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AFFINELY REGULAR POLYGONS IN AN AFFINE PLANE

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Dedicated to the centenary of the birth of Ferenc Kárteszi (1907–1989).

Abstract. In this paper we survey results about affinely regular polygons. First, the definitions and classification of affinely regular polygons are given. Then the theory of Bachmann–Schmidt is outlined. There are several classical theorems about regular polygons, some of them having analogues in finite planes, such as the Napoleon–Barlotti theorem. Such analogues, variants of classical theorems are also collected. Affinely regular polygons occur in many combinatorial problems for sets in a finite plane. Some of these results about sharply focused arcs, internal and external nuclei, complete arcs are collected. Finally, bounds on the number of chords of an affinely regular polygon through a point are discussed.

1. Introduction

Regular polygons of the Euclidean plane have a long and interesting history including the birth of the Galois theory, see [22] and [26]. In the late sixties, Bachmann and Schmidt [7] considered regular polygons in an affine plane $\mathrm{AG}(2,\mathbb{F})$ coordinatized over a field $\mathbb{F}$. Their algebraic point of view was originated by a result in the Euclidean plane, namely if A is a point other than the centre of a rotation ρ of order n and $A_i = \rho^i(A)$, then $A_1A_2\cdots A_n$ is a regular n-gon. This was a far-reaching generalization as the order of every finite multiplicative subgroup of norm 1 elements of the quadratic extension of $\mathbb{F}$ is the order of a rotation group of $\mathrm{AG}(2,\mathbb{F})$. Actually, some more might have been done developing an idea of Coxeter, see [22, 3.41]: in the Euclidean plane, if a congruent transformation ρ has finite order n, the transforms under ρ of a point A of general position, namely the points $A_i = \rho^i(A)$ with $i = 1,\ldots,n$, may be regarded as the vertices of a "generalized" regular polygon. In the classical case, ρ is a rotation, A may be any point other than the centre of ρ, and $A_1A_2\cdots A_n$ is a regular, possibly star, n-gon.

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In 1970 Kárteszi introduced a more geometric idea that works in any affine plane, see [39].

Let π be an affine plane defined by the three Hilbert incidence axioms. The usual terminology on point-line incidences extends from the Euclidean plane to π. An *n-gon* in an affine plane π is a set of n pairwise distinct points arranged in a cyclic order. An n-gon is *non–degenerate* if it has no three collinear vertices. Now, let $A_1A_2\cdots A_n$ be a regular n-gon in the Euclidean plane. Kárteszi called a (non-degenerate) n-gon $B_1B_2\cdots B_n$ in an affine plane *affinely regular* if the bijection $A_i \mapsto B_i$ preserves all parallelisms between chords (i.e. sides, and diagonals), that is,

$$A_iA_j \parallel A_kA_m \Longleftrightarrow B_iB_j \parallel B_kB_m\,,$$

for all $1 \le i < j \le n$, and $1 \le k < m \le n$.

Later, Kárteszi's idea was developed by his Ph.D. students Nguyen Mong Hy [36] and G. Korchmáros [47, 48, 49, 50]. Affinely regular n-gons have also played an important role in finite geometry. Our aim is to survey the results in this direction.

2. Classification of affinely regular polygons in $\mathrm{AG}(2,\mathbb{F})$.

We start by describing some examples of affinely regular polygons. Let $\mathcal{C}$ be a non-degenerate conic in $\mathrm{AG}(2,\mathbb{F})$ and choose the affine coordinate system in $\mathrm{AG}(2,\mathbb{F})$ in such a way that $\mathcal{C}$ is in canonical form.

If $\mathcal{C}$ is the hyperbola of equation $XY = 1$ and G is a finite multiplicative subgroup of $\mathbb{F}$ of order n, choose a non-zero element $a \in \mathbb{F}$ and a generator g of G. For $i = 1,\dots,n$, let $A_i = (g^ia, (g^ia)^{-1})$. Then $A_1A_2\cdots A_n$ is an affinely regular n-gon inscribed in $\mathcal{C}$.

If $\mathcal{C}$ is the parabola of equation $Y = X^2$, $p > 0$ is the characteristic of $\mathbb{F}$ and $a \in \mathbb{F}$, let $A_i = (a+i, (a+i)^2)$ for $i = 1,\dots,p$. Then $A_1A_2\cdots A_p$ is an affinely regular p-gon inscribed in $\mathcal{C}$.

If $\mathcal{C}$ is the ellipse of equation $X^2 + Y^2 - sXY = 1$ where the polynomial $x^2 - sx + 1$ is irreducible over $\mathbb{F}$, a parametrization of the points of $\mathcal{C}$ is possible by certain elements of the quadratic extension $\mathbb{K} = \mathbb{F}(\theta)$ of $\mathbb{F}$ where $\theta^2 - s\theta + 1 = 0$. To do this, $\mathrm{AG}(2,\mathbb{F})$ is viewed as the Gauss-Argand plane, the points of $\mathrm{AG}(2,\mathbb{F})$ being identified by the elements of $\mathbb{K}$. If $z = x + \theta y \in \mathbb{K}$, then its conjugate $\bar{z}$ is $x + \theta^{-1}y$, and hence the points of $\mathcal{C}$ are those elements in $\mathbb{K}$ which satisfy the equation $z\bar{z} = 1$. Now, choose any non-zero element $z \in \mathbb{K}$ and a generator g of a finite multiplicative subgroup G of $\mathbb{K}$ of order n. For $i = 1,\dots,n$, let A_i be the point with parameter g^iz. Then $A_1A_2\cdots A_n$ is an affinely regular n-gon inscribed in $\mathcal{C}$. A concrete example in the finite plane $\mathrm{AG}(2,9)$ is worked out in [36].

All affinely regular polygons in $\mathrm{AG}(2,\mathbb{F})$ were classified, see [50, Teorema 2],[49].

Theorem 2.1. *In an affine plane* $\mathrm{AG}(2, \mathbb{F})$ *coordinatized by a field* $\mathbb{F}$*, the affinely regular polygons are exactly the above examples and their images under the affinities of* $\mathrm{AG}(2, \mathbb{F})$*.*

A key step in the proof is the following lemma due to Korchmáros [47] and independently to Van de Craats and Simonis [81].

Lemma 2.2. *In an affine plane* $\mathrm{AG}(2, \mathbb{F})$ *coordinatized by a field* $\mathbb{F}$*, every affinely regular polygon is inscribed in an irreducible conic.*

In the special case when $\mathbb{F}$ is the finite field $GF(q)$ of order $q = p^a$, the following result holds, see [47].

Theorem 2.3. *In the finite affine plane* $\mathrm{AG}(2, q)$ *coordinatized by* $GF(q)$*, an affinely regular* n*-gon exists if and only if either* $n \mid (q+1)$ *and the* n*-gon is inscribed in an ellipse, or* $n \mid (q-1)$ *and the* n*-gon is inscribed in a hyperbola, or* $n = p$ *and the* n*-gon is inscribed in a parabola.*

Affinely regular polygons in $\mathrm{AG}(2, \mathbb{F})$ may also be defined using variations of Coxeter's approach. Fisher and Jamison [25] presented seven apparently different but equivalent definitions. Their paper also deals with many related properties and gives an interesting historical account.

3. A summary of the theory of n-gons of Bachmann and Schmidt

Let $\mathbf{V}_2$ be the two-dimensional vector space over $\mathbb{F}$. For any positive integer n, Bachmann and Schmidt called an n-gon any ordered n-ple ($\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_{n-1}$) of vectors from $\mathbf{V}_2$. With the usual addition and scalar multiplication, such n-gons constitute a vector space, the *space of* n*-gons* $\mathcal{E}_n$ with the Bachmann–Schmidt terminology. The natural map from $\mathcal{E}_n$ to the set of all n-gons (including the degenerate ones and those with multiple vertices) of the affine plane arising from $\mathbf{V}_2$ is surjective but injective. In fact, two distinct n-gons in $\mathcal{E}_n$ have the same image under this map if and only if their coordinate vectors are the same but in a different order. A theory of n-gons from this point of view was thoroughly developed in the book [7]. Here, we limit ourselves to affinely regular polygons.

To introduce the concept of affinely regular n-gon, some elementary results on cyclotomic polynomials are needed. Let $d, n \geq 3$ be positive integers prime to the characteristic of $\mathbb{F}$. The d-th cyclotomic polynomial

$$\varphi_d(x) = \prod_w (x - w)$$

with w ranging over all primitive d-th roots of unity has only integer coefficients, and may be viewed as an element of $\mathbb{F}[X]$. Then

$$x^n - 1 = \prod_{d \mid n} \varphi_d(x)$$

holds in $\mathbb{F}[X]$. For a primitive d-th root w of unity (lying in $\mathbb{F}$ or a finite extension of $\mathbb{F}$), the polynomial $x^2-(w^k+w^l)x+w^{k+l}$ is symmetric if and only if $k+l\equiv 0 \pmod n$. From this, $\varphi_d(x)$ splits into the product of its symmetric divisors of degree 2. For a fixed primitive d-th root w of unity, set $c_k=w^k+w^{n-k}$ with $k=1,\dots,[(n-1)/2]$. Then

$$\varphi_d(x)=\prod(x^2-c_kx+1) \tag{3.1}$$

where k ranges over the multiples of n/d which are prime to d but do not exceed n. In particular,

$$\varphi_n(x)=\prod_{\substack{k<n/2,\\(k,n)=1}}(x^2-c_kx+1).$$

This leads to the following factorization of x^n-1 over $\mathbb{F}$ (or a finite extension of $\mathbb{F}$):

$$x^n-1=\begin{cases}(x-1)\prod_{k=1}^{(n-1)/2}(x^2-c_kx+1) & \text{for } n \text{ odd,}\\ (x-1)(x+1)\prod_{k=1}^{(n-2)/2}(x^2-c_kx+1) & \text{for } n \text{ even.}\end{cases} \tag{3.2}$$

Now assume that (3.1) holds in $\mathbb{F}[X]$ that is $c_k\in F$, and define, for every c_k, the circular system of equations

$$\mathbf{v}_0+c_k\mathbf{v}_1+\mathbf{v}_2=\mathbf{0},\quad \dots,\quad \mathbf{v}_{n-1}+c_k\mathbf{v}_0+\mathbf{v}_1=\mathbf{0} \tag{3.3}$$

Following Bachmann and Schmidt, see [7, Chapter 12.3], an affinely regular n-gon is defined to be a non-trivial solution $(\mathbf{v}_0,\dots,\mathbf{v}_{n-1})$ of (3.3). For $c\in\mathbb{F}$, the same authors proved that the circular system of equations

$$\mathbf{v}_0+c\,\mathbf{v}_1+\mathbf{v}_2=\mathbf{0},\quad \dots,\quad \mathbf{v}_{n-1}+c\,\mathbf{v}_0+\mathbf{v}_1=\mathbf{0} \tag{3.4}$$

has a non-trivial solution if and only if either $c=\pm2$ or $(x^2-cx+1)\mid(x^n-1)$. They also discussed these possibilities showing that $c=2$ only occurs when the vertices (i.e. the vectors $\mathbf{v}_0,\dots,\mathbf{v}_{n-1}$) all coincide and that this holds true for $c=-2$ as long as n is odd. Moreover, for n even, $c=-2$ if and only the n-gon has either one or two distinct vertices, as its vertices with even index (and those with odd index) coincide. Going on in their discussion, they thoroughly investigated the case $(x^2-c_kx+1)\mid(x^n-1)$. From (3.2), x^2-cx+1 divides φ_d for a divisor d of n. In particular, $d\le n$ and $c=c_k$ for a fixed primitive d-th root of unity, w. For $d<n$, every solution of (3.4) (or, equivalently, (3.3)) has the property that

$$\mathbf{v}_0=\mathbf{v}_d,\quad \mathbf{v}_1=\mathbf{v}_{d+1},\quad \mathbf{v}_2=\mathbf{v}_{d+2},\quad \dots.$$

Furthermore,

$$\mathbf{v}_0+c_k\mathbf{v}_1+\mathbf{v}_2=\mathbf{0},\quad \dots,\quad \mathbf{v}_{d-1}+c_k\mathbf{v}_0+\mathbf{v}_1=\mathbf{0} \tag{3.5}$$

which shows that $(\mathbf{v}_0,\mathbf{v}_1,\dots,\mathbf{v}_{d-1})$ is an affinely regular d-gon. The n-gon $(\mathbf{v}_0,\dots,\mathbf{v}_{n-1})$ can be viewed as an affinely regular d-gon whose vertices are n/d-times covered. With the terminology of Bachmann and Schmidt, $(\mathbf{v}_0,\dots,\mathbf{v}_{n-1})$ is an n/d-folded affinely regular n-gon.

It is worth mentioning that the affinely regular n-gons arising from (3.5) for a given c form a subspace $\mathcal{C}(c)$ in the space of n-gons whose dimension equals to 2 with the only exception for $c = \pm 2$ in which case the dimension is 1. Also, any two such subspaces intersect trivially. A main result in the Bachmann–Schmidt theory of n-gons, see [7, Satz 2, Chapter 12.3], consists in proving that the space of n-gons is the direct sum of these subspaces, that is,

$$\mathcal{E}_n = \begin{cases} \mathcal{C}(2) \bigoplus \mathcal{C}(-2) \bigoplus \cdots \bigoplus \mathcal{C}(c_{(n-2)/2}) & \text{for } n \text{ even,} \\ \mathcal{C}(2) \bigoplus \cdots \bigoplus \mathcal{C}(c_{(n-1)/2}) & \text{for } n \text{ odd.} \end{cases} \tag{3.6}$$

An interpretation of (3.6) in terms of sums of cyclic matrices is given in [48].

Now, look at the Euclidean plane, $\mathbb{F}$ being the real field,

$$c_k = 2\cos k\,\frac{2\pi}{n} = e^{ik(2\pi/n)} + e^{-ik(2\pi/n)} \quad (k = 0, \ldots, [n/2];\ i^2 = -1). \tag{3.7}$$

In particular, $c_0 = 2$ and, for n even, $c_{n/2} = 1$. Take a regular n-gon $R_0 \cdots R_{n-1}$ with centre at the origin. For $i = 0, \ldots, n-1$, let $\mathbf{v}_i$ be the vector represented by the oriented segment $\overrightarrow{OR_i}$. Then $(\mathbf{v}_0, \ldots, \mathbf{v}_{n-1})$ is a solution of (3.3) for $k = 1$ showing that the n-gon $(\mathbf{v}_0, \ldots, \mathbf{v}_{n-1})$ is an affinely regular n-gon with the Bachmann–Schmidt terminology. More generally for every k with c_k as in (3.7), $R_0 \cdots R_{n-1}$ gives rise to solutions $(\mathbf{v}_0, \ldots, \mathbf{v}_{n-1})$ of (3.3) given by

$$\mathbf{v}_0^{(k)} = OR_0, \quad \mathbf{v}_1^{(k)} = OR_k, \quad \ldots, \quad \mathbf{v}_{n-1}^{(k)} = OR_{(n-1)k}$$

where the indices are taken modulo n. If k is prime to n, the vertices

$$R_0,\ R_k,\ R_{2k},\ \ldots,\ R_{(n-1)k}$$

are pairwise distinct, and $R_0R_k \cdots R_{(n-1)k}$ is a regular n-gon, in particular a star regular n-gon for $k > 1$. Otherwise, $d = n/\gcd(n,k)$ is a proper divisor of n,

$$R_0 = R_{dk},\ R_d = R_{(d+1)k},\ \ldots,$$

and $R_0R_k \cdots R_{(d-1)k}$ is a regular (in general span) d-gon, such that

$$(\mathbf{v}_0,\ \mathbf{v}_k,\ \ldots,\ \mathbf{v}_{(d-1)k},\ \ldots,\ \mathbf{v}_{(n-1)k})$$

is an n/d-covered regular n-gon with the Bachmann–Schmidt terminology.

The above argument still works if the regular n-gon $R_0R_1 \cdots R_{n-1}$ is replaced by its image under any affinity. The main result of Bachmann and Schmidt stated that the above procedure provide all solutions of (3.5) in the Euclidean plane.

The question whether or not the two definitions of affinely regular polygons, namely Kárteszi's synthetic one and Bachmann–Schmidt's algebraic one, coincide was completely solved in [49, 50].

An n-gon which is affinely regular under the algebraic definition is inscribed either in hyperbola or an ellipse and it is also an affinely regular under the synthetic definition. Conversely, any affinely regular n-gon under the synthetic definition which is inscribed either in a hyperbola or an ellipse is also an affinely regular n-gon under the algebraic definition.

Therefore, in characteristic zero, two or three, the algebraic and synthetic definitions are equivalent. This does not hold true in characteristic $p \geq 5$, as there exist affinely regular p-gons which are inscribed in a parabola. Nevertheless, some circular system of equations can still be associated to the exceptional cases. To do this, take a non-degenerate p-gon $R_0 \cdots R_{p-1}$. If O is the origin and $\mathbf{v}_i$ is the vector represented by the oriented segment $\overrightarrow{OR_i}$, then the circular system of equations

$$\mathbf{v}_0 - 3\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}, \quad \ldots, \quad \mathbf{v}_{n-1} - 3\mathbf{v}_0 + 3\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}, \tag{3.8}$$

is satisfied if and only if $R_0 \cdots R_{p-1}$ is an affinely regular p-gon under the synthetic definition. Furthermore, any two non-degenerate affinely regular p-gons with $p \geq 5$ are equivalent under an affinity. This feather contrasts with the fact that the number of classes of pairwise non-degenerate affinely equivalent n-gons for a given $n \neq p$ is as many as the integers d prime to n in the range $1 < d \leq [(n-1)/2]$. According to (3.6), this number becomes $[(n+2)/2]$ when the algebraic definition of affinely regular n-gons is considered.

The formula (3.6) happens to be a very useful tool in dealing with problems from discrete geometry. In this spirit, the interested reader may give a proof of the following theorem originally conjectured by László Fejes Tóth [79] for the special case $\lambda_0 = \lambda_1 = \lambda_2 = \cdots = 1/2$ and $\lambda = 1/2$. For $n = 4$ the conjecture was shown by Fejes Tóth himself [79], and for $n = 5$ by Kárteszi [38].

Let $\lambda_0, \lambda_1, \lambda_2, \ldots$ be a sequence of numbers $(0 < \lambda_i < 1)$, and let P^0, P^1, $P^2, \ldots$ be a sequence of convex n-gons in the Euclidean plane such that the vertices of P^{i+1} divide the edges of P^i cyclically in the ratio $\lambda_i\colon 1$. Assume that $\sum \lambda_i(1-\lambda_i)$ diverges. Then there is a sequence of affine transformations A_i such that the transformed n-gons $Q^i = \alpha_i(P^i)$ converge to a regular n-gon. Also, if the numerical sequence alternates between λ and $1-\lambda$ then there is a sequence of dilatations δ_i such that the two sequences $E^i = \delta_i(P^i)$ (i even) and $O^i = \delta(P^i)$ (i odd) each converge to a regular n-gon.

This theorem extends to any (non-convex or degenerate) n-gons in the Euclidean plane, but the limit n-gons happen to be star, degenerate or n/d–folded regular n-gons, as it appears from the theory of Bachmann-Schmidt. Results around the above Fejes-Tóth conjecture are also found in [16, 46, 57, 64, 65].

Ratios among the diagonals of an affinely regular polygon $\mathrm{AG}(2,\mathbb{F})$ inscribed in an ellipse or a hyperbola can be investigated as in the Euclidean plane. Fisher and Jamison [25] pointed out that such ratios are provided by the Chebysev polynomials of the second kind.

4. Affinely regular and shape-regular polygons

In this section we collect some classical theorems concerning regular polygons which have an analogue in finite geometry. There are quite a few such theorems, we picked those which are related to the work of Kárteszi and his students. Most of this section is based on the papers [43], [44], [4].

Let $P_0P_1\cdots P_{n-1}$ be a regular n-gon in the classical Euclidean plane. Let us consider the k-th diagonals, that is the lines P_0P_k, $P_1P_{k+1}, \dots, P_{n-1}P_{k-1}$. These diagonals form the sides of a new n-gon called the k-th n-gon of the original one. The following theorem was proved in the classical case by Kárteszi, its analogue in a finite plane by Kárteszi and Kiss, see [41], [42], [43].

Theorem 4.1. *Let $P_0P_1\cdots P_{n-1}$ be an affinely regular n-gon in $\mathrm{AG}(2,\mathbb{F})$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathrm{GF}(q)$, q odd. Then the ℓ-th diagonals of the m-th n-gon and the m-th diagonals of the ℓ-th n-gon are the same sets of lines for any $\ell, m \in \{2,3,\dots,\lfloor\frac{n-2}{2}\rfloor\}$*

For some interesting theorems the so-called shape-regularity is needed which looks more a metric type regularity than affine regularity. The notion of shape in the Euclidean plane coordinatized by complex numbers was introduced by Lester [54] and extended by Artzy [3]. Artzy and Kiss [4] extended the notion further to affine Galois planes of odd order. As above let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathrm{GF}(q)$, q odd. Let $\mathbb{K}$ denote the quadratic extension of $\mathbb{F}$, that is either $\mathbb{C}$ or $\mathrm{GF}(q^2)$. It is well-known that the affine plane $\mathrm{AG}(2,\mathbb{F})$ can be represented by $\mathbb{K}$ (this was called Gauss–Argand plane in Section 2).

Definition 4.2. *Let $u, v, w \in \mathbb{K}$ be the vertices of an ordered triangle called (uvw). The* shape *of this triangle is defined as*

$$S(uvw) = (u-w)(u-v)^{-1} \in \mathbb{K}.$$

Definition 4.3. *Let $u_1, u_2, \dots, u_n$ be distinct points of $\mathrm{AG}(2,\mathbb{F}) = \mathbb{K}$. The ordered n-gon $(u_1u_2\cdots u_n)$ is called* shape-regular *with shape $s \in \mathbb{K}$ if $S(u_{k+1}u_ku_{k+2}) = s \in \mathbb{K}$ for all integers k, where k and the addition is considered modulo n.*

The paper of Artzy and Kiss [4] studies shape-regular n-gons for $\mathbb{K} = \mathrm{GF}(q)$, q odd. They show that there are no shape-regular triangles for $q \equiv 0, 1 \pmod 3$, while there are two types for $q \equiv 1 \pmod 3$. The main result about the existence of shape-regular n-gons is the following.

Theorem 4.4. *There exists a shape-regular n-gon with shape s if and only if $s \notin \mathbb{F}$, $(-s)^n = 1$ and $(-s)^m \neq 1$ for $1 \leq m < n$.*

This means that $\mathrm{AG}(2,q)$ contains a shape-regular n-gon precisely when n divides q^2-1 but does not divide $q-1$. If $n = m^k$ with m being an odd prime then $\mathrm{AG}(2,q)$ contains a shape regular m^k-gon if and only if m^k divides $q+1$. Artzy and Kiss also investigated the relationship between affine and shape regularity and found the following criterion.

Theorem 4.5. *A shape-regular n-gon with shape $s \in \mathrm{GF}(q^2)$ is affinely regular if and only if $s + 1/s \in \mathrm{GF}(q)$.*

As a corollary this gives that shape-regular m^k-gons of $\mathrm{AG}(2,q)$ are affinely regular. The previous results show that not every affinely regular polygon is shape-regular. There are also examples showing that in general it is not true that a shape-regular polygon is affinely regular.

There is a theorem, attributed to Napoleon, about triangles in the Euclidean plane. As before, the affine plane $\mathrm{AG}(2,\mathbb{F})$ will be identified with $\mathbb{K}$, where $\mathbb{F} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

Theorem 4.6. *Let $\mathcal{A} = A_1A_2A_3$ be a triangle and $\mathcal{B} = B_1B_2B_3$ be the triangle whose vertices are the centers of the equilateral triangles all erected externally (or internally) on the sides of $\mathcal{A}$. Then $\mathcal{B}$ is an equilateral triangle.*

The theorem was generalized to polygons by Barlotti [9], still in the Euclidean case.

Theorem 4.7. *Let $\mathcal{A} = A_0A_1 \cdots A_{n-1}$ be an n-gon and $\mathcal{B} = B_0B_1 \cdots B_{n-1}$ be the n-gon whose vertices are the centers of regular n-gons all erected externally (or all internally) on the sides of $\mathcal{A}$. Then $\mathcal{B}$ is regular if and only if $\mathcal{A}$ is affinely regular.*

Another generalization of Napoleon's theorem is due to Neumann [62].

Theorem 4.8. *Let $\mathcal{B}^{(0)} = A_0A_1 \cdots A_{n-1}$ be an n-gon in the Euclidean plane, and let σ be any permutation of the set $\{0, 1, \ldots, n-1\}$. The n-gons $\mathcal{B}^{(j)}$ are defined by induction on j. Let $\mathcal{B}^{(j)}$ be the n-gon whose vertices are the centers of the regular n-gons all erected externally on the sides of $\mathcal{B}^{(j-1)}$ such that the sides of $\mathcal{B}^{(j-1)}$ are the $\sigma(j)$-th diagonals of $\mathcal{B}^{(j)}$. Then $\mathcal{B}^{(n-1)}$ is a point.*

Ruoff and Shilleto [66] proved a finite analogue of Barlotti's theorem for a class of finite planes. Here $\mathbb{F} = \mathrm{GF}(q)$, $\mathbb{K} = \mathrm{GF}(q^2)$.

Theorem 4.9. *Let n be a positive integer and s a primitive n-th root of unity such that s belongs to the quadratic extension $\mathrm{GF}(q^2)$ of the finite field $\mathrm{GF}(q)$, where $q \equiv 3 \pmod 4$. Then the centers of the shape-regular n-gons with shape s erected on the sides of a given n-gon $\mathcal{A}$ form a shape-regular n-gon with shape s if and only if $\mathcal{A}$ is the image of a shape-regular n-gon with shape s under an affinity.*

These results were generalized by Kiss [44] using the theory of recursive polygons, see the previous section and [27], [7]. In this theory, the following notation is used. Let $\mathcal{A} = A_0A_1 \cdots A_{n-1}$ be any n-gon. Multiplication by x^k will denote a cyclic shift by k places to the left, that is, $x^k(A_0A_1 \cdots A_{n-1}) = (A_kA_{k+1} \cdots A_{k-1})$. $\mathcal{O}$ will denote $(0, \ldots, 0)$ and equations of the form $p(x)\mathcal{A} = \mathcal{O}$, $p(x) = x^d - c_{d-1}x^{d-1} - \cdots - c_0$, will abbreviate

$$a_{i+d} = c_{d-1}a_{i+d-1} + c_{d-2}a_{i+d-2} + \cdots + c_0a_i, \tag{4.1}$$

where the indices are taken modulo n. The following lemma comes from [27].

Lemma 4.10. (Fisher, Ruoff, Shilleto, [27, 66])

(1) *For any n-gon $\mathcal{A}$ we have $(x^n - 1)\mathcal{A} = \mathcal{O}$.*
(2) *If $(x-1)\mathcal{A} = \mathcal{O}$, then $\mathcal{A}$ is a single vertex.*
(3) *If $(x-1)p(x)\mathcal{A} = \mathcal{O}$, then $\mathcal{A}$ has a translate satisfying $p(x)\mathcal{A} = \mathcal{O}$.*
(4) *$\mathcal{A}$ is shape-regular with shape s if $(x-1)(x-s)\mathcal{A} = \mathcal{O}$.*

First the centre of a shape-regular polygon is defined. The point $c \in \mathbb{K}$ is called the *centre* of $(u_1u_2\cdots u_n)$ if $S(cu_ku_{k+1}) = S(cu_{k+1}u_{k+2})$ for every $k \pmod n$. Then it is shown that any shape-regular polygon has a unique centre. The generalization of Neumann's theorem is the following.

Theorem 4.11. (Kiss [44]) *Let $\mathcal{B}^{(0)} = A_0A_1\cdots A_{n-1}$ be an n-gon and σ be any permutation of $\{0, 1, \dots, n-1\}$. The n-gons $\mathcal{B}^{(j)}$ are defined by induction on j. Let $\mathcal{B}^{(j)}$ be the n-gon whose vertices are the centers of the regular n-gons all erected on the sides of $\mathcal{B}^{(j-1)}$ such that the sides of $\mathcal{B}^{(j-1)}$ are the $\sigma(j)$-th diagonals of $\mathcal{B}^{(j)}$. Then $\mathcal{B}^{(n-1)}$ is a point.*

In the proof, an important step is to find a relation between an n-gon $\mathcal{A}$ and the n-gon $\mathcal{B}^{(j)}$ whose vertices are the centers of shape-regular n-gons with shape s all erected such that the sides of $\mathcal{A}$ are the j-th diagonals of $\mathcal{B}^{(j)}$. This is

$$(-1 + (-s)^j)\mathcal{B}^{(j)} = ((-s)^j x - 1)\mathcal{A}.$$

Barlotti's theorem can also be generalized.

Theorem 4.12. (Kiss [44]) *Let $\mathcal{A} = A_0A_1\cdots A_{n-1}$ be an n-gon and $\mathcal{B} = B_0B_1\cdots B_{n-1}$ be the n-gon whose vertices are the centers of shape-regular n-gons with shape s all erected on the sides of $\mathcal{A}$. Then $\mathcal{B}$ is shape-regular with shape $-t$ if and only if $\mathcal{A}$ satisfies the equation*

$$(x-1)(sx+1)(x+t)\mathcal{A} = \mathcal{O}.$$

Hence $a_j \in \mathbb{K}$, the elements corresponding to the points A_j satisfy

$$a_j = \frac{d}{e-f}((k-f)e^j - (k-e)f^j) + g,$$

where $0 \neq d, g \in \mathbb{F}$, $e, k, f \in \mathbb{K}$. $\mathcal{A}$ is affinely regular in two particular cases:

(1) *$\mathcal{A}$ is shape-regular and n divides $q+1$,*
(2) *$s = t$.*

Napoleon's theorem and its generalizations are very attractive classical results, so there are several papers discussing it even in the Euclidean plane. Some generalize it, some papers give new, elementary proofs. One such paper is Szabó [74] which discusses two other elementary theorems for affinely regular polygons in the Euclidean plane, one is the propeller theorem by Bankoff, Erdős and Klamkin [8]. There are several variants of Napoleon's theorem, generalizations to higher dimensions, other spaces, see Martini and Spirova [59], Hajja, Martini, Spirova [32], Martini and Weissbach [60], Spirova [72], Martini [58]. In the last paper the reader can find hundreds of references to Napoleon's theorem and its many generalizations and extensions.

5. Finite geometry and affinely regular polygons

In this section we collect combinatorial problems for sets in a finite plane that are related to affinely regular polygons or their generalizations. Throughout this section $\mathbb{F}$ is the finite field $\mathrm{GF}(q)$ of order $q = p^a$, p prime.

Let us begin with the basic definitions. A *k-arc* in a projective plane of order q is a set of k points no three of which are collinear. A k-arc is *complete* if it is not contained in a $(k+1)$-arc, that is when it is maximal subject to inclusion. A line meeting a k-arc in 2 points is called a *chord* (or secant), a line meeting it in precisely 1 point is called a *tangent*, and the remaining lines are called *exterior lines* or passants. By a classical theorem of Bose (see [33]), the maximum number of points a k-arc can have is $q+1$ for odd, and $q+2$ for even q. $(q+1)$-arcs are called *ovals*, $(q+2)$-arcs are called *hyperovals*.

Motivated by secret sharing schemes Simmons [70, 71] has presented the following construction: Suppose that the secret is a point X on a given line s in $\mathrm{PG}(3, q)$. There are participants on two levels, they receive certain pieces of information (called *shadows*). The shadows of the participants on the top level is a subset of points $\{P_1, \dots, P_m\}$ of a line ℓ which intersects s in X, and the set of shadows for the participants on the lower level is a subset $\mathcal{S}$ of points of a plane π which intersects s in X and contains ℓ. The secret can be reconstructed by any two shadows of $\mathcal{I} = \{X, P_1, \dots, P_m\}$ and any three shadows in $\mathcal{S} \cup \mathcal{I}$ but two shadows from $\mathcal{S}$ are not enough. Simmons showed that $\mathcal{S}$ must be an arc and the chords of the arc $\mathcal{S}$ cannot contain X or a point P_i. This leads to the following definition.

Definition 5.1. *Let $\mathcal{S}$ be a k-arc and ℓ be an exterior line. $\mathcal{S}$ is called* sharply focused *on ℓ if the chords of $\mathcal{S}$ cover exactly k points of ℓ. $\mathcal{S}$ is called* very sharply focused *on ℓ if the chords of $\mathcal{S}$ cover exactly $k-1$ points of ℓ.*

In this section the following generalization of affinely regular polygon occurs frequently.

Definition 5.2. *Let $\mathcal{C}$ be the parabola of equation $Y = X^2$ in $\mathrm{AG}(2, q)$ with $q = p^a$. Let $u \in GF(q)$ and $H = \{h_1, \dots, h_n\}$ be a subgroup of the additive group of $GF(q)$. If $A_i = (u + h_i, (u + h_i)^2)$, for $i = 1, \dots, n$, then the n-gon $A_1 A_2 \cdots A_n$ is called a* generalized affinely regular polygon.

Wettl proved the following characterization of sharply focused sets in planes of odd order.

Theorem 5.3. (Wettl [82]) *Let $q = p^a$ be odd and $\mathcal{S}$ be sharply focused on ℓ. Then $|\mathcal{S}| = k$ divides $q+1, q-1$, or q, and $\mathcal{S}$ is an affinely regular k-gon if*

p^2 does not divide k. If p^2 divides k, then $\mathcal{S}$ is a generalized affinely regular k-gon.

For planes of even order Wettl proved an analogous result if $\mathcal{S}$ is a subset of an oval $\mathcal{O}$ and the line ℓ is a Pascal-line of $\mathcal{O}$. Namely, k divides $q \pm 1$ or $k+1$ divides q. For q even, very sharply focused sets exist as well; they correspond to cosets of the additive group of $\mathrm{GF}(q)$. This is what we called a generalized affinely regular polygon; in the above example $\mathcal{O}$ is a translation hyperoval and ℓ is a tangent line. Very sharply focused sets were later called supersharply focused by Holder [35] and then *hyperfocused* arcs by Cherowitzo and Holder [20] and Giulietti and Montanucci [31]. From the results of Bichara, Korchmáros [12], hyperfocused arcs only exist when q is even. The next result is implicit in [82]. Chapter 5 of the Ph.D. thesis of Wen-Ai Jackson [37] contains several results on sharply focused and hyperfocused sets with particular attention to ones contained in a conic, such as the next one.

Theorem 5.4. *Let $\mathcal{S}$ be a hyperfocused arc on ℓ (hence q is even). Suppose that $\mathcal{S}$ is contained in a conic $\mathcal{C}$. Then $|\mathcal{S}| = k$ divides q, and $\mathcal{S}$ is a generalized affinely regular k-gon.*

It was already noticed by Wettl [82] that there were hyperfocused arcs not contained in a conic, see the example above. Cherowitzo and Holder [20] constructed hyperfocused arcs that were contained in a hyperoval or a subplane. They also classified small hyperfocused arcs and gave a negative answer to a question of Drake and Keating [23] on possible sizes of hyperfocused arcs. Recently, Giulietti and Montanucci [31] constructed hyperfocused translation arcs that were not contained in a hyperoval or a subplane. Moreover, these arcs are complete in the sense that every point not on the distinguished line ℓ belongs to some chord of the arc. Giulietti and Montanucci [31] also introduced the notion of *generalized hyperfocused k-arcs* where a set of $k-1$ points that block the chords of the arc was not collinear. They showed the existence of a generalized hyperfocused 8-arc which was not hyperfocused. It should also be noted that Aguglia, Korchmáros, and Siciliano [2] proved that in Desarguesian planes any generalized hyperfocused arc contained in a conic is hyperfocused. In addition, Giulietti and Montanucci provided a complete list of examples for size up to 10.

The results of Wettl come from more general results about internal nuclei. Let $\mathcal{K}$ be a k-set in $\mathrm{PG}(2,q)$. A point $P \in \mathcal{K}$ is called an *internal nucleus* of $\mathcal{K}$ if each line through P meets $\mathcal{K}$ in at most two points including P. This notion was introduced for $k = q+2$ by Bichara and Korchmáros [12] and generalized by Wettl [82]. The set of internal nuclei of $\mathcal{K}$ is denoted by $IN(\mathcal{K})$. If $\mathcal{S}$ is sharply focused on ℓ and $\mathcal{N}$ denotes the set of points of ℓ not covered by a chord of $\mathcal{S}$ then $\mathcal{K} = \mathcal{S} \cup \mathcal{N}$ is a set of $q+1$ points and the points of $\mathcal{S}$ are internal nuclei of $\mathcal{K}$. If one starts from a very sharply focused set then the corresponding $\mathcal{K}$ is a $(q+2)$-set.

From our point of view the main result of Bichara, Korchmáros, and Wettl are the following.

Theorem 5.5. (Bichara, Korchmáros [12]) *If $\mathcal{K}$ is a set of $q+2$ points in* $\mathrm{PG}(2,q)$ *and* $|IN(\mathcal{K})| \geq 3$*, then q is even. If q is even and* $|IN(\mathcal{K})| > q/2$*, then $\mathcal{K}$ is a hyperoval.*

Theorem 5.6. (Wettl [82]) *If $\mathcal{K}$ is a set of $q+1$ points in* $\mathrm{PG}(2,q)$*, q odd, and* $|IN(\mathcal{K})| > (q+1)/2$ *then* $\mathcal{K} = IN(\mathcal{K})$*, that is $\mathcal{K}$ is an oval.*

At an internal nucleus P, a line t is a tangent to $\mathcal{K}$ if it meets $\mathcal{K}$ in just P. For the set of tangents, Segre's Lemma of tangents can be used and for $|\mathcal{K}| = q+1$, q odd, it gives that $IN(\mathcal{K})$ is contained in a conic. Affinely regular $(q+1)/2$-gons together with the ideal points not covered by the chords show that the bound $(q+1)/2$ in the theorem of Wettl is sharp. Similarly, for q even, generalized affinely regular $q/2$-gons show the sharpness of the bound in the theorem of Bichara, Korchmáros. For q even, the above trick with Segre's Lemma of tangents gives that $(q+1)$-sets are contained in $(q+2)$-sets whose internal nuclei set cannot be smaller. More general embedding theorems can be found in [77]. Szőnyi [77] also proved that for $q > 121$, $(q+1)$-sets in $\mathrm{PG}(2,q)$, q odd, having $(q+1)/2$ internal nuclei are projectively equivalent to $\mathcal{K} = \mathcal{S} \cup \mathcal{N}$, where $\mathcal{S}$ is an affinely regular $(q+1)/2$-gon, and the points of $\mathcal{N}$ are collinear. Internal nuclei in non-Desarguesian planes were investigated by Biscarini, Conti [13], Szőnyi [76] and Wettl [83].

Recent improvements of the embedding theorems of Szőnyi can be found in the paper Beato, Faina, Giulietti [10], where the connection of nuclei and arcs in Desarguesian r-nets is also discussed.

Another type of nuclei was defined by Mazzocca. If $\mathcal{B}$ is a set of $q+1$ points in a projective plane Π of order q, then a point A of Π is called a *nucleus* of $\mathcal{B}$ if every line through A intersects $\mathcal{B}$ in precisely one point. The set of nuclei will be denoted by $N(\mathcal{B})$. For the plane $\mathrm{PG}(2,q)$, Blokhuis and Wilbrink [15] proved that $|N(\mathcal{B})| \leq q-1$. The motivation for introducing the concept of nucleus was the following result by Segre and Korchmáros [69], for the case when q is odd, and Bruen and Thas [18], when q is even:

Theorem 5.7. *Let $\mathcal{C}$ be a conic and $\mathcal{B}$ a set of $q+1$ points in* $\mathrm{PG}(2,q)$*. If every line joining two points of $\mathcal{B}$ is an exterior line of $\mathcal{C}$, then $\mathcal{B}$ is a line.*

In the terminology of nuclei, this theorem says that the set $N(\mathcal{B})$ cannot contain a conic, unless $\mathcal{B}$ is a line. We shall always suppose that the set $\mathcal{B}$ is not a line. The situation is similar but easier for lines instead of conics. The set $N(\mathcal{B})$ cannot contain a line; for non-collinear $\mathcal{B}$ it can either contain $q-1$ or at most $q-\sqrt{q}$ collinear points, see Fisher [24], Cameron [19], Bruen [17]. So it is natural to ask how many nuclei of a non-collinear set $\mathcal{B}$ can lie on a conic. The examples of Fisher [24] use affinely regular polygons. Let $\mathcal{S}$ be an affinely regular k-gon contained in the conic $\mathcal{C}$ and let $\mathcal{N}$ denote the set of points of ℓ not covered by the chords of $\mathcal{S}$, exactly as in case of sharply focused sets. If $\mathcal{B} = (\mathcal{C} \setminus \mathcal{S}) \cup \mathcal{N}$, then the points of $\mathcal{S}$ are nuclei of $\mathcal{B}$. Similar examples can be constructed from a generalized affinely regular k-gon, and we call them *Fisher type* examples. The main result of [14] is the following.

Theorem 5.8 (Blokhuis–Szőnyi). *Let $\mathcal{B}$ be a set of $q+1$ points in* $\mathrm{PG}(2,q)$*, q odd, $q > 10^7$ and suppose that $\mathcal{B}$ has more than $0.326q$ nuclei on a certain conic. Then $\mathcal{B}$ is of Fisher type.*

Some of the Fisher type examples were characterized by Mazzocca [61] who also posed the problem of finding blocking sets with respect to special line sets. This problem has received a lot of attention in recent years.

In a series of papers [51, 75, 52] it was shown that if d is big enough with respect to n, then the chords of any affinely regular n-gon cover almost all points of $\mathrm{AG}(2,q)$, the uncovered points being the remaining points of the circumscribed conic and, in some cases, the centre of the conic when this is either an ellipse or a hyperbola. The exact formulation is the following.

Theorem 5.9. *Let $\mathcal{S}$ be an affinely regular k-gon inscribed in the conic $\mathcal{C}$ of* $\mathrm{AG}(2,q)$*. If $q/k < q^{1/4}/2$, then the chords of $\mathcal{S}$ cover all points of the affine plane except the points of $\mathcal{C}$ and possibly the centre of $\mathcal{C}$.*

Let $P \in \mathrm{AG}(2,q)$ be a point outside $\mathcal{C}$ and distinct from the centre of $\mathcal{C}$. The essential idea in the proof is to find a low-degree, absolutely irreducible polynomial $f(X,Y) \in GF(q^2)[X,Y]$ such that if $f(X,Y)$ has a zero over $GF(q^2)$ then at least one chord of $\mathcal{S}$ passes through P. This can be done for degree $2s$ where $s = (q+1)/k$, $(q-1)/k$ or q/k according to whether $\mathcal{C}$ is an ellipse, hyperbola or parabola. Then the hypothesis $q/k < q^{1/4}/2$ together with the Hasse–Weil lower bound ensures the existence of such a root showing that some chord of S passes through P.

Theorem 5.9 has been an essential tool for the proof of the completeness of k-arcs containing many points from a conic $\mathcal{C}$. In the applications to such k-arcs, the generalisation of the concept of an affinely regular n-gon plays a role, see Definition 5.2. The special case where $n = q/d$ with $d = p^v$, $v < h$ and H consists of all elements of $GF(q)$ which can be written as $x^d - x$ with $x \in GF(q)$, was investigated in [75]. He proved that the chords of such a generalized affinely regular polygon cover all points of $\mathrm{AG}(2,q)$ outside $\mathcal{C}$. So, Theorem 5.9 holds true in a larger context. For more results on k-arcs containing many points from a conic, see the survey paper [78].

6. Algebraic curves and affinely regular polygons

From the above discussion, a natural question on chords of an affinely regular n-gon $\mathcal{A}$ in $\mathrm{AG}(2,q)$ arises, namely to determine (or at least give an upper bound on) the number of chords of $\mathcal{A}$ passing through a given point P.

The case $n = (q+1)/2$ is related to the notion of a regular point introduced by B. Segre, see [67, 68]. Let $\mathcal{A}$ be an affinely regular $(q+1)/2$–gon in

AG$(2, q)$ with q odd. Denote by $\mathcal{E}$ the ellipse which $\mathcal{A}$ is inscribed in. Segre's definition of a regular point in terms of affinely regular $(q+1)/2$–gons can be stated as follows. A point $P \notin \mathcal{E}$ is regular when every chord of $\mathcal{E}$ through P and a vertex of $\mathcal{A}$ contains another vertex of $\mathcal{A}$. He proved that $\mathcal{A} = A_1 A_2 \cdots A_{(q+1)/2}$ admits at most one regular point, namely the centre of $\mathcal{E}$ and this only happens when $q \equiv 3 \pmod 4$.

In the above definition, replacing $(q+1)/2$ by any proper divisor $n \geq 4$ of $q+1$ gives a straightforward extension of the notion of a regular point of an affinely regular n-gon inscribed in an ellipse $\mathcal{E}$. It may be noted that the centre of $\mathcal{E}$ is regular if and only if n is even. However, Segre's result on the non-existence of regular points distinct from the centre of the ellipse does not extend to every n as the following examples show.

Assume that $\bar{q} = p^b$ with $r = \rho b$ and $\rho > 1$ odd. Let $GF(\bar{q})$ be the subfield of $GF(q)$ of order $\bar{q}$. Choose an irreducible polynomial $x^2 - ux + 1$ over $GF(\bar{q})$. Since ρ is odd, this polynomial remains irreducible over $GF(q)$. In AG$(2, \bar{q})$, the conic $\bar{\mathcal{E}}$ of equation $x^2 + y^2 - uxy = 1$ is an ellipse which extends to an ellipse $\mathcal{E}$ in AG$(2, q)$. Let $\mathcal{A}$ be an affinely regular $(p^b + 1)$–gon whose vertices are the points of $\mathcal{E}$. Obviously, every point P of AG$(2, \bar{q})$ which does not lie on $\bar{\mathcal{E}}$ is regular with respect to $\mathcal{A}$, but no point of AG$(2, q)$ outside AG$(2, \bar{q})$ has this property.

For an affinely regular n-gon $\mathcal{A}$ inscribed in $\mathcal{E}$, the number $N_n(P)$ of chords passing through a point P is trivially upper bounded by $n/2$; more precisely, if $T_n(P)$ denotes the number of tangents to $\mathcal{E}$ through P whose tangency point is a vertex of $\mathcal{A}$, then

$$N_n(P) \leq \begin{cases} \frac{1}{2}\, n & \text{if } T_n(P) = 0, \\ \frac{1}{2}\,(n-1) & \text{if } T_n(P) = 1, \\ \frac{1}{2}\,(n-2) & \text{if } T_n(P) = 2. \end{cases} \tag{6.1}$$

It may be noted that (6.1) is attained by the affinely regular n-gons described in the above examples. Actually, these turn out to be the unique affinely regular n-gons with a regular point distinct from the centre of $\mathcal{E}$. This was proven using the classification of subgroups of the projective linear group PGL$(2, q)$, see [1, Teorema 1].

The problem of determining $N_n(P)$ appears to be rather difficult. In [1, Proposizione 3.2], it was shown that

$$\left| N_n(P) - \frac{1}{2} \left(\frac{n}{(q-1)(q+1)} \right)^2 \mu \right| \leq 2, \tag{6.2}$$

where μ denotes the number of points in AG$(2, q^2)$ of the plane irreducible algebraic curve $\mathcal{C}$ of affine equation

$$t^q x^k y^k - (x^k + y^k) + t = 0. \tag{6.3}$$

where $k = (q-1)(q+1)/d$ while $t \in GF(q^2)$ is a non-zero constant uniquely determined by the position of P with respect to $\mathcal{A}$. This curve has singular points at infinity, all ordinary, and its genus is equal to k^2.

Therefore the problem of determining $N_n(P)$ and μ are equivalent. Unfortunately, $\mathcal{C}$ does not belong to the meagre family of curves whose number of points over a finite field is known or can be computed using the technique developed by Vanvider and others, see [55, Chapters 5,6].

So, $N_n(P)$ is not deducible from known results about algebraic curves defined over a finite field, but the approach works when the aim is an improvement on (6.2). In this direction, the key results from algebraic geometry are the upper bounds on the number $M(q^2)$ of points in $\mathrm{AG}(2, q^2)$ lying on a plane irreducible algebraic curve of degree d defined over $GF(q^2)$. A well known and widely applied upper bound of this kind derives from the famous Hasse-Weil theorem: $M(q^2) \le q^2 + 1 + (d-1)(d-2)q$, see [5, 6, 53] and [34, Chapter 10]. But this bound does not produce any improvement on (6.2). In fact, the Hasse-Weil bound only depends on the degree of the curve while any useful result for our purpose must provide different bounds according to the value of the parameter t in (6.3). This is apparent from the above example: if $t \in GF(\bar{q}^2)$ then equality holds in (6.1) but this is not the case when $t \in GF(q^2) \setminus GF(\bar{q}^2)$.

The technical tool adequate to our purpose is the Stöhr–Voloch theory on non–classical curves defined over a finite field, see [73] and [34, Chapter 8]. It turns out that equality holds in (6.2) if and only if the curve $\mathcal{C}$ is Frobenius non-classical with respect to the linear series cut out by conics. For Frobenius classical curves, the Stöhr–Voloch theory provides a good upper bound for μ which in turn gives a significant improvement on (6.2), see [1]:

Theorem 6.1. *For a proper divisor n of $q+1$, with an odd prime-power q, let $\mathcal{A}$ be an affinely regular n-gon in $\mathrm{AG}(2, q)$. Let $\mathcal{E}$ be the ellipse in which $\mathcal{A}$ is inscribed. Then for a point P outside $\mathcal{E}$ and distinct from the centre of $\mathcal{E}$, the number $N_n(P)$ of chords of $\mathcal{A}$ through P is bounded above by*

$$N_n(P) \le \frac{2}{5}\, d + 2 \tag{6.4}$$

unless $n = \bar{q} + 1$, with $\bar{q}^\rho = q$ for some odd integer $\rho > 1$, and P together with all vertices of $\mathcal{A}$ lie in an affine subplane of $\mathrm{AG}(2, q)$ of order $\bar{q}$.

Very recently, Giulietti [30] developed the Stöhr–Voloch approach further using an "ad hoc" linear series on $\mathcal{C}$. He was able to improve (6.4) proving that

$$N_n(P) \le \frac{1}{3}\, d + 2. \tag{6.5}$$

As a matter of fact, his result also holds for affinely regular n-gons inscribed in a hyperbola.

Similar results can be proven for generalized affinely regular polygons. In [45], the special case mentioned at the end of Section 5 was investigated.

The upper bound found for the number of chords through a point P of $\mathrm{AG}(2,q)$ outside the parabola $\mathcal{C}$ was

$$N_d(P) \leq \frac{q+3}{3d} + 1,$$

with some exceptions occurring when $a = d^2$, $b \in GF(d)$, $a^d + d = 0$ and (a,b) are the coordinates of the point P. In the exceptional cases,

$$N_d(P) = \frac{1}{2}\left(\frac{q}{d} - 1\right).$$

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