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INTERNAL AND EXTERNAL DUALITY IN ABSTRACT POLYTOPES

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Abstract. We define an abstract regular polytope to be internally self-dual if its self-duality can be realized as one of its symmetries. This property has many interesting implications on the structure of the polytope, which we present here. Then, we construct many examples of internally self-dual polytopes. In particular, we show that there are internally self-dual regular polyhedra of each type $\{p, p\}$ for $p \geq 3$ and that there are both infinitely many internally self-dual and infinitely many externally self-dual polyhedra of type $\{p, p\}$ for p even. We also show that there are internally self-dual polytopes in each rank, including a new family of polytopes that we construct here.

1. Introduction

Whenever a polytope is invariant under a transformation such as duality, we often describe this as an "external" symmetry of the polytope. In the context of abstract regular polytopes, self-duality manifests as an automorphism of the symmetry group of the polytope. For example, the n-simplex is self-dual, and this fact is reflected by an automorphism of its symmetry group S_{n+1}. Since the symmetric group S_{n+1} has no nontrivial outer automorphisms (unless $n + 1 = 6$), we see that in general the self-duality of the simplex must somehow correspond to an ordinary rank-preserving symmetry. What does this mean combinatorially? What other polytopes have this property that self-duality yields an inner automorphism?

In this paper, we define a regular self-dual abstract polytope to be *internally self-dual* if the group automorphism induced by its self-duality is an inner automorphism. Otherwise, a regular self-dual polytope is *externally self-dual*. Our search for internally self-dual regular polytopes started with the atlas of regular polytopes with up to 2000 flags [2]. Using a computer algebra system, we found the following results on the number of regular polytopes (up to isomorphism and duality).

The internally self-dual polyhedra we found include examples of type $\{p, p\}$ for $3 \leq p \leq 12$ and $p = 15$. In rank 4, the only two examples are the

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Rank	Total	Self-dual	Internally self-dual	Externally self-dual
3	3571	242	54	188
4	2016	156	2	154
5	258	15	1	14

Table 1. Data on dualities of small regular polytopes

simplex $\{3, 3, 3\}$ and a toroid of type $\{4, 3, 4\}$. In rank 5, the only example is the simplex $\{3, 3, 3, 3\}$.

The data suggest that there are many internally self-dual polyhedra (although many more externally self-dual). Indeed, we will show that there are internally self-dual polyhedra of type $\{p, p\}$ for each $p \geq 3$. The data in ranks 4 and 5 seem less promising as far as the existence of internally self-dual polytopes. We will show, however, that there are several families of internally self-dual polytopes in every rank. Furthermore, due to the ubiquity of symmetric groups, it seems likely that in every rank, there are many polytopes that are internally self-dual for the same reason as the simplex.

In addition to the existence results mentioned above, we explore the consequences of a self-dual polytope being either externally or internally self-dual.

The paper is organized as follows. In Section 2 we briefly outline the necessary definitions and background on abstract polytopes. Sections 3 and 4 contain the definition and some basic structural results about internally self-dual polytopes. Section 5 includes existence results regarding both self-dual regular polyhedra and self-dual polytopes of higher ranks. Finally, in Section 6, we highlight a few open questions and variants of the problems considered here.

2. Background

Most of our background, along with many more details can be found in [10]. An *abstract n-polytope* $\mathcal{P}$ is a ranked partially ordered set of *faces* with four defining properties. First, each maximal totally ordered subset of $\mathcal{P}$ contains $n+2$ faces. These maximal totally ordered subsets of $\mathcal{P}$ are called *flags*. Second, $\mathcal{P}$ has a unique least face F_{-1} of rank -1, and a unique greatest face F_n of rank n; here if an $F \in \mathcal{P}$ has $\mathrm{rank}(F) = i$, then F is called an *i-face*. Faces of rank 0, 1, and $n-1$ are called *vertices*, *edges*, and *facets*, respectively. Third, if $F < G$ with $\mathrm{rank}(F) = i-1$ and $\mathrm{rank}(G) = i+1$, then there are exactly two i-faces H with $F < H < G$. Finally, $\mathcal{P}$ is strongly connected, which is defined as follows. For any two faces F and G of $\mathcal{P}$ with $F < G$, we call $G/F := \{H \mid H \in \mathcal{P},\ F \leq H \leq G\}$ a *section* of $\mathcal{P}$. If $\mathcal{P}$ is a partially ordered set satisfying the first two properties, then $\mathcal{P}$ is *connected* if either $n \leq 1$, or $n \geq 2$ and for any two faces F and G of $\mathcal{P}$ (other than F_{-1} and F_n) there is a sequence of faces $F = H_0, H_1, \ldots, H_{k-1}, H_k = G$, not containing F_{-1} and F_n, such that H_i and H_{i-1} are comparable for $i = 1, \ldots, k$. We say that $\mathcal{P}$ is *strongly connected* if each section of $\mathcal{P}$ (including $\mathcal{P}$ itself) is connected. Due to the relationship between abstract polytopes and incidence geometries, if two faces F and G are comparable in the partial order, then it is said that F is *incident on* G.

Two flags of an n-polytope $\mathcal{P}$ are said to be *adjacent* if they differ by exactly one face. If Φ is a flag of $\mathcal{P}$, the third defining property of an abstract polytope tells us that for $i = 0, 1, \ldots, n-1$ there is exactly one flag that differs from Φ in its i-face. This flag is denoted Φ^i and is *i-adjacent* to Φ. We extend this notation recursively and let $\Phi^{i_1 \cdots i_k}$ denote the flag $(\Phi^{i_1 \cdots i_{k-1}})^{i_k}$. If w is a finite sequence $(i_1, \ldots, i_k)$, then Φ^w denotes $\Phi^{i_1 \cdots i_k}$. Note that $\Phi^{ii} = \Phi$ for each i, and $\Phi^{ij} = \Phi^{ji}$ if $|i-j| > 1$. An n-polytope ($n \geq 2$) is called *equivelar* if for each $i = 1, 2, ..., d-1$, there is an integer p_i so that every section G/F defined by an $(i-2)$-face F incident on an $(i+1)$-face G is the partial order induced by a p_i-gon. In this case we say that the polytope has (*Schläfli*) *type* $\{p_1, p_2, \ldots, p_{n-1}\}$.

The automorphism group of an n-polytope $\mathcal{P}$ (consisting of the order-preserving bijections from $\mathcal{P}$ to itself) is denoted by $\Gamma(\mathcal{P})$. For any flag Φ, any finite sequence w, and any automorphism φ, we have $\Phi^w \varphi = (\Phi\varphi)^w$. An n-polytope $\mathcal{P}$ is called *regular* if its automorphism group $\Gamma(\mathcal{P})$ has exactly one orbit when acting on the flags of $\mathcal{P}$, or equivalently, if for some *base flag* and each $i = 0, 1, \ldots, n-1$ there exists a unique involutory automorphism $\rho_i \in \Gamma(\mathcal{P})$ such that $\Phi\rho_i = \Phi^i$. For a regular n-polytope $\mathcal{P}$, its group $\Gamma(\mathcal{P})$ is generated by the involutions $\rho_0, \ldots, \rho_{n-1}$ described above. These generators satisfy the relations

$$(\rho_i \rho_j)^{p_{ij}} = \epsilon \quad (0 \leq i, j \leq n-1), \tag{1}$$

where $p_{ii} = 1$ for all i, $2 \leq p_{ji} = p_{ij}$ if $j = i-1$, and

$$p_{ij} = 2 \text{ for } |i-j| \geq 2. \tag{2}$$

Any group $\langle \rho_0, \ldots, \rho_{n-1} \rangle$ satisfying properties 1 and 2 is called a *string group generated by involutions*. Moreover, $\Gamma(\mathcal{P})$ has the following *intersection property*:

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle \text{ for } I, J \subseteq \{0, \ldots, n-1\}. \tag{3}$$

Any string group generated by involutions that has this intersection property is called a *string C-group*. The group $\Gamma(\mathcal{P})$ of an abstract regular polytope $\mathcal{P}$ is a string C-group. Conversely, it is known (see [10, Sec. 2E]) that an abstract regular n-polytope can be constructed uniquely from any string C-group. This correspondence between string C-groups and automorphism groups of abstract regular polytopes allows us to talk simulatneously about the combinatorics of the flags of abstract regular polytopes as well as the structure of their groups.

Let $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ be a string group generated by involutions acting as a permutation group on a set $\{1, \ldots, k\}$. We can define the *permutation representation graph* $\mathcal{X}$ as the r-edge-labeled multigraph with k vertices, and with a single i-edge $\{a, b\}$ whenever $a\rho_i = b$ with $a < b$. Note that, since each of the generators is an involution, the edges in our graph are not directed. When Γ is a string C-group that acts faithfully on $\{1, \ldots, k\}$, the multigraph $\mathcal{X}$ is called a *CPR graph*, as defined in [12].

If $\mathcal{P}$ and $\mathcal{Q}$ are regular polytopes, then we say that $\mathcal{P}$ *covers* $\mathcal{Q}$ if there is a well-defined surjective homomorphism from $\Gamma(\mathcal{P})$ to $\Gamma(\mathcal{Q})$ that respects the canonical generators. In other words, if $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ and $\Gamma(\mathcal{Q}) = \langle \rho_0', \ldots, \rho_{n-1}' \rangle$, then $\mathcal{P}$ covers $\mathcal{Q}$ if there is a homomorphism that sends each ρ_i to ρ_i'.

Suppose $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a string group generated by involutions, and Λ is a string C-group such that Γ covers Λ. If the covering is one-to-one on the subgroup $\langle \rho_0, \ldots, \rho_{n-2} \rangle$, then the *quotient criterion* says that Γ is itself a string C-group [10, Thm. 2E17].

Given regular polytopes $\mathcal{P}$ and $\mathcal{Q}$, the *mix* of their automorphism groups, denoted $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$, is the subgroup of the direct product $\Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q})$ that is generated by the elements (ρ_i, ρ_i'). This group is the minimal string group generated by involutions that covers both $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ (where again, we only consider homomorphisms that respect the generators). Using the procedure in [10, Sec. 2E], we can build a poset from this, which we call the mix of $\mathcal{P}$ and $\mathcal{Q}$, denoted $\mathcal{P} \diamond \mathcal{Q}$. This definition naturally extends to any family of polytopes (even an infinite family); see [11, Section 5] for more details. It is also possible to mix an n-polytope $\mathcal{P}$ with an edge e. To do so, we take $\Gamma(e) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ with defining relations $\rho_0^2 = \epsilon$ and $\rho_i = \epsilon$ for $1 \leq i \leq n-1$, and then use the same definition as before.

The *comix* of $\Gamma(\mathcal{P})$ with $\Gamma(\mathcal{Q})$, denoted $\Gamma(\mathcal{P}) \,\square\, \Gamma(\mathcal{Q})$, is the largest string group generated by involutions that is covered by both $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$ [3]. A presentation for $\Gamma(\mathcal{P}) \,\square\, \Gamma(\mathcal{Q})$ can be obtained from that of $\Gamma(\mathcal{P})$ by adding all of the relations of $\Gamma(\mathcal{Q})$, rewriting the relations to use the generators of $\Gamma(\mathcal{P})$ instead.

The dual of a poset $\mathcal{P}$ is the poset $\mathcal{P}^*$ with the same underlying set as $\mathcal{P}$, but with the partial order reversed. Clearly, the dual of a polytope is itself a polytope. If $\mathcal{P}^* \cong \mathcal{P}$, then $\mathcal{P}$ is said to be *self-dual*. A *duality* $d : \mathcal{P} \to \mathcal{P}$ is an order-reversing bijection. If $w = (i_1, \ldots, i_k)$, we define w^* to be the sequence $(n - i_1 - 1, \ldots, n - i_k - 1)$. For any flag Φ, finite sequence w, and duality d, we have $\Phi^w d = (\Phi d)^{w*}$.

If $\mathcal{P}$ is a self-dual regular n-polytope, then there is a group automorphism of $\Gamma(\mathcal{P})$ that sends each ρ_i to ρ_{n-i-1}. If $\varphi \in \Gamma(\mathcal{P})$, then we will denote by φ^* the image of φ under this automorphism. In particular, if $\varphi = \rho_{i_1} \cdots \rho_{i_k}$, then we define φ^* to be $\rho_{n-i_1-1} \cdots \rho_{n-i_k-1}$. Thus, if Φ is the base flag of $\mathcal{P}$ and $\Phi\varphi = \Phi^w$, then $\Phi\varphi^* = \Phi^{w*}$.

3. Internal self-duality

3.1. **Basic notions.** As just noted, if $\mathcal{P}$ is a self-dual regular n-polytope, then there is a group automorphism of $\Gamma(\mathcal{P})$ that sends each ρ_i to ρ_{n-i-1}. When this automorphism is inner, there is a polytope symmetry $\alpha \in \Gamma(\mathcal{P})$

such that $\alpha\rho_i = \rho_{n-i-1}\alpha$ for $0 \le i \le n-1$. We use this idea to motivate the following definitions.

Definition 3.1. *Suppose $\mathcal{P}$ is a regular self-dual polytope. An automorphism $\alpha \in \Gamma(\mathcal{P})$ is a* dualizing automorphism *(or simply* dualizing*) if $\alpha\rho_i = \rho_{n-i-1}\alpha$ for $0 \le i \le n-1$ (equivalently, if $\alpha\varphi = \varphi^*\alpha$ for every $\varphi \in \Gamma(\mathcal{P})$). A regular self-dual polytope is* internally self-dual *if it has a dualizing automorphism. Otherwise, it is* externally self-dual*.*

Depending on the automorphism group of a polytope, we can sometimes determine internal self-duality without knowing anything deep about the polytope's structure.

Proposition 3.2. *Suppose $\mathcal{P}$ is a regular self-dual polytope, and that $\Gamma(\mathcal{P})$ is a symmetric group. Then $\mathcal{P}$ is internally self-dual unless $\Gamma(\mathcal{P}) \cong S_6$ and $\mathcal{P}$ has type $\{6,6\}$ or $\{4,4,4\}$.*

Proof. For $k \neq 6$, the symmetric group S_k has no nontrivial outer automorphisms, and so a self-dual polytope with this automorphism group must be internally self-dual. Up to isomorphism, there are 11 regular polytopes with automorphism group S_6; eight of them are not self-dual, one of them (the 5-simplex) is internally self-dual, and the remaining two (denoted {6,6}*720a and {4, 4, 4}*720 in [8]) are externally self dual. $\square$

It would perhaps be possible to find other abstract results of this type, but they could never tell the whole story. After all, it is possible for the automorphism that sends each ρ_i to ρ_{n-i-1} to be inner, even if $\Gamma(\mathcal{P})$ has nontrivial outer automorphisms. So let us shift our focus away from the abstract groups.

What does it mean combinatorially to say that $\mathcal{P}$ is internally self-dual? Let Φ be the base flag of $\mathcal{P}$, and let $\Psi = \Phi\alpha$ for some dualizing automorphism α. Let $\varphi \in \Gamma(\mathcal{P})$ and suppose that $\Phi\varphi = \Phi^w$. Then

$$\begin{aligned}\Psi\varphi &= \Phi\alpha\varphi \\ &= \Phi\varphi^*\alpha \\ &= \Phi^{w^*}\alpha \\ &= (\Phi\alpha)^{w^*} \\ &= \Psi^{w^*}.\end{aligned}$$

In other words, every automorphism acts on Ψ dually to how it acts on Φ.

Definition 3.3. *We say that flags Φ and Ψ are* dual *(to each other) if every automorphism acts dually on Φ and Ψ. That is, for every w such that $\Phi\varphi = \Phi^w$, it follows that $\Psi\varphi = \Psi^{w^*}$.*

Note that if $\mathcal{P}$ is regular and Ψ is dual to Φ, then in particular $\Psi\rho_i = \Psi^{n-i-1}$.

Proposition 3.4. *A regular polytope $\mathcal{P}$ is internally self-dual if and only if its base flag Φ has a dual flag Ψ.*

Proof. The discussion preceding Definition 3.3 shows that if $\mathcal{P}$ is internally self-dual, then Φ has a dual flag. Conversely, suppose that Φ has a dual flag Ψ. Since $\mathcal{P}$ is regular, there is an automorphism $\alpha \in \Gamma(\mathcal{P})$ such that $\Psi = \Phi\alpha$. Then for $0 \leq i \leq n-1$,

$$\begin{aligned} \Phi\alpha\rho_i\alpha^{-1} &= \Psi\rho_i\alpha^{-1} \\ &= \Psi^{n-i-1}\alpha^{-1} \\ &= (\Psi\alpha^{-1})^{n-i-1} \\ &= \Phi^{n-i-1}. \end{aligned}$$

So $\alpha\rho_i\alpha^{-1}$ acts on Φ the same way that ρ_{n-i-1} does, and since the automorphism group acts regularly on the flags, it follows that $\alpha\rho_i\alpha^{-1} = \rho_{n-i-1}$ for each i. Thus α is a dualizing automorphism, and $\mathcal{P}$ is internally self-dual. □

Proposition 3.4 provides an intuitive way to determine whether a regular polytope is internally self-dual: try to find a flag that is dual to the base flag. Let us consider some simple examples. Let $\mathcal{P} = \{p\}$, the abstract p-gon, with $3 \leq p \leq \infty$. Fix a base flag Φ. In order for Ψ to be dual to Φ, we need for $\Psi\rho_0 = \Psi^1$, which in particular means that ρ_0 fixes the vertex of Ψ. Now, whenever p is even or infinite, the reflection ρ_0 does not fix any vertices, and so $\mathcal{P}$ must be externally self dual (See Figure 1). When p is odd, then there is a unique vertex v fixed by ρ_0. Furthermore, there is an edge incident to v that is fixed by ρ_1. The flag Ψ consisting of this vertex and edge is dual to Φ, and so in this case, $\mathcal{P}$ is internally self-dual. Indeed, when p is odd, then the automorphism $(\rho_0\rho_1)^{(p-1)/2}\rho_0$ is dualizing. The following result is then clear.

Proposition 3.5. *The p-gon $\mathcal{P} = \{p\}$ is internally self-dual if and only if p is odd.*

Next consider $\mathcal{P} = \{3, 3\}$, the simplex. Since $\Gamma(\mathcal{P}) \cong S_4$, which has no nontrivial outer automorphisms, it follows that $\mathcal{P}$ is internally self-dual. Nevertheless, let us see what the dual to the base flag looks like. Consider the labeled simplex in Figure 2, with vertices $\{1, 2, 3, 4\}$, edges $\{a, b, c, d, e, f\}$, and facets $\{L, F, R, D\}$. Let us pick the triple $\Phi = (1, a, L)$ as the base flag.

To find the dual of the base flag of this simplex, consider the action of each of the distinguished generators of the automorphism group on the vertices, edges, and facets. We summarize the actions in Table 2.

Since ρ_1 and ρ_2 both fix the base vertex, we need for their duals, ρ_1 and ρ_0, to fix the vertex of the dual flag. The only possibility is vertex 3. Next, since ρ_0 and ρ_2 fix the base edge, we need for their duals to fix the edge of the dual flag. The only edges fixed by both ρ_0 and ρ_2 are edges a and f, and the only one of those incident on vertex 3 is f. Finally, since ρ_0 and ρ_1

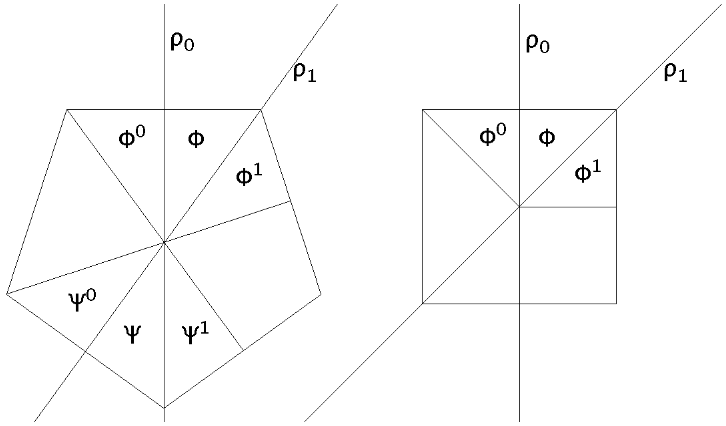


FIGURE 1. Adjacent flags and dual flags in a pentagon and a square

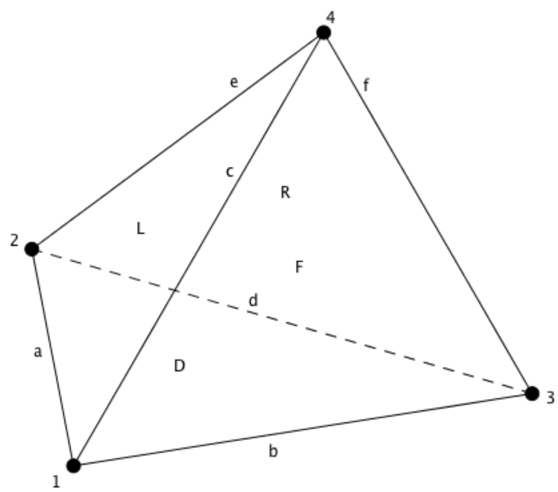


FIGURE 2. A simplex with labeled faces

	Vertices	Edges	Facets
ρ_0	$(1,2)(3)(4)$	$(a)(b,d)(c,e)(f)$	$(L)(F,R)(D)$
ρ_1	$(1)(2,4)(3)$	$(a,c)(b)(d,f)(e)$	$(L)(F,D)(R)$
ρ_2	$(1)(2)(3,4)$	$(a)(b,c)(d,e)(f)$	$(L,D)(F)(R)$

TABLE 2. Symmetries of the labeled simplex

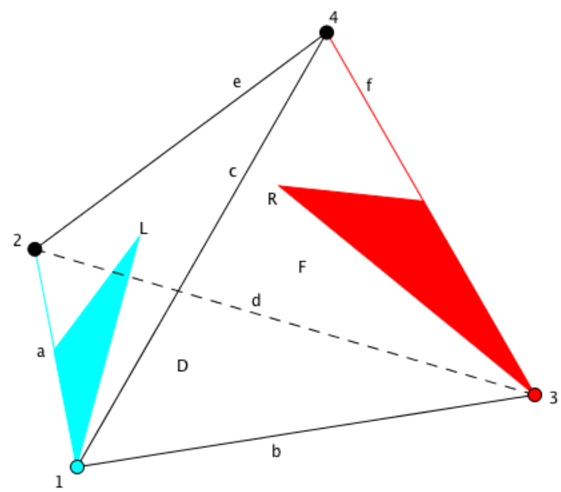


FIGURE 3. A base and dual flag of a simplex

fix the base facet, we need for ρ_2 and ρ_1 to fix the facet of the dual flag, and so the only possibility is R. So the flag that is dual to $(1, a, L)$ is $(3, f, R)$ (shown in Figure 3).

The process just described is a good illustration of the general process of finding a dual flag. Let us now describe that process. Suppose $\mathcal{P}$ is a regular, self-dual polytope, with base flag Φ and $\Gamma(\mathcal{P}) = \langle \rho_0, \dots, \rho_{n-1} \rangle$. A flag Ψ will be dual to Φ if and only if, for $0 \le i \le n-1$, the i-face of Ψ is fixed by $\langle \rho_j \mid j \neq n-i-1 \rangle$. So to find Ψ, we start by looking for a vertex F_0 that is fixed by $\langle \rho_0, \dots, \rho_{n-2} \rangle$. If no such vertex exists, then Φ does not have a dual flag. Otherwise, once we have found F_0, we now need an edge F_1 that is incident to F_0 and fixed by $\langle \rho_0, \dots, \rho_{n-3}, \rho_{n-1} \rangle$. Since ρ_{n-1} does not fix F_0, the only way that it will fix F_1 is if it interchanges the endpoints. Thus, F_1 must be incident on the vertices F_0 and $F_0\rho_{n-1}$. Similar reasoning then shows that F_2 must be incident on all of the edges in $F_1\langle \rho_{n-2}, \rho_{n-1} \rangle$. Continuing in this way, we arrive at the following algorithm:

Algorithm 3.6. *Suppose $\mathcal{P}$ is a regular n-polytope with base flag Φ and $\Gamma(\mathcal{P}) = \langle \rho_0, \dots, \rho_{n-1} \rangle$. We pick a flag $\Psi = (F_0, F_1, \dots, F_{n-1})$ as follows.*

1. *Find all vertices of $\mathcal{P}$ that are fixed by $\langle \rho_0, \dots, \rho_{n-2} \rangle$. Call this set A_0.*
2. *For each i in the range $0 \le i \le n-2$:*
   a. *If A_i is empty, then stop and output the empty set.*
   b. *Otherwise, for each chain $(F_0, \dots, F_i)$ in A_i, find all $(i+1)$-faces F_{i+1} that are incident to every i-face in the orbit $F_i\langle \rho_{n-i-1}, \dots, \rho_{n-1} \rangle$. For each such $(i+1)$-face, add the chain $(F_0, \dots, F_i, F_{i+1})$ to A_{i+1}.*

3. *Output the (possibly empty) set A_{n-1}, which consists of flags of $\mathcal{P}$.*

For simplicity, let us assume for now that $\mathcal{P}$ is *vertex-describable*, meaning that each face of $\mathcal{P}$ is uniquely determined by its vertex-set.

Proposition 3.7. *Suppose $\mathcal{P}$ is a regular, self-dual, vertex-describable polytope with base flag Φ. Then the output A_{n-1} of Algorithm 3.6 is the set of flags that is dual to Φ. In particular, $\mathcal{P}$ is internally self-dual if and only if Algorithm 3.6 outputs a non-empty set of flags.*

Proof. First, suppose that the flag Ψ is dual to the base flag Φ. Let $\Psi = (F_0, \ldots, F_{n-1})$. Then for each i, the face F_i is fixed by $\langle \rho_j \mid j \neq n-i-1 \rangle$. In particular, since $F_i < F_{i+1}$, it follows that for each $\varphi \in \langle \rho_{n-i-1}, \ldots, \rho_{n-1} \rangle$, we have $F_i\varphi < F_{i+1}\varphi = F_{i+1}$.

Thus, once we have built the chain $(F_0, \ldots, F_i)$, we can extend it to a chain $(F_0, \ldots, F_{i+1})$ with the desired properties. It follows that each set A_i is nonempty, and that Ψ is in A_{n-1}.

Conversely, suppose that the algorithm produced a nonempty set of flags, and consider one such flag $\Psi = (F_0, \ldots, F_{n-1})$. To show that this flag is dual to Φ, it suffices to show that ρ_j fixes every face except for F_{n-j-1}. By construction, the vertex-set of each face F_i is $F_0\langle \rho_{n-i}, \ldots, \rho_{n-1} \rangle$. For $j < n-i-1$, the automorphism ρ_j commutes with $\langle \rho_{n-i}, \ldots, \rho_{n-1} \rangle$ and fixes F_0, and so

$$F_0\langle \rho_{n-i}, \ldots, \rho_{n-1} \rangle \rho_j = F_0\rho_j\langle \rho_{n-i}, \ldots, \rho_{n-1} \rangle = F_0\langle \rho_{n-i}, \ldots, \rho_{n-1} \rangle.$$

For $j > n-i-1$, we have

$$F_0\langle \rho_{n-i}, \ldots, \rho_{n-1} \rangle \rho_j = F_0\langle \rho_{n-i}, \ldots, \rho_{n-1} \rangle.$$

So for $j \neq n-i-1$, the automorphism ρ_j fixes the vertex set of F_i. Since $\mathcal{P}$ is vertex-describable, it follows that ρ_j fixes F_i itself. Thus ρ_j fixes every face except (possibly) for F_{n-j-1}. But it is clear that ρ_j cannot also fix F_{n-j-1}, because the only automorphism that fixes any flag is the identity. This shows that Ψ is dual to Φ. □

Corollary 3.8. *Let $\mathcal{P}$ be a self-dual regular polytope with automorphism group $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$. If no vertices are fixed by the facet subgroup $\langle \rho_0, \ldots, \rho_{n-2} \rangle$, then $\mathcal{P}$ is externally self-dual.*

3.2. **Properties of dualizing automorphisms.** Now let us return to the algebraic point of view to determine some properties of dualizing automorphisms.

Proposition 3.9. *If $\alpha \in \Gamma(\mathcal{P})$ is dualizing, then $\alpha = \alpha^*$.*

Proof. If α is dualizing, then for all $\varphi \in \Gamma(\mathcal{P})$, we have that $\alpha\varphi = \varphi^*\alpha$. Taking $\varphi = \alpha$ yields the desired result. □

Proposition 3.10. *If $\alpha \in \Gamma(\mathcal{P})$ is dualizing, then α is not in $\langle \rho_0, \ldots, \rho_{n-2} \rangle$ or in $\langle \rho_1, \ldots, \rho_{n-1} \rangle$.*

Proof. Since α is dualizing, $\rho_{n-1} = \alpha^{-1}\rho_0\alpha$. If $\alpha \in \langle \rho_0, \dots, \rho_{n-2} \rangle$, then this gives us that ρ_{n-1} is in $\langle \rho_0, \dots, \rho_{n-2} \rangle$, which violates the intersection condition (Equation 3). Similarly, if $\alpha \in \langle \rho_1, \dots, \rho_{n-1} \rangle$, then the equation $\rho_0 = \alpha^{-1}\rho_{n-1}\alpha$ shows that ρ_0 is in $\langle \rho_1, \dots, \rho_{n-1} \rangle$, which again violates the intersection condition. □

The following properties are straightforward to verify.

Proposition 3.11. *Let $\mathcal{P}$ be a self-dual regular polytope.*

(1) *If α and β are dualizing automorphisms, then $\alpha\beta$ is central, and $\alpha\beta = \beta\alpha$.*
(2) *If φ is central and α is dualizing, then $\varphi\alpha$ is dualizing.*
(3) *If α is dualizing, then any even power of α is central, and any odd power of α is dualizing. In particular, α has even order.*
(4) *If α is dualizing, then α^{-1} is dualizing.*

Proposition 3.12. *Let $\mathcal{P}$ be an internally self-dual regular polytope such that $\Gamma(\mathcal{P})$ has a finite center. Then the number of dualizing automorphisms of $\Gamma(\mathcal{P})$ is equal to the order of the center of $\Gamma(\mathcal{P})$.*

Proof. Proposition 3.11 implies that the central and dualizing automorphisms of $\Gamma(\mathcal{P})$ together form a group in which the center of $\Gamma(\mathcal{P})$ has index 2. The result follows immediately. □

3.3. **Internal self-duality of nonregular polytopes.** Generalizing internal self-duality to nonregular polytopes is not entirely straightforward. When $\mathcal{P}$ is a self-dual regular polytope, then $\Gamma(\mathcal{P})$ always has an automorphism (inner or outer) that reflects this self-duality. This is not the case for general polytopes.

One promising way to generalize internal self-duality is using the notion of dual flags. Indeed, Definition 3.3 does not require the polytope to be regular, and makes sense even for nonregular polytopes. Let us determine some of the simple consequences of this definition.

Proposition 3.13. *Suppose that Ψ is dual to Φ. Then Ψ^{n-i-1} is dual to Φ^i.*

Proof. Let $\varphi \in \Gamma(\mathcal{P})$. We need to show that φ acts dually on Φ^i and Ψ^{n-i-1}. Suppose that $\Phi\varphi = \Phi^w$, from which it follows that $\Psi\varphi = \Psi^{w^*}$. Then:

$$(\Phi^i)\varphi = (\Phi\varphi)^i = \Phi^{wi} = (\Phi^i)^{iwi},$$

whereas

$$(\Psi^{n-i-1}\varphi) = (\Psi\varphi)^{n-i-1} = \Psi^{w^*(n-i-1)} = (\Psi^{n-i-1})^{(n-i-1)w^*(n-i-1)},$$

and the claim follows. □

Since polytopes are flag-connected, the following is an immediate consequence.

Corollary 3.14. *If $\mathcal{P}$ has one flag that has a dual, then every flag has a dual.*

Thus we see that the existence of dual flags is in fact a global property, not a local one. Here are some further consequences of the definition of dual flags.

Proposition 3.15. *Let $\mathcal{P}$ be a polytope, and let Φ and Ψ be flags of $\mathcal{P}$. If Φ and Ψ are dual, then $\mathcal{P}$ is self-dual, and there is a duality $d : \mathcal{P} \to \mathcal{P}$ that takes Φ to Ψ.*

Proof. We attempt to define the duality d by $\Phi d = \Psi$ and then extend it by $\Phi^w d = \Psi^{w^*}$. To check that this is well-defined, suppose that $\Phi^w = \Phi$; we then need to show that $\Psi^{w^*} = \Psi$. Indeed, if $\Phi^w = \Phi$, then taking φ to be the identity we have that $\Phi\varphi = \Phi^w$, whence $\Psi = \Psi\varphi = \Psi^{w*}$, with the last equality following since Φ and Ψ are dual. □

We see that dual flags have several nice properties, even in the nonregular case. We could define a polytope to be internally self-dual if every flag has a dual (equivalently, if any single flag has a dual). It is not entirely clear if this is the "right" definition. In any case, we do not pursue the nonregular case any further here.

4. Properties of internally self-dual polytopes

4.1. **Basic structural results.** Internally self-dual polytopes have a number of structural restrictions. Many of them are consequences of the following simple property.

Proposition 4.1. *If $\mathcal{P}$ is an internally self-dual regular polytope, then any regular polytope covered by $\mathcal{P}$ is also internally self-dual.*

Proof. Since $\mathcal{P}$ is internally self-dual, there is an automorphism $\alpha \in \Gamma(\mathcal{P})$ such that $\alpha\rho_i = \rho_{n-i-1}\alpha$ for $0 \leq i \leq n-1$. If $\mathcal{P}$ covers $\mathcal{Q}$, then the image of α in $\Gamma(\mathcal{Q})$ has the same property, and so $\mathcal{Q}$ is internally self-dual. □

Proposition 4.1 makes it difficult to construct internally self-dual polytopes with mixing, since the mix of any two polytopes must cover them both. The next proposition characterizes which pairs of polytopes can be mixed to create an internally self-dual polytope.

Proposition 4.2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be regular n-polytopes such that $\mathcal{P} \diamond \mathcal{Q}$ is polytopal. Then $\mathcal{P} \diamond \mathcal{Q}$ is internally self-dual if and only if*

1. *$\mathcal{P}$ and $\mathcal{Q}$ are internally self-dual, and*
2. *There are dualizing automorphisms $\alpha \in \Gamma(\mathcal{P})$ and $\beta \in \Gamma(\mathcal{Q})$ such that the images of α and β in $\Gamma(\mathcal{P}) \,\square\, \Gamma(\mathcal{Q})$ coincide.*

Proof. Let $\Gamma(\mathcal{P}) = \langle \rho_0, \dots, \rho_{n-1} \rangle$ and $\Gamma(\mathcal{Q}) = \langle \rho'_0, \dots, \rho'_{n-1} \rangle$, and let $\Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q}) = \langle \sigma_0, \dots, \sigma_{n-1} \rangle$, where $\sigma_i = (\rho_i, \rho'_i)$. Let $\varphi = (\alpha, \beta) \in \Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$. Then φ is dualizing if and only if α and β are both dualizing, since $\varphi\sigma_i =$

$\sigma_{n-i-1}\varphi$ if and only if $(\alpha\rho_i, \beta\rho'_i) = (\rho_{n-i-1}\alpha, \rho'_{n-i-1}\beta)$. Therefore, $\mathcal{P} \diamond \mathcal{Q}$ is internally self-dual if and only if there are dualizing automorphisms $\alpha \in \Gamma(\mathcal{P})$ and $\beta \in \Gamma(\mathcal{Q})$ such that $(\alpha, \beta) \in \Gamma(\mathcal{P}) \diamond \Gamma(\mathcal{Q})$. By [3, Prop. 3.7], this occurs if and only if the images of α and β in $\Gamma(\mathcal{P}) \,\square\, \Gamma(\mathcal{Q})$ coincide. $\square$

As a corollary of Proposition 4.2, if $\mathcal{P}$ and $\mathcal{Q}$ are internally self-dual regular n-polytopes with the "same" dualizing element, then their mix is internally self-dual. More formally:

Corollary 4.3. *Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are internally self-dual regular n-polytopes, and let w be a word in the free group on $\rho_0, \ldots, \rho_{n-1}$. If the images of w in $\Gamma(\mathcal{P})$ and in $\Gamma(\mathcal{Q})$ are both dualizing, then $\mathcal{P} \diamond \mathcal{Q}$ is internally self-dual.*

If we have a presentation for $\Gamma(\mathcal{P})$, it is often simple to show that $\mathcal{P}$ is not internally self-dual. Indeed, because of Proposition 4.1, all we need to do is find some non-self-dual quotient of $\Gamma(\mathcal{P})$. For example, if $\mathcal{P}$ is internally self-dual, then adding a relation that forces $\rho_0 = \epsilon$ must also force $\rho_{n-1} = \epsilon$. Here are some similar results that are easily applied.

Proposition 4.4. *If $\mathcal{P}$ is an internally self-dual regular n-polytope, then in the abelianization of $\Gamma(\mathcal{P})$, the image of each ρ_i is equal to the image of ρ_{n-i-1}.*

Proof. By the argument given in Proposition 4.1, the abelianization of $\Gamma(\mathcal{P})$ must have a dualizing automorphism. Since such an automorphism also must commute with the image of every ρ_i, it follows that the images of ρ_i and ρ_{n-i-1} must coincide. $\square$

Suppose $\mathcal{P}$ is a regular n-polytope, with m-faces isomorphic to $\mathcal{K}$. We say that $\mathcal{P}$ has the *Flat Amalgamation Property (FAP) with respect to its m-faces* if adding the relations $\rho_i = \epsilon$ to $\Gamma(\mathcal{P})$ for $i \geq m$ yields a presentation for $\Gamma(\mathcal{K})$ (rather than a proper quotient).

Proposition 4.5. *If $\mathcal{P}$ has the FAP with respect to its m-faces for some m with $1 \leq m \leq n-1$, then $\mathcal{P}$ is not internally self-dual.*

Proof. If $\mathcal{P}$ is internally self-dual, then adding the relation $\rho_{n-1} = \epsilon$ to $\Gamma(\mathcal{P})$ must force $\rho_0 = \epsilon$, and this precludes $\mathcal{P}$ from having the FAP with respect to its m-faces for any $m \geq 1$. $\square$

Recall that if $\mathcal{P}$ is an n-polytope, and e is the unique 1-polytope, then we define $\mathcal{P} \diamond e$ to be the polytope whose group is

$$\Gamma(\mathcal{P}) \diamond \langle \rho_0, \ldots, \rho_{n-1} \mid \rho_0^2 = \epsilon, \rho_i = \epsilon \text{ for } 1 \leq i \leq n-1 \rangle.$$

(The fact that the mix is a polytope and not just a poset is proved by [10, Thm. 7A7].)

Corollary 4.6. *If $\mathcal{P}$ is internally self-dual and e is the unique 1-polytope, then $\Gamma(\mathcal{P} \diamond e) \cong \Gamma(\mathcal{P}) \times C_2$.*

Proof. The group $\Gamma(\mathcal{P} \diamond e)$ is isomorphic to either $\Gamma(\mathcal{P})$ or $\Gamma(\mathcal{P}) \times C_2$, by [10, Thm. 7A8]. From Proposition 4.5 and [10, Thm. 7A11] we know that $\Gamma(\mathcal{P} \diamond e) \not\cong \Gamma(\mathcal{P})$. □

Proposition 4.7. *If $\mathcal{P}$ is self-dual, then $(\mathcal{P} \diamond e)^* \diamond e$ is self-dual.*

Proof. First, note that the automorphism group of $(\mathcal{P} \diamond e)^*$ is naturally isomorphic to

$$\Gamma(\mathcal{P}^*) \diamond \langle \rho_0, \dots, \rho_{n-1} \mid \rho_{n-1}^2 = \epsilon, \rho_i = \epsilon \text{ for } 0 \leq i \leq n-2 \rangle.$$

Therefore, the automorphism group of $(\mathcal{P} \diamond e)^* \diamond e$ is naturally isomorphic to the mix of $\Gamma(\mathcal{P}^*)$ with

$$\langle \rho_0, \dots, \rho_{n-1} \mid \rho_{n-1}^2 = \epsilon, \rho_i = \epsilon \text{ for } 0 \leq i \leq n-2 \rangle$$

and

$$\langle \rho_0, \dots, \rho_{n-1} \mid \rho_0^2 = \epsilon, \rho_i = \epsilon \text{ for } 1 \leq i \leq n-1 \rangle.$$

Taking the dual of this mix amounts to taking the dual in each component separately, which fixes the first factor while interchanging the other two factors. □

Corollary 4.8. *If $\mathcal{P}$ is internally self-dual and e is the unique 1-polytope, then $(\mathcal{P} \diamond e)^* \diamond e$ is externally self-dual, with automorphism group $\Gamma(\mathcal{P}) \times C_2$ or $\Gamma(\mathcal{P}) \times C_2^2$.*

Proof. By Proposition 4.7, the polytope $(\mathcal{P} \diamond e)^* \diamond e$ is self-dual. Furthermore, if you mix $(\mathcal{P} \diamond e)^* \diamond e$ with an edge again, then the automorphism group does not change. Thus, by Corollary 4.6, $(\mathcal{P} \diamond e)^* \diamond e$ must be externally self-dual. Finally, the automorphism group of $(\mathcal{P} \diamond e)^*$ is abstractly isomorphic to the automorphism group of $\mathcal{P} \diamond e$, which Proposition 4.6 says is $\Gamma(\mathcal{P}) \times C_2$. The result then follows. □

Corollary 4.9. *If $\mathcal{P}$ is an internally self-dual polyhedron of type $\{p, p\}$, then $(\mathcal{P} \diamond e)^* \diamond e$ is an externally self-dual polyhedron of type $\{q, q\}$, where $q = \mathrm{lcm}(p, 2)$.*

Proof. Let $\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2 \rangle$ and let $\Gamma(e) = \langle \lambda_0 \rangle$. For $0 \leq i \leq 2$, let $\sigma_i = (\rho_i, \lambda_i) \in \Gamma(\mathcal{P}) \times \Gamma(e)$, where we take $\lambda_1 = \lambda_2 = \epsilon$. Then $(\sigma_0\sigma_1)^p = ((\rho_0\rho_1)^p, \lambda_0^p) = (\epsilon, \lambda_0^p)$. If p is even, then this gives us (ϵ, ϵ), and so $\sigma_0\sigma_1$ has order p. Otherwise $\sigma_0\sigma_1$ has order $2p$. So $\mathcal{P} \diamond e$ is of type $\{q, p\}$, and by taking the dual and mixing with e again, we get a polyhedron of type $\{q, q\}$. □

In some sense, Corollary 4.8 says that externally self-dual polytopes are at least as common as internally self-dual polytopes.

4.2. **Internal self-duality of universal polytopes.** A natural place to start looking for internally self-dual polytopes is the universal polytopes $\{p_1, \ldots, p_{n-1}\}$ whose automorphism groups are string Coxeter groups. Let us start with those polytopes with a 2 in their Schläfli symbol. Recall that a polytope is *flat* if every vertex is incident on every facet.

Proposition 4.10. *There are no flat, regular, internally self-dual polytopes.*

Proof. Suppose $\mathcal{P}$ is a flat regular polytope, with $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$. The stabilizer of the base facet is $\Gamma_{n-1} = \langle \rho_0, \ldots, \rho_{n-2} \rangle$, which acts transitively on the vertices of the base facet. Since $\mathcal{P}$ is flat, it follows that Γ_{n-1} acts transitively on all the vertices of $\mathcal{P}$. In particular, Γ_{n-1} does not fix any vertices, and thus Corollary 3.8 implies that $\mathcal{P}$ is not internally self-dual. □

Corollary 4.11. *If $\mathcal{P}$ is a regular internally self-dual polytope of type $\{p_1, \ldots, p_{n-1}\}$, then each p_i is at least* 3.

Proof. Proposition 2B16 in [10] proves that if some p_i is 2, then $\mathcal{P}$ is flat. □

Next, we can rule out infinite polytopes.

Proposition 4.12. *If $\mathcal{P}$ is a self-dual regular polytope such that $\Gamma(\mathcal{P})$ is an infinite string Coxeter group, then $\mathcal{P}$ is externally self-dual.*

Proof. Lemma 2.14 in [6] proves that an infinite Coxeter group with no finite irreducible components has no nontrivial inner automorphisms that realize any graph automorphisms of the Coxeter diagram. Since self-duality is induced by a graph automorphism, it follows that $\mathcal{P}$ cannot be internally self-dual. □

We can now easily cover the remaining self dual string Coxeter groups.

Theorem 4.13. *The only internally self-dual regular polytopes such that $\Gamma(\mathcal{P})$ is a string Coxeter group are simplices and p-gons with p odd.*

Proof. In light of Corollary 4.11 and Proposition 4.12, the only possibilities left to consider are simplices, polygons, and the 24-cell $\{3, 4, 3\}$. Propositions 3.2 and 3.5 establish the claim for simplices and p-gons. Using a computer algebra system, we can verify that $\{3, 4, 3\}$ is not internally self-dual. □

4.3. **Restrictions on the automorphism group.** Based on the data in Table 1 and the restrictions from the previous section, it seems that regular internally self-dual polytopes could be relatively rare, especially in ranks other than three. Before exploring some existence results about internally self-dual polytopes, let us discuss a few natural questions that arise while looking for examples. First, one might want to know if the existence of internally or externally self-dual polytopes with a certain group Γ as an automorphism group might depend on the abstract class of Γ. In a simple way, the answer to this question is yes, in that if Γ has no outer automorphisms, then there can be no external dualities of the polytope (as seen in Proposition 3.2). Otherwise, it seems that the abstract structure of the group does not provide insight into whether self-dual regular polytopes will be either externally or internally self-dual. In particular, there are both internally and externally self-dual regular polytopes with simple groups as their automorphism groups; examples of which can be easily found by considering alternating groups.

Second, while looking for internally self-dual polytopes of higher rank, a natural question is whether they are built from only internally self-dual polytopes of lower rank. For example, must the medial sections of an internally self-dual polytope be internally self-dual themselves? (The *medial section* of a polytope is a section F/v where F is a facet and v is a vertex.) Consider the unique self-dual regular polytope of rank four with an alternating group acting on nine points as its automorphism group (see Figure 4 of [4]). This is easily seen to be internally self-dual, as the duality is realized as an even permutation on the nine points. However, this polytope is of type $\{5, 6, 5\}$, and so its medial sections, being hexagons, are not internally self-dual; see Proposition 3.5.

Finally, Proposition 4.1 says that if a regular polytope is internally self-dual, then the regular polytopes that it covers are also internally self-dual. This is a stringent requirement, so one might hope that the converse would be true. However, there are externally self-dual polytopes that only cover internally-self dual polytopes. For example, the unique polyhedron $\mathcal{P}$ of type $\{5, 5\}$ with 320 flags is externally self-dual, and it double-covers the unique polyhedron $\mathcal{Q}$ of type $\{5, 5\}$ with 160 flags, which is internally self-dual. Furthermore, every quotient of the former (polyhedral or not) filters through the latter, since the kernel of the covering from $\mathcal{P}$ to $\mathcal{Q}$ was the unique minimal normal subgroup of $\Gamma(\mathcal{P})$. So $\mathcal{P}$ only covers internally self-dual polyhedra, despite being externally self-dual itself.

5. Examples of internally self-dual polytopes

In this section we will prove the existence of internally and externally self-dual polytopes with various characteristics. We mainly focus on rank 3, but higher rank polytopes are also constructed.

5.1. **Rank three.**

First we construct a few families of internally self-dual regular polyhedra. Our main result is the following.

Theorem 5.1. *For each $p \geq 3$ (including $p = \infty$), there is an internally self-dual regular polyhedron of type $\{p, p\}$, and for each $p \geq 4$ (including $p = \infty$), there is an externally self-dual regular polyhedron of type $\{p, p\}$. Furthermore, if p is even, then there are infinitely many internally and externally self-dual regular polyhedra of type $\{p, p\}$.*

We will focus on constructing internally self-dual regular polyhedra; then Theorem 4.13 and Corollary 4.9 will take care of the rest. First we will show that there is an internally self-dual polyhedron of type $\{p, p\}$ for each $p \geq 3$.

The data from [2] provides examples for $3 \leq p \leq 12$. We will construct a family that covers $p \geq 7$. We start with a simple lemma.

Lemma 5.2. *Suppose π_1 and π_2 are distinct permutations that act cyclically on n points and that $\pi_1^{d_1} = \pi_2^{d_2}$ for some d_1 and d_2. Suppose that for some positive integer k, there is a unique point i such that $i\pi_1^j = i\pi_2^j$ for all j with $1 \leq j \leq k$. Then $d_1 = d_2 = 0$ (mod n).*

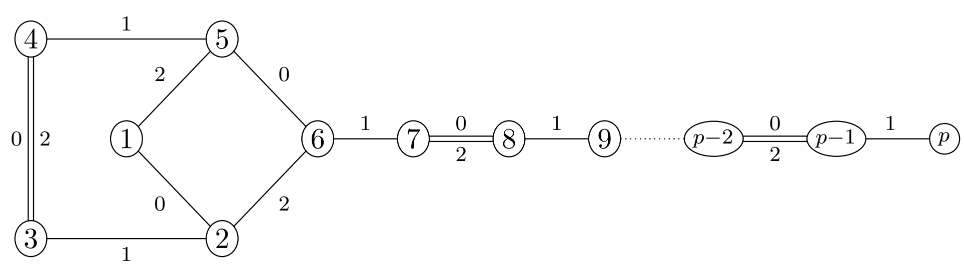
Proof. Since $\pi_1^{d_1} = \pi_2^{d_2}$, it follows that π_2 and $\pi_1^{d_1}$ commute. Then, for $1 \leq j \leq k$,

$$\begin{aligned}(i\pi_1^{d_1})\pi_2^j &= (i\pi_2^j)\pi_1^{d_1} \\ &= (i\pi_1^j)\pi_1^{d_1} \\ &= (i\pi_1^{d_1})\pi_1^j.\end{aligned}$$

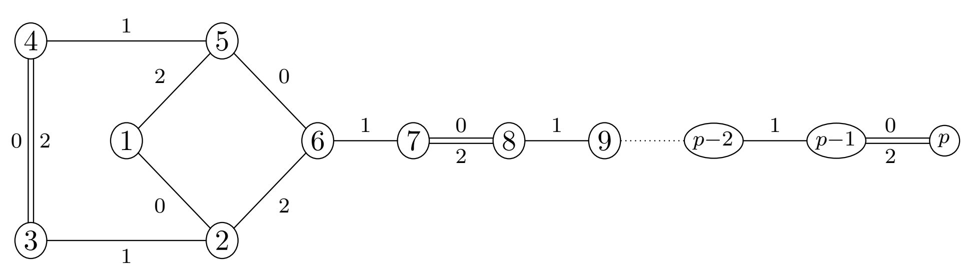
That is, π_1^j and π_2^j act the same way on $(i\pi_1^{d_1})$ for $1 \leq j \leq k$. By assumption, i was the only point such that π_1^j and π_2^j act the same way on that point for $1 \leq j \leq k$. It follows that $\pi_1^{d_1}$ (which is equal to $\pi_2^{d_2}$) fixes i, which implies that $d_1 = d_2 = 0$ (mod n). ☐

Theorem 5.3. *For each $p \geq 7$, there is an internally self-dual polyhedron of type $\{p, p\}$ such that $(\rho_0\rho_2\rho_1)^6$ is dualizing.*

Proof. We will construct a permutation representation graph, and then show that it is a CPR graph. If p is odd, then consider the following permutation representation graph:



If p is even, then instead consider the following.



In both cases, it is easy to see that the group is a string group generated by involutions. To verify this, we only need to notice that the subgraph induced by edges of labels 0 and 2, consists of connected components that are either isolated vertices, double edges, or squares with alternating labels.

If p is odd, then

$$\rho_0\rho_2\rho_1 = (1,7,6)(2,4,5,3)(8,9)(10,11)\cdots(p-1,p),$$

and if p is even, then

$$\rho_0\rho_2\rho_1 = (1,7,6)(2,4,5,3)(8,9)(10,11)\cdots(p-2,p-1).$$

In both cases, it follows that $(\rho_0\rho_2\rho_1)^6 = (2,5)(3,4)$. It is simple to show that this is in fact a dualizing automorphism. In other words, $(\rho_0\rho_2\rho_1)^6\rho_i$ acts the same on every vertex as $\rho_{2-i}(\rho_0\rho_2\rho_1)^6$.

It remains to show that $\langle\rho_0,\rho_1,\rho_2\rangle$ is a string C-group. Following [10, Prop. 2E16], it suffices to show that $\langle\rho_0,\rho_1\rangle\cap\langle\rho_1,\rho_2\rangle = \langle\rho_1\rangle$.

Let φ be in the intersection, $\varphi \notin \langle\rho_1\rangle$. Without loss of generality, we may assume that $\varphi = (\rho_0\rho_1)^{d_1}$ for some d_1 (since if φ is an odd product of factors ρ_i, then $\varphi\rho_1$ is also in the intersection and can be written like that). Now, φ is also in $\langle\rho_1,\rho_2\rangle$. If $\varphi = \rho_1(\rho_2\rho_1)^{d_2}$ for some d_2, then $\varphi^2 = \epsilon$, and thus $(\rho_0\rho_1)^{2d_1} = \epsilon$. If p is odd, then this only happens if φ is itself the identity, contrary to our assumption. So in that case we may assume that $\varphi = (\rho_2\rho_1)^{d_2}$ for some d_2. If p is even, then in principle, it is possible that $\varphi = (\rho_0\rho_1)^{p/2}$, and so it could happen that $\varphi = \rho_1(\rho_2\rho_1)^{d_2}$.

Suppose p is odd, $p \geq 9$. Then

$$\rho_0\rho_1 = (1,3,5,7,9,\dots,p,p-1,p-3,\dots,6,4,2)$$

and

$$\rho_2\rho_1 = (1,4,2,7,9,\dots,p,p-1,p-3,\dots,6,3,5).$$

We note that these cycles act the same way on 3, on 4, and on 7 through p. Indeed, 7 is the start of a unique longest sequence of points on which $\rho_0\rho_1$ and $\rho_2\rho_1$ act, and so we can apply Lemma 5.2 with $i = 7$ and $k = p-6$. It follows that there is no nontrivial equation of the form $(\rho_0\rho_1)^{d_1} = (\rho_2\rho_1)^{d_2}$.

Now, suppose p is even, $p \geq 10$. Then

$$\rho_0\rho_1 = (1,3,5,7,9,\dots,p-1,p,p-2,p-4,\dots,6,4,2)$$

and

$$\rho_2\rho_1 = (1,4,2,7,9,\dots,p-1,p,p-2,p-4,\dots,6,3,5).$$

As in the odd case, $(\rho_0\rho_1)^{d_1}$ cannot equal $(\rho_2\rho_1)^{d_2}$, by Lemma 5.2 with $i = 7$ and $k = p-6$. We still need to rule out the case $(\rho_0\rho_1)^{p/2} = \rho_1(\rho_2\rho_1)^{d_2}$. Note that $(\rho_0\rho_1)^{p/2}$ always sends 1 to p. In order for $\rho_1(\rho_2\rho_1)^{d_2}$ to do the same thing, we would need $d_2 = p/2$ as well. But then $(\rho_0\rho_1)^{p/2}$ sends 3 to $p-2$, whereas $\rho_1(\rho_2\rho_1)^{p/2}$ sends 3 to $p-4$. So that rules out this case, proving that the intersection condition holds.

The remaining cases where $p = 7, 8$ can be verified using a computer algebra system. □

Using the polyhedra built in Theorem 5.3 as a base, we can construct an infinite polyhedron that is internally self-dual.

Theorem 5.4. *Consider a family of internally self-dual regular polyhedra* $\{\mathcal{P}_i\}_{i=1}^{\infty}$*, with infinitely many distinct polytopes. Let* $\Gamma(\mathcal{P}_i) = \langle \rho_0^{(i)}, \rho_1^{(i)}, \rho_2^{(i)} \rangle$*. Suppose that there is a finite sequence* $j_1, \ldots, j_m$ *such that* $\rho_{j_1}^{(i)} \cdots \rho_{j_m}^{(i)}$ *is dualizing in each* $\Gamma(\mathcal{P}_i)$*. Then* $\mathcal{P} = \mathcal{P}_1 \diamond \mathcal{P}_2 \diamond \cdots$ *is an infinite internally self-dual regular polyhedron.*

Proof. It is clear that $\mathcal{P}$ is infinite. Let

$$\rho_0 = (\rho_0^{(1)}, \rho_0^{(2)}, \ldots), \rho_1 = (\rho_1^{(1)}, \rho_1^{(2)}, \ldots), \rho_2 = (\rho_2^{(1)}, \rho_2^{(2)}, \ldots).$$

To show that $\mathcal{P}$ is a polyhedron, we need to show that $\langle \rho_0, \rho_1, \rho_2 \rangle$ satisfies the intersection condition. In particular, by Proposition 2E16 of [10], the only intersection that is nontrivial to check is $\langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$. Suppose $\varphi \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$. For each i, let φ_i be the natural projection of φ in $\Gamma(\mathcal{P}_i)$, where we send each ρ_j to $\rho_j^{(i)}$. Since $\mathcal{P}_i$ is a polyhedron, each φ_i is either ϵ or $\rho_1^{(i)}$. Now, since $\langle \rho_0, \rho_1 \rangle$ is dihedral, the automorphism φ is either even or odd. Furthermore, its projection φ_i into the dihedral group $\langle \rho_0^{(i)}, \rho_1^{(i)} \rangle$ must have the same parity as φ itself. Therefore, every φ_i must have the same parity, and so either $\varphi_i = \epsilon$ for every i, or $\varphi_i = \rho_1^{(i)}$ for every i. In the first case, $\varphi = \epsilon$, and in the second, $\varphi = \rho_1$. This proves the intersection condition.

Finally, it is clear that $\rho_{j_1} \cdots \rho_{j_m}$ is dualizing in $\Gamma(\mathcal{P})$, and so $\mathcal{P}$ is internally self-dual. □

Corollary 5.5. *There is an infinite internally self-dual regular polyhedron of type* $\{\infty, \infty\}$*, with dualizing automorphism* $(\rho_0\rho_2\rho_1)^6$*.*

Proof. Apply the construction of Theorem 5.4 to the polyhedra in Theorem 5.3. □

The infinite polyhedron of Corollary 5.5 is a little difficult to work with; we have neither a permutation representation nor a presentation for the automorphism group. With this example, however, we can now build a simpler example.

Corollary 5.6. *Let* Γ *be the quotient of* $[\infty, \infty]$ *by the three relations*

$$(\rho_0\rho_2\rho_1)^6\rho_i = \rho_{2-i}(\rho_0\rho_2\rho_1)^6, \text{ where } 0 \le i \le 2.$$

Then Γ *is the automorphism group of an infinite internally self-dual regular polyhedron of type* $\{\infty, \infty\}$*.*

Proof. First, note that Γ covers the automorphism group of any polyhedron with dualizing automorphism $(\rho_0\rho_2\rho_1)^6$. Therefore, Γ covers the automorphism group of the polyhedron in Corollary 5.5. Then the quotient criterion (see [10, Thm. 2E17]) shows that this is a polyhedron, since it covers the polyhedron in Corollary 5.5 without any collapse of the facet subgroup $\langle \rho_0, \rho_1 \rangle$. □

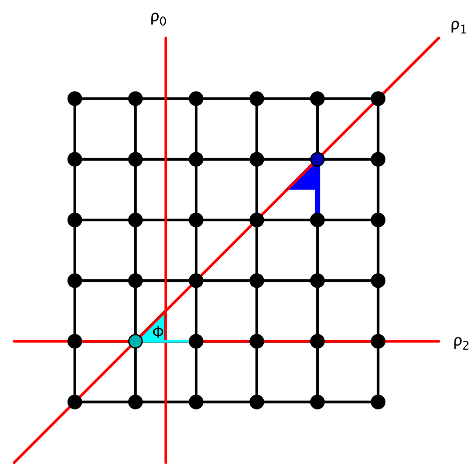


FIGURE 4. Base and dual flags in the regular polyhedron $\{4,4\}_{(5,0)}$

Now let us prove that there are infinitely many internally self-dual polyhedra of type $\{p,p\}$ when $p \geq 4$ and p is even. We first cover the case $p = 4$.

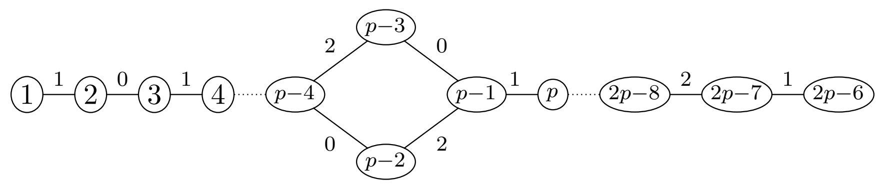
Proposition 5.7. *The polyhedron $\{4,4\}_{(s,0)}$ is internally self-dual if and only if s is odd.*

Proof. Let $\mathcal{P} = \{4,4\}_{(s,0)}$, and let us identify the vertices of $\mathcal{P}$ with $(\mathbb{Z}/s\mathbb{Z})^2$. Let us choose the base flag to consist of the origin, the edge from the origin to $(1,0)$, and the square $[0,1] \times [0,1]$. Then ρ_0 sends each vertex (x,y) to $(1-x,y)$, ρ_1 sends (x,y) to (y,x), and ρ_2 sends (x,y) to $(x,-y)$. If s is even, then ρ_0 does not fix any vertex, and so by Corollary 3.8, $\mathcal{P}$ is externally self-dual. If s is odd, say $s = 2k-1$, then the unique vertex fixed by $\langle \rho_0, \rho_1 \rangle$ is (k,k). Continuing with Algorithm 3.6, we want an edge that contains (k,k) and $(k,k)\rho_2 = (k,-k) = (k,k-1)$. Finally, we want a square that contains that edge and its images under $\langle \rho_1, \rho_2 \rangle$, which consists of the 4 edges bounding the square with corners (k,k) and $(k-1,k-1)$. Thus, the flag that is dual to the base flag consists of the vertex (k,k), the edge to $(k,k-1)$, and the square that also includes $(k-1,k-1)$. See Figure 4 for the case $k = 3$. $\square$

We now construct a family of examples to cover the remaining cases.

Theorem 5.8. *For each even $p \geq 6$, there are infinitely many internally self-dual polyhedra of type $\{p,p\}$.*

Proof. For each even $p \geq 6$, we will construct a family of permutation representation graphs, and show that each one is the CPR graph of a distinct internally self-dual polyhedron of type $\{p, p\}$. First, consider the following permutation representation graph G.



Let ρ_0, ρ_1, ρ_2 be the permutations induced by edges of the appropriate label. We have

$$\rho_0 = (2,3)(4,5)\cdots(p-6,p-5)(p-4,p-2)(p-3,p-1),$$
$$\rho_1 = (1,2)(3,4)\cdots(p-5,p-4)(p-1,p)(p+1,p+2)\cdots(2p-7,2p-6),$$
$$\rho_2 = (p-4,p-3)(p-2,p-1)(p,p+1)(p+2,p+3)\cdots(2p-8,2p-7).$$

It is again clear that $\langle \rho_0, \rho_1, \rho_2 \rangle$ is a string group generated by involutions. Next, note that $\rho_0\rho_2\rho_1$ interchanges $p-2$ with $p-3$, while cyclically permuting the remaining $2p-8$ points. Now consider $\sigma = (\rho_0\rho_2\rho_1)^{p-4}$. This interchanges 1 with $2p-6$ and thus $1(\rho_0\rho_2\rho_1)^j$ with $(2p-6)(\rho_0\rho_2\rho_1)^j$, for each j. Since the action of $(\rho_0\rho_2\rho_1)$ on vertices on the left is the mirror of the action on the right, it follows that σ interchanges i with $2p-5-i$ for $1 \le i \le p-4$, while fixing $p-2$ and $p-3$.

We will now build a larger graph $\mathcal{X}$ using the one above as a building block. Let k be an odd positive integer, and take $N := k(p-4)$ copies of the above graph, labeled $G_1, G_2, \ldots, G_N$, and arrange them cyclically. Let us use (i,j) to mean vertex i in G_j (and where j is considered modulo N if necessary). We connect the graphs G_j by adding edges labeled 1 from $(p-2,j)$ to $(p-3,j+1)$. By Theorem 4.5 in [12], this is the CPR graph of a polyhedron. Furthermore, if we erase the edges labeled 2, then the connected components either have 2 vertices or p vertices. The latter consists of the first $p-4$ vertices, then the bottom of a diamond, then the top of the next diamond, the right of that diamond, and one more vertex. The same happens if we erase the edges labeled 0, and so we get a polyhedron of type $\{p, p\}$.

Let us now redefine ρ_0, ρ_1, and ρ_2 as the permutations induced by edges of $\mathcal{X}$, and let $\sigma = (\rho_0\rho_2\rho_1)^{p-4}$ as before. The new σ acts in exactly the same way on every vertex in every copy of the original CPR graph except for the top and bottom of the diamonds. Indeed, σ takes $(p-3,j)$ to $(p-3,j-p+4)$ and it takes $(p-2,j)$ to $(p-2,j+p-4)$. Then the order of σ is $2k$, since σ to any odd power interchanges every $(1,j)$ with $(2p-6,j)$, and σ^k is the smallest power of σ that fixes every $(p-3,j)$ and $(p-2,j)$.

We claim that σ^k is dualizing. To prove that, we need to show that $\rho_i\sigma^k = \sigma^k\rho_{2-i}$ for $i = 0, 1, 2$. That is clearly true for all vertices other than the tops and bottoms of diamonds, because σ^k acts as a reflection through the middle of the diagram, and this reflection also dualizes every label. Checking that $\rho_i \sigma_k$ and $\sigma^k \rho_{2-i}$ act the same on the top and bottom of every diamond is then easy.

Finally, we claim that the constructed polyhedra are distinct for each k. For this, it suffices to show that in each polyhedron, k is the smallest positive integer such that σ^k is dualizing. (In principle, if σ^k is dualizing, it might also be true that σ^m is dualizing for some m dividing k.) In order for a power of σ to act like a dualizing automorphism on most vertices, it must be odd, since σ^2 fixes every vertex other than the tops and bottoms of diamonds. So consider σ^m for some odd $m < k$. The permutation $\sigma^m \rho_1$ sends $(p-3, 1)$ to $(p-2, m(p-4))$, whereas $\rho_1 \sigma^m$ sends $(p-3, 1)$ to $(p-2, N-m(p-4)) = (p-2, (k-m)(p-4))$. In order for these two points to be the same, we need $k - m = m$, so that $k = 2m$. But k is odd, so this is impossible. So we get infinitely many internally self-dual polyhedra of type $\{p, p\}$. $\square$

We can now prove our main result.

Proof of Theorem 5.1. The data from [2], combined with Theorem 5.3 and Corollary 5.5, show that there are internally self-dual polyhedra of type $\{p, p\}$ for all $3 \leq p \leq \infty$. Theorem 4.13 then shows that there are externally self-dual polyhedra of type $\{p, p\}$ for $4 \leq p \leq \infty$. Proposition 5.7 and Theorem 5.8 show that there are infinitely many internally self-dual polyhedra of type $\{p, p\}$ for p even, and combining with Corollary 4.9, we get infinitely many externally self-dual polyhedra of type $\{p, p\}$ for p even as well. $\square$

5.2. **Higher ranks.** Now that the rank three case is well established, let us consider internally self-dual regular polytopes of higher ranks. We will start by showing that there are infinitely many internally self-dual polytopes in every rank. By Theorem 4.13, we already know that the n-simplex is internally self-dual. It is instructive to actually show this constructively, using Algorithm 3.6.

Consider the representation of the regular n-simplex $\mathcal{P}$ as the convex hull of the points $e_1, e_2, \ldots, e_{n+1}$ in $\mathbb{R}^{n+1}$. Each i-face of $\mathcal{P}$ is the convex hull of $i+1$ of the vertices, and each flag of $\mathcal{P}$ can be associated to an ordering of the vertices $(e_{i_1}, e_{i_2}, \ldots, e_{i_{n+1}})$, where the i-face of the flag is the convex hull of the first $i+1$ vertices.

Let us set the base flag Φ to be the flag corresponding to the ordering $(e_1, e_2, \ldots, e_{n+1})$. Then each automorphism ρ_i acts by switching coordinates $i+1$ and $i+2$, corresponding to a reflection in the hyperplane $x_{i+1} = x_{i+2}$. In order to find a flag dual to Φ, we need to first find a vertex that is fixed by $\langle \rho_0, \ldots, \rho_{n-2} \rangle$. The only such vertex is e_{n+1}. Next, we need an edge that is incident to e_{n+1} and fixed by $\langle \rho_0, \ldots, \rho_{n-3} \rangle \times \langle \rho_{n-1} \rangle$. Since ρ_{n-1} interchanges e_n and e_{n+1} and it must fix the edge, it follows that the edge must be incident on e_n. Continuing in this manner, it is easy to see that the dual flag must be $(e_{n+1}, e_n, \ldots, e_1)$.

Now let us find the dualizing automorphism of $\Gamma(\mathcal{P})$. We can identify $\Gamma(\mathcal{P})$ with the symmetric group on $n+1$ points, where ρ_i is the transposition

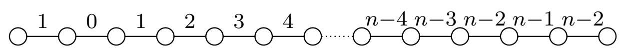
$(i+1, i+2)$. We noted above that the dual of the base flag simply reversed the order of the vertices. So the dualizing automorphism of $\Gamma(\mathcal{P})$ can be written as

$$(\rho_0\rho_1\cdots\rho_{n-1})(\rho_0\rho_1\cdots\rho_{n-2})\cdots(\rho_0\rho_1)(\rho_0),$$

as this "bubble sorts" the list $(1, 2, \ldots, n+1)$ into its reverse.

Here is another example of high-rank internally self-dual polytopes. That they are internally self-dual follows from Proposition 3.2.

Proposition 5.9. *For each $n \geq 5$ the regular n-polytope with group S_{n+3} described in* [5, Prop. 4.10]*, is internally self dual.*



The CPR graph of these polytopes is shown above. Each of these polytopes is obtained from a rank $(n+2)$ simplex, by first taking the Petrie contraction and then dualizing and taking another Petrie contraction. (The Petrie contraction of a string group generated by involutions $\langle\rho_0, \ldots, \rho_{n-1}\rangle$ is the group generated by $\langle\rho_1, \rho_0\rho_2, \rho_3, \ldots, \rho_{n-1}\rangle$.)

The cubic toroids (described in [10, Section 6D]) provide an infinite family of internally self-dual polytopes with automorphism groups other than the symmetric group.

Theorem 5.10. *The regular $(n+1)$-polytope $\{4, 3^{n-2}, 4\}_{(s,0^{n-1})}$ is internally self-dual if and only if s is odd.*

Proof. Let us take the vertex set of $\mathcal{P}$ to be $(\mathbb{Z}/s\mathbb{Z})^n$. For $0 \leq i \leq n$, let G_i be the i-face of $\mathcal{P}$ containing vertices $\mathbf{0}, e_1, e_1+e_2, \ldots, e_1+e_2+\cdots+e_i$, where $\{e_i\}_{i=1}^n$ is the standard basis. Let $\Phi = (G_0, \ldots, G_n)$ be our base flag. Then the generators of $\Gamma(\mathcal{P})$ can be described geometrically as follows (taken from [10, Eq. 6D2]): ρ_0 sends $(x_1, \ldots, x_n)$ to $(1-x_1, x_2, \ldots, x_n)$, ρ_n sends $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_{n-1}, -x_n)$, and for $1 \leq i \leq n-1$, ρ_i interchanges x_i and x_{i+1}.

We now try to build a flag that is dual to Φ, using Algorithm 3.6. First, we need to find a vertex that is fixed by $\langle\rho_0, \ldots, \rho_{n-1}\rangle$. In order for $(x_1, \ldots, x_n)$ to be fixed by ρ_0, we need $x_1 \equiv 1-x_1 \pmod{s}$; in other words, $2x_1 \equiv 1 \pmod{s}$. That has a solution if and only if s is odd, so that already establishes that when s is even, the polytope $\mathcal{P}$ is not internally self-dual. On the other hand, if s is odd, then we can take $x_1 = (s+1)/2$ as a solution. Now, in order for $\rho_1, \ldots, \rho_{n-1}$ to also fix this vertex, we need all of the coordinates to be the same. So we pick $F_0 = ((s+1)/2, \ldots, (s+1)/2)$.

Next we need an edge incident on F_0 and $F_0\rho_n$. The latter is simply $((s+1)/2, \ldots, (s+1)/2, (s-1)/2)$, which is indeed adjacent to F_0, and that gives us our edge F_1. To pick F_2, we look at the orbit $G_1\langle\rho_{n-1}, \rho_n\rangle$; this gives us the square whose 4 vertices are $((s+1)/2, \ldots, (s+1)/2, \pm(s+1)/2, \pm(s+1)/2)$. In general, we take F_i to be the i-face such that its vertices are obtained from F_0 by any combination of sign changes in the last i coordinates.

Then it is clear that F_i is fixed by $\langle \rho_0, \ldots, \rho_{n-i-1}, \rho_{n-i+1}, \ldots, \rho_n \rangle$, and thus we have a dual flag to Φ. $\square$

In the remaining part of this section we will show that there are examples of internally self-dual polytopes other than the toroids and simplices (and self-dual petrie contracted relatives) in each rank $n \geq 5$; we give a family of examples in Theorem 5.17 that are string C-groups Γ which are internally self-dual. (There are other examples in rank 4 as well, such as the polytope of type $\{5, 6, 5\}$ that was mentioned at the end of Section 4.3.)

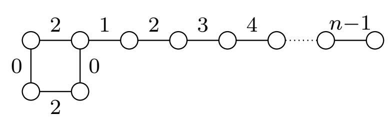
We will take advantage of the fact that Γ will self-dual by design, so the structure of each of its parabolic subgroups is the same as the "dual" subgroup. For any subset S of $\{0, 1, \ldots, n-1\}$, we define $\Gamma_S = \langle \rho_i \mid i \notin S \rangle$. The structure of Γ_S is determined by the structure of the given permutation representation graph after we delete all edges with labels in S.

To simplify the proof of the the theorem, we first provide some lemmas about other string C-groups.

Lemma 5.11. *Let $\Gamma = \langle \rho_0, ..., \rho_{n-1} \rangle$ be a string group generated by involutions. If $\Gamma_0 := \langle \rho_1, ..., \rho_{n-1} \rangle$ and $\Gamma_{n-1} := \langle \rho_0, ..., \rho_{n-2} \rangle$ are string C-groups, $\rho_{n-1} \notin \Gamma_{n-1}$, and $\Gamma_{0,n-1} := \langle \rho_1, ..., \rho_{n-2} \rangle$ is maximal in Γ_0 or Γ_{n-1}, then Γ is itself a string C-group.*

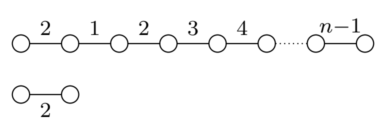
Proof. This is a restatement of [4, Lemma 2.2]. $\square$

Lemma 5.12. *For each $n \geq 4$, the following permutation representation graph is the CPR graph of a regular n-polytope with automorphism group Γ isomorphic to the symmetric group on $n+3$ points. Furthermore, Γ_0 is isomorphic to the direct product of a group of order two and a symmetric group on $n+1$ points.*



Proof. For $n \geq 6$ the first part of this claim is shown in [5, Prop. 4.8]. The remaining cases for the first part of the claim can either be checked by hand or using a computer algebra system.

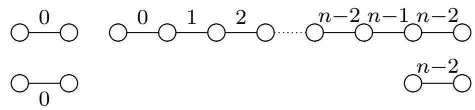
Consider the permutation representation graph for Γ_0 below.



It is easy to show that Γ_0 must either be isomorphic to S_{n+1} or $C_2 \times S_{n+1}$. To determine which of the two structures is correct, we need to understand if the transposition represented by the isolated edge labeled 2 is in the group. The element $(\rho_2\rho_1\rho_2\rho_3\rho_2)^5$ is equal to that transposition, and thus $\Gamma_0 \cong C_2 \times S_{n+1}$.

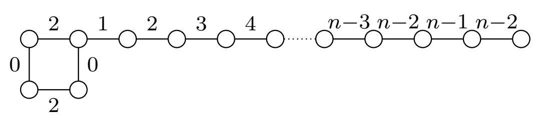
$\square$

Lemma 5.13. *For each $n \geq 4$, the following permutation representation graph is the CPR graph of a regular n-polytope with automorphism group Γ isomorphic to the direct product of the symmetric group on $n+2$ points with two groups of order 2.*



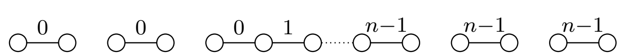
Proof. Let $\Lambda = \langle \rho_0, \ldots, \rho_n \rangle$ be the automorphism group of the dual of an $(n+1)$-polytope obtained from Lemma 5.12, and let Γ be a group obtained from the permutation representation above. Then, Γ is a string C-group as it is isomorphic to $\Lambda_n \diamond e$. The action of Γ on the larger orbit can easily be shown to be the symmetric group S_{n+2}. Furthermore, $(\rho_{n-2}\rho_{n-1}\rho_{n-2}\rho_{n-3}\rho_{n-2})^5$ yields the transposition represented by the isolated edge labeled $n-2$, and $(\rho_0\rho_1)^3$ yields the involution represented by the pair of isolated edges labeled 0. Thus the group Γ is isomorphic to $S_{n+2} \times C_2 \times C_2$. $\square$

Lemma 5.14. *For each $n \geq 6$, the following permutation representation graph is the CPR graph of a regular n-polytope with automorphism group Γ isomorphic to the symmetric group on $n+4$ points. Furthermore, the element $(\rho_{n-2}\rho_{n-1}\rho_{n-2}\rho_{n-3}\rho_{n-2})$ is a five cycle with the last five points in its support.*



Proof. For $n \geq 7$ the fact that this is a CPR graph is shown in [5, Prop. 4.14]. The remaining case can either be checked by hand or using a computer algebra system. The structure of $(\rho_{n-2}\rho_{n-1}\rho_{n-2}\rho_{n-3}\rho_{n-2})$ can easily be checked by hand. $\square$

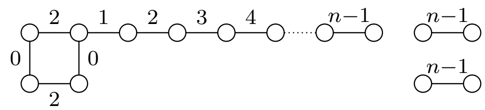
Lemma 5.15. *For each $n \geq 3$, the following permutation representation graph is the CPR graph of a regular n-polytope with automorphism group Γ isomorphic to the direct product of the symmetric group on $n+1$ points with two groups of order 2.*



Proof. This is the group obtained from an n-simplex by mixing with an edge, dualizing, and mixing with another edge and it is thus a string C-group. To show that the group is isomorphic to $S_{n+1} \times C_2 \times C_2$ we notice that $(\rho_0\rho_1)^3$ and $(\rho_{n-1}\rho_{n-2})^3$ yield the involutions represented by the isolated edges of the graph of labels 0 and $n-1$. Thus both involutions are in the group, along with the full symmetric action on the larger orbit. $\square$

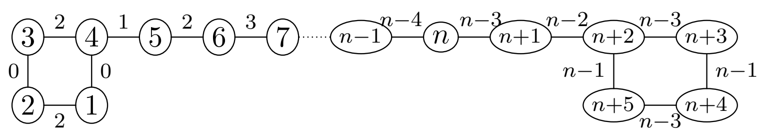
Lemma 5.16. *For each $n \geq 4$, the following permutation representation graph is the CPR graph of a regular n-polytope with automorphism group Γ*

isomorphic to the direct product of the symmetric group on $n+3$ points with a group of order 2.

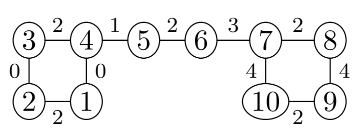


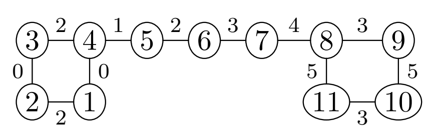
Proof. This group is the mix of a group from Lemma 5.12 with an edge, and is thus a string C-group. It can be shown to be isomorphic to $S_{n+3} \times C_2$ since $(\rho_{n-2}\rho_{n-1})^3$ is the involution represented by the isolated edges labeled $n-1$. □

Theorem 5.17. *For each $n \geq 5$, the following permutation representation graph is the CPR graph of an internally self-dual regular n-polytope. Furthermore, when $n \geq 6$ the polytope has automorphism group Γ isomorphic to the symmetric group acting on $n+5$ points.*



Proof. First we notice that these graphs have slightly different structure depending on whether n is odd or even; the two smallest cases of ranks 5 and 6 are seen below. Using a computer algebra system, these groups can be verified to be string C-groups. The $n = 5$ case yields a group of order 28800, and the $n = 6$ case yields the symmetric group on 11 points. In the first case, we can verify that conjugating $\rho_2\rho_1\rho_2\rho_1\rho_3\rho_2\rho_1\rho_3\rho_2\rho_1\rho_2$ by $\rho_4\rho_2\rho_3\rho_2\rho_0\rho_1\rho_0\rho_2\rho_3\rho_2\rho_4$ yields a dualizing automorphism of Γ. In the second case, the permutation $\pi_1 = \rho_4\rho_3\rho_2\rho_3\rho_4\rho_3\rho_2\rho_3\rho_4$ swaps nodes 5 and 7 while fixing everything else. Conjugating π_1 by $\rho_1\rho_4$ yields a permutation π_2 that swaps 4 and 8 while fixing everything else. By further conjugating π_2, we can find permutations that swap 3 with 9, 2 with 10, and 1 with 11, and then the product of these five permutations is a dualizing automorphism.





In the rest of the proof, we will assume that $n \geq 7$. By design, Γ is self-dual, since interchanging every edge label i with $n-1-i$ is a symmetry of the graph, corresponding to a reflection through a line that goes through the middle edge (if n is odd) or through the middle node (if n is even). Let us show that Γ is internally self-dual.

First, when n is odd and $n = 2k+1$, the permutation $\pi_1 = \rho_k$ interchanges the two nodes that are incident on the middle edge, while fixing everything else. Then, setting $\pi_2 = (\rho_{k-1}\rho_{k+1})\pi_1(\rho_{k+1}\rho_{k-1})$, we get that π_2 interchanges the next two nodes from the center, while fixing everything else. Continuing this way, we can find permutations $\pi_1, \ldots, \pi_{k+3}$ that each interchange a node with its dual while fixing everything else. The product of all of these will be a dualizing automorphism in Γ.

When n is even and $n = 2k$, the middle node is incident to edges labeled $k-1$ and k. The permutation $\pi_1 = \rho_{k-1}\rho_k\rho_{k-1}$ is easily seen to interchange the two nodes that are incident on the middle node, while fixing everything else. Then, setting $\pi_2 = (\rho_{k-2}\rho_{k+1})\pi_1(\rho_{k+1}\rho_{k-2})$, we get that π_2 interchanges the two nodes at a distance of 2 from the middle node, while fixing everything else. As in the odd case, we can continue to define permutations $\pi_1, \ldots, \pi_{k+2}$ that each interchange a node with its dual, and the product of all of these will be a dualizing automorphism in Γ.

Now we need to show that each Γ is a string C-group. It is clear that it is a string group generated by involutions, so we only need to show that it satisfies the intersection condition. We will do this by utilizing Lemma 5.11 and [10, Prop. 2E16] by showing that its facet group and vertex figure group are string C-groups, and that their intersection is what is needed.

First, let us show that $\Gamma_{0,1,n-1} \cap \Gamma_{0,n-2,n-1} \cong \Gamma_{0,1,n-2,n-1}$. Following Lemma 5.15, $\Gamma_{0,1,n-2,n-1} \cong S_{n-3} \times C_2 \times C_2$. Consider the action of $\varphi \in \Gamma_{0,1,n-1} \cap \Gamma_{0,n-2,n-1}$. Since $\varphi \in \Gamma_{0,1,n-1}$, we know that $\{1, 2\}\varphi = \{1, 2\}$ and $\{3, 4\}\varphi = \{3, 4\}$. Also, since $\varphi \in \Gamma_{0,1,n-1}$, either φ fixes all of $\{1, 2, 3, 4\}$ or it sends 1 to 2 and sends 3 to 4. Thus when restricted to the set $\{1, 2, 3, 4\}$, $\Gamma_{0,1,n-1} \cap \Gamma_{0,n-2,n-1}$ acts like a group of order two. Dually the same thing can be said about how $\Gamma_{0,1,n-1} \cap \Gamma_{0,n-2,n-1}$ acts on the set $\{n-3, n-2, n-1, n\}$. We conclude that $\Gamma_{0,1,n-1} \cap \Gamma_{0,n-2,n-1} \leq S_{n-3} \times C_2 \times C_2$. On the other hand, $S_{n-3} \times C_2 \times C_2 \cong \Gamma_{0,1,n-1,n-2} \leq \Gamma_{0,1,n-1} \cap \Gamma_{0,n-2,n-1}$. Thus, $\Gamma_{0,1,n-1} \cap \Gamma_{0,n-2,n-1} \cong \Gamma_{0,1,n-2,n-1}$. Since, by Lemma 5.13, both $\Gamma_{0,1,n-1}$ and $\Gamma_{0,n-2,n-1}$ are string C-groups isomorphic to $S_{n-1} \times C_2 \times C_2$, it follows from [10, Prop. 2E16], that $\Gamma_{0,n-1}$ is also a string C-group.

We can now show that Γ_0 (and thus Γ_{n-1}) is a string C-group. The group $\Gamma_{0,1}$ is a string C-group isomorphic to $S_{n+1} \times C_2$ as seen in Lemma 5.16, and the group $\Gamma_{0,n-1}$ was just shown to be a string C-group. Furthermore, since $\Gamma_{0,1,n-1}$ is isomorphic to $S_{n-1} \times C_2 \times C_2$, it follows from the O'Nan–Scott Theorem that $\Gamma_{0,1,n-1}$ is maximal in $\Gamma_{0,1}$. Then by Lemma 5.11, it follows that Γ_0 is a string C-group. By the self-duality of the construction, we see that Γ_{n-1} is also a string C-group.

To finally show that Γ itself is a string C-group we need to understand the structure of Γ_0, Γ_{n-1}, and $\Gamma_{0,n-1}$. We can show that $\Gamma_{0,n-1} \cong S_{n+1} \times C_2 \times C_2$, by proving that the transpositions $(1, 2)$ and $(n+5, n+4)$ are both in the group; the symmetric action on the remaining $(n+1)$ points follows from Proposition 5.9. The elements $(\rho_2\rho_1\rho_2\rho_3\rho_2)^5$ and $(\rho_{n-3}\rho_{n-2}\rho_{n-3}\rho_{n-4}\rho_{n-3})^5$ give these two transpositions and thus $\Gamma_{0,n-1} \cong S_{n+1} \times C_2 \times C_2$.

Following Lemma 5.14, Γ_0 is a symmetric group extended by a single transposition. Furthermore, since the element $(\rho_2\rho_1\rho_2\rho_3\rho_2)^5$ fixes all the nodes of the large connected component of the graph of Γ_0, and interchanges the nodes of the isolated edge labeled 2, it follows that this single transposition is in the group Γ_0. Thus $\Gamma_0 \cong S_{n+3} \times C_2$.

Lemma 5.11 then shows that Γ is a string C-group, since Γ_0 and Γ_{n-1} are string C-groups, and $\Gamma_{0,n-1}$ is maximal in Γ_0. Finally, note that $\Gamma_0 < \Gamma \leq S_{n+5}$, and that Γ_0 is a maximal subgroup of S_{n+5} (again by the O'Nan–Scott Theorem), and thus $\Gamma \cong S_{n+5}$.

□

6. Related problems and open questions

Some problems on the existence of internally self-dual polytopes remain open. Here are perhaps the most fundamental.

Problem 1. *For each odd $p \geq 5$, describe an infinite family of internally self-dual regular polyhedra of type $\{p, p\}$, or prove that there are only finitely many internally self-dual regular polyhedra of that type.*

Problem 2. *Determine the values of p and q such that there is a finite internally self-dual regular 4-polytope of type $\{p, q, p\}$.*

Problem 3. *Determine whether each self-dual $(n-2)$-polytope occurs as the medial section of an internally self-dual regular n-polytope.*

To our knowledge, these problems are open even if we consider all self-dual regular polytopes, rather than just the internally self-dual ones. Problem 3, for general self-dual polytopes, was posed as a positive conjecture by Schulte in Section 9 of [13].

To what extent does the theory we have developed apply to transformations other than duality? For example, the *Petrie dual* of a polyhedron $\mathcal{P}$, denoted $\mathcal{P}^\pi$, is obtained from $\mathcal{P}$ by interchanging the roles of its facets and its Petrie polygons (see [10, Sec. 7B]). If $\mathcal{P}^\pi \cong \mathcal{P}$, then we say that $\mathcal{P}$ is *self-Petrie*. If $\mathcal{P}$ is regular, with $\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2 \rangle$, then $\mathcal{P}$ is self-Petrie if and only if there is a group automorphism of $\Gamma(\mathcal{P})$ that sends ρ_0 to $\rho_0\rho_2$, while fixing ρ_1 and ρ_2. We can then say that $\mathcal{P}$ is *internally self-Petrie* if this automorphism is inner.

Working with internally self-Petrie polyhedra is not substantially different from working with internally self-dual polyhedra, due to the following result.

Proposition 6.1. *A regular polyhedron $\mathcal{P}$ is internally self-Petrie if and only if $(\mathcal{P}^*)^\pi$ is internally self-dual.*

Proof. Suppose that $\mathcal{P}$ is (internally or externally) self-Petrie, and let the group $\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2 \rangle$. Then there is some $\pi \in \mathrm{Aut}(\Gamma(\mathcal{P}))$ such that $\rho_0\pi = \rho_0\rho_2$, $\rho_1\pi = \rho_1$, and $\rho_2\pi = \rho_2$. Now, $\Gamma(\mathcal{P}^*) = \langle \rho_2, \rho_1, \rho_0 \rangle$, and then

$$\Gamma((\mathcal{P}^*)^\pi) = \langle \rho_0\rho_2, \rho_1, \rho_0 \rangle =: \langle \lambda_0, \lambda_1, \lambda_2 \rangle.$$

Then it is easy to show that $\lambda_i\pi = \lambda_{2-i}$ for $i = 0, 1, 2$. It follows that $(\mathcal{P}^*)^\pi$ is self-dual. Furthermore, since $\Gamma(\mathcal{P}) \cong \Gamma((\mathcal{P}^*)^\pi)$ as abstract groups and π induces the self-Petriality of the former and the self-duality of the latter, it follows that π is either inner for both groups or outer for both. The result then follows. $\square$

Perhaps it would be possible to discuss internal versus external invariance under other polytope transformations, such as those studied in [9].

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