_f(G), ldim_f(H)\}$.*

The bound given in Theorem 3.10 is sharp for $H = K_n$ as follows:

Theorem 3.12. *Let G be any graph with $|V(G)| < n$, for all $n \geq 3$. Then $ldim_f(G \square K_n) = \frac{n}{2}$.*

Proof. Let $|V(G)| = m$ with $m < n$. Let $V(G) = \{u_1, u_2, \dots, u_m\}$ and $V(K_n) = \{v_1, v_2, \dots, v_n\}$. Since by Theorem 2.3, $ldim_f(K_n) = \frac{n}{2}$, then by Theorem 3.10, $ldim_f(G \square K_n) \geq \frac{n}{2}$. We claim that

$$|L_{G \square K_n}((u_i, v_r)(u_j, v_s))| \geq 2m$$

for all $(u_i, v_r)(u_j, v_s) \in E(G \square K_n)$. For $(u_i, v_r)(u_j, v_s) \in E(G \square K_n)$, we have two cases. If $i = j$, then $r \neq s$ and by Lemma 3.6, we have

$$L_{G \square K_n}((u_i, v_r)(u_i, v_s)) = \{(u_t, v_r) : 1 \leq t \leq m\} \cup \{(u_t, v_s) : 1 \leq t \leq m\}.$$

So $|L_{G \square K_n}((u_i, v_r)(u_i, v_s))| = 2m$. If $r = s$, then $i \neq j$ and by Lemma 3.6, we have $\{(u_i, v_t) : 1 \leq t \leq n\} \cup \{(u_j, v_t) : 1 \leq t \leq n\} \subseteq L_{G \square K_n}((u_i, v_r)(u_j, v_r))$. So $|L_{G \square K_n}((u_i, v_r)(u_j, v_r))| \geq 2n > 2m$.

Now the function $f : V(G \square K_n) \rightarrow [0, 1]$ defined by $f((u, v)) = \frac{1}{2m}$ for all $(u, v) \in V(G \square K_n)$ is a local resolving function of $G \square K_n$ with $|f| = \frac{|V(G \square K_n)|}{2m} = \frac{n}{2}$ and $ldim_f(G \square K_n) \leq \frac{n}{2}$. Hence, $ldim_f(G \square K_n) = \frac{n}{2}$. \square

From Corollary 3.11, we have the following result.

Theorem 3.13. *For $2 \leq k \leq n$, $n \geq 3$, $ldim_f(K_k \square K_n) = \frac{n}{2}$.*

Proof. The result follows from Theorem 3.12, when $k < n$. Consider the case when $k = n$. Since by Theorem 2.3, $ldim_f(K_n) = \frac{n}{2}$, then by Theorem

3.10, $ldim_f(K_k \square K_n) \geq \frac{n}{2}$. Let $V(K_k) = \{u_1, u_2, \dots, u_k\}$ and $V(K_n) = \{v_1, v_2, \dots, v_n\}$. We claim that $|L_{K_k \square K_n}((u_i, v_r)(u_j, v_s))| \geq 2n$ for all

$$(u_i, v_r)(u_j, v_s) \in E(K_k \square K_n).$$

For $(u_i, v_r)(u_j, v_s) \in E(G \square K_n)$, we have similar cases as in the proof of Theorem 3.12 and we have $|L_{K_k \square K_n}((u_i, v_r)(u_j, v_s))| \geq 2n$.

Now the function $f : V(K_k \square K_n) \rightarrow [0, 1]$ defined by $f((u, v)) = \frac{1}{2^n}$ for all $(u, v) \in V(K_k \square K_n)$ is a local resolving function of $K_k \square K_n$ with $|f| = \frac{n}{2}$ and $ldim_f(K_k \square K_n) \leq \frac{n}{2}$. Hence, $ldim_f(K_k \square K_n) = \frac{n}{2}$. \square

4. SUMMARY AND CONCLUSION

In this paper, the concept of fractional local metric dimension of graphs has been introduced. Graphs with $ldim_f(G) = \frac{|V(G)|}{2}$ have been characterized. The fractional local metric dimension of some families of graphs have been studied. Differences between the fractional metric dimension and the fractional local metric dimension of graphs have also been investigated. The fractional local metric dimension of strong and Cartesian product of graphs have been studied and established some bounds on their fractional local metric dimension. However, it remains to determine the fractional local metric dimension of several other graph products.

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Centre for Advanced Studies in Pure and Applied Mathematics
E-mail address: imran.javaid@bzu.edu.pk

Centre for Advanced Studies in Pure and Applied Mathematics
E-mail address: hira.benish@yahoo.com

Centre for Advanced Studies in Pure and Applied Mathematics
E-mail address: mahru830@gmail.com