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GENERALIZED CPR-GRAPHS AND APPLICATIONS

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Abstract. We give conditions for oriented labeled graphs that must be satisfied in order that such are permutations representations for groups of automorphisms of chiral or orientably regular polytopes. We develop a technique for construction of highly symmetric polytopes using such graphs. In particular, we construct chiral polytopes with the automorphism group S_n for each $n > 5$, an infinite family of finite chiral polytopes of rank 4, a polytope of rank 5, as well as several infinite chiral polyhedra.

Regular and chiral polytopes are highly symmetric combinatorial structures admitting full symmetry by (abstract) reflections and rotations respectively. The study of such objects is usually done by investigating the properties of their automorphism groups. In the case of regular polytopes, the automorphism groups are in a one-to-one correspondence with certain quotients of Coxeter groups called string C-groups (see Section 1). In an earlier paper [13] the concept of C-group permutation representation graph (or CPR graph) has been introduced as a graph with labeled edges that encodes all information about the automorphism group of a regular polytope (string C-group). In this paper we shall extend this idea to construct oriented labeled graphs (GPR graphs) which will encode all information about the automorphism group of a chiral polytope, or about the rotation subgroup of the automorphism group of a regular polytope. Furthermore, these ideas can be generalized to enable us to work with other less symmetric polytopes. Although in [13] and several subsequent publications only CPR graphs were used (to work with regular polytopes), in some cases it may be easier to work with GPR graphs instead. In any case we can derive a CPR graph of a given regular polytope from a GPR graph of the same polytope and vice-versa. We illustrate the effectiveness of GPR graphs by using them in solving several problems related to existence of certain type of chiral and regular polytopes which could not have been solved by previously used techniques.

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In [12] and [13] the first author constructed regular polyhedra (that is, polytopes of rank 3) with symmetric or alternating automorphism group. Chiral maps with alternating automorphism group have been considered by Nedela (see [10] for example). In Section 3 GPR graphs are used to revisit chiral polyhedra with small alternating groups as automorphism groups, as well as chiral polyhedra with symmetric automorphism groups. In particular we construct a chiral abstract polyhedron with symmetric automorphism group S_n for each $n \geq 6$ and show that none exist with alternating automorphism group A_n for $n \leq 7$.

The authors do not know of any previously published proof of the chirality in the abstract sense of an infinite polyhedron. In [6] Hubard describes constructions of two-orbit finite or infinite polyhedra from groups, but did not provide examples of the latter. Some two-orbit polyhedra which are not chiral were known to exist, for example, the tessellation of the Euclidean plane with regular hexagons and equilateral triangles where every edge lies between a hexagon and a triangle. In [16] and [17] Schulte classifies all geometrically chiral polyhedra in the Euclidean 3-space $\mathbb{E}^3$. They are all infinite and belong to six families. The polyhedra in three of these families have finite faces whereas the polyhedra in the remaining three families have infinite faces. However, it was not known whether these polyhedra were regular or chiral in the abstract sense. We proved in [15] that all geometrically chiral polyhedra in $\mathbb{E}^3$ with infinite faces are combinatorially regular, whereas those with finite faces are combinatorially chiral. In Section 4 we provide three additional examples of infinite chiral abstract polyhedra with all possible types (finite or infinite) of faces and vertex-figures.

Chiral maps have been extensively studied (see for example [2]). Most known rank 4 chiral polytopes were constructed as quotients of regular tessellations of the hyperbolic space (see [19]). This restricted the possible facet and vertex-figure types of the polytopes. An approach using a computer software was given in [3] where not only chiral 4-polytopes, but also some chiral 5-polytopes are constructed. In [20] a construction is given for the universal $(d+1)$-chiral polytope with the property that its facets are isomorphic to a given chiral d-polytope whose facets are regular. Recently the existence of chiral d-polytopes for $d \geq 7$ has been announced. In Section 5 we provide a method to construct orientably regular or chiral polytopes with a preassigned facet type. We illustrate this method by constructing an infinite family of chiral 4-polytopes none of which is a quotient of a regular tessellation of the hyperbolic space. Furthermore, we also show an example of a chiral 5-polytope with regular facets.

1. Definitions

We start by introducing abstract polytopes with emphasis on the regular and 2-orbit ones, referring to [7], [8], [9], and [18] for details.

An *abstract d-polytope* (or *abstract polytope of rank d*) $\mathcal{K}$ is a partially ordered set that satisfies the below introduced properties. Its elements are called *faces* and its maximal totally ordered subsets are called *flags*. $\mathcal{K}$ contains a minimum F_{-1} and a maximum F_d, and every flag of $\mathcal{K}$ contains precisely $d+2$ elements (including F_{-1} and F_d). This induces a rank function from $\mathcal{K}$ to $\{-1, 0, \ldots, d\}$ such that $\text{rank}(F_{-1}) = -1$ and $\text{rank}(F_d) = d$. The faces of rank i are called *i-faces*, the 0-faces are called *vertices*, the 1-faces are called *edges* and the $(d-1)$-faces are called *facets*. Whenever the rank of the face G is i we shall abuse notation and refer to the *section* $G/F_{-1} := \{H \mid H \leq G\}$ as an i-face of $\mathcal{K}$ and denote it simply by G. Given a vertex G, the section $F_d/G := \{H \mid H \geq G\}$ is called the *vertex figure* of $\mathcal{K}$ at G. For every incident faces F and G such that $\text{rank}(F) - \text{rank}(G) = 2$ there exist precisely two faces H_1 and H_2 such that $G < H_1, H_2 < F$. This property is referred to as the *diamond condition*. As a consequence of the diamond condition, for any flag Φ and any $i \in \{0, \ldots, d-1\}$ there exists a unique flag Φ^i that coincides with Φ in the j-face for $j \neq i$ but has a different i-face. Such a flag is called the *i-adjacent* flag of Φ. Finally, $\mathcal{K}$ must be *strongly flag-connected*, that is, for any two flags Φ, Φ' there exist a sequence of flags $\Psi_0 = \Phi, \Psi_1, \ldots, \Psi_m = \Phi'$ such that $\Phi \cap \Phi' \subseteq \Psi_k$ and Ψ_{k-1} is adjacent to Ψ_k for $k = 1, \ldots, m$.

The *dual* of a polytope $\mathcal{K}$ consists of the set of faces of $\mathcal{K}$ with the order reversed.

Whenever a d-polytope $\mathcal{K}$ has the property that for $i = 1, \ldots, d-1$ the (polygonal) section $G/F := \{H \mid F \leq H \leq G\}$ between an $(i-2)$-face F and an $(i+1)$-face G incident to F depends only on i and not on F and G, we say that $\mathcal{K}$ is *equivelar*. In this case, for any incident faces F and G of ranks $i-2$ and $i+1$ respectively, the section G/F is isomorphic to an abstract p_i-gon for a fixed number $p_i \leq \infty$. We define the *Schläfli type* of an equivelar polytope $\mathcal{K}$ as $\{p_1, \ldots, p_{d-1}\}$. Although any number p_i can be 2, we shall consider only the cases when $p_i \geq 3$ for $i = 1, \ldots, d-1$. Regular and chiral polytopes defined below are equivelar.

An *automorphism* of a polytope $\mathcal{K}$ is an order preserving permutation of its faces. We denote by $\Gamma(\mathcal{K})$ the automorphism group of $\mathcal{K}$.

We say that a d-polytope $\mathcal{K}$ is *regular* if $\Gamma(\mathcal{K})$ is transitive on the flags of $\mathcal{K}$. In this case, $\Gamma(\mathcal{K})$ is generated by permutations $\rho_0, \ldots, \rho_{d-1}$ where ρ_i is the (unique) automorphism of $\mathcal{K}$ mapping a fixed *base flag* to its i-adjacent flag. This set of generators satisfy the relations

$$\begin{array}{rcll} \rho_i^2 & = & \varepsilon, & \\ (\rho_i\rho_j)^2 & = & \varepsilon & \text{whenever } |i-j| \geq 2, \\ (\rho_{i-1}\rho_i)^{p_i} & = & \varepsilon & \text{for } p_i \neq \infty \end{array}$$

where ε represents the identity element, as well as the intersection conditions given by

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle. \tag{1.1}$$

The groups generated by involutions $\rho_0, \ldots, \rho_d$ which satisfy the relation $(\rho_i\rho_j)^2 = \varepsilon$, for $|i-j| \geq 2$, and the intersection condition (1.1) are called *string C-groups* and they are in a one-to-one correspondence with the automorphism groups of regular polytopes. We shall say that the elements ρ_i are *abstract reflections*.

We define the *rotation subgroup* of (the automorphism group of) a regular d-polytope $\mathcal{K}$ as the subgroup of $\Gamma(\mathcal{K})$ consisting of words of even length on the generators $\rho_0, \dots, \rho_{d-1}$ and it is denoted by $\Gamma^+(\mathcal{K})$. For $i = 1, \dots, d-1$ we define the *abstract rotation* σ_i to be $\rho_{i-1}\rho_i$, that is, the automorphism of $\mathcal{K}$ mapping the base flag Φ to $(\Phi^i)^{i-1}$. It is easy to see that $\Gamma^+(\mathcal{K}) = \langle \sigma_1, \dots, \sigma_{d-1} \rangle$ and that the relations

$$(\sigma_i \sigma_{i+1} \cdots \sigma_j)^2 = \varepsilon \quad \text{for } i < j \tag{1.2}$$

hold. The involutions $\tau_{i,j} := \sigma_i \cdots \sigma_j$ are called *abstract half-turns* for $i < j$. Note that $\tau_{i,i} = \sigma_i$ is not an involution when $p_i \geq 3$. The abstract rotations and half-turns also satisfy the intersection condition given by

$$\begin{aligned} \langle \tau_{i,j} \,|\, i \leq j; i-1, j \in I \rangle \cap \langle \tau_{i,j} \,|\, i \leq j; i-1, j \in J \rangle \\ = \langle \tau_{i,j} \,|\, i \leq j; i-1, j \in I \cap J \rangle \end{aligned} \tag{1.3}$$

for $I, J \subseteq \{0, \dots, d-1\}$.

For any regular polytope $\mathcal{K}$, the index of $\Gamma^+(\mathcal{K})$ on $\Gamma(\mathcal{K})$ is at most 2. Whenever $\Gamma^+(\mathcal{K})$ has index 2 in $\Gamma(\mathcal{K})$ we say that $\mathcal{K}$ is *orientably regular*, otherwise $\mathcal{K}$ is said to be *non-orientably regular*. Every non-orientably regular polytope $\mathcal{K}$ admits an *orientably regular double cover* $\mathcal{P}$, which can be obtained by applying the construction in [18] to the (rotation subgroup of the) automorphism group of $\mathcal{K}$. The rotation subgroups of $\mathcal{K}$ and $\mathcal{P}$ are isomorphic.

We say that the d-polytope $\mathcal{K}$ is *chiral* if its automorphism group induces two orbits on the flags with the property that adjacent flags belong to different orbits. In this case $\Gamma(\mathcal{K}) = \langle \sigma_1, \dots, \sigma_{d-1} \rangle$, where σ_i maps a base flag Φ to $(\Phi^i)^{i-1}$, that is, σ_i cyclically permutes the i- and $(i-1)$-faces of $\mathcal{K}$ incident with the k-faces of Φ for $k \notin \{i-1, i\}$. Furthermore, the generators σ_i also satisfy (1.2) as well as the intersection conditions in (1.3). Because of the obvious similarities between the automorphism group of a chiral polytope and the rotation subgroup of a regular polytope we shall also refer to the generators σ_i of the automorphism group of a chiral polytope as *abstract rotations*, and to the automorphism group of a chiral polytope as its *rotation subgroup*.

Each chiral polyhedron occurs in two *enantiomorphic forms*, in a sense in a right and left handed version which can be thought of as mirror images of each other. For details we refer to [19].

It was proved in [18] that any group Γ generated by the set $\{\sigma_1, \dots, \sigma_{d-1}\}$ satisfying (1.2) and the intersection conditions in (1.3) is either the rotation subgroup of the automorphism group of a regular polytope, or the automorphism group of a chiral polytope. Furthermore, Γ is the automorphism group of a regular polytope if and only if there is a group automorphism $\widehat{\alpha}$

of Γ such that

$$\sigma_i\widehat{\alpha} = \begin{cases} \sigma_i^{-1} & \text{for } i = 1, \\ \sigma_1^2\sigma_i & \text{for } i = 2, \\ \sigma_i & \text{for } i > 2, \end{cases} \tag{1.4}$$

or dually,

$$\sigma_i\widehat{\alpha} = \begin{cases} \sigma_i^{-1} & \text{for } i = d-1, \\ \sigma_i\sigma_{d-1}^2 & \text{for } i = d-2, \\ \sigma_i & \text{for } i < d-2, \end{cases} \tag{1.5}$$

Alternatively, a polyhedron is regular if and only if there is some $\widehat{\alpha}$ such that

$$\sigma_i\widehat{\alpha} = \sigma_i^{-1} \quad \text{for } i = 1, 2. \tag{1.6}$$

As a consequence of (1.2) we have

Corollary 1.1. *Let $\Gamma = \langle \sigma_1, \ldots, \sigma_{d-1}\rangle$ be the rotation subgroup of an orientably regular or chiral polytope. Then*

(a) $\sigma_i\sigma_{i+1}\cdots\sigma_j = \sigma_j^{-1}\sigma_{j-1}^{-1}\cdots\sigma_i^{-1}$ *for* $i < j$,

(b) $\sigma_i\sigma_{i+1}\cdots\sigma_j\sigma_i = \sigma_{i+1}\sigma_{i+2}\cdots\sigma_j$ *for* $i + 1 < j$.

The following lemma follows directly from [18, Lemma 10].

Lemma 1.2. *Let $n \geq 4$ and let $\Gamma = \langle \sigma_1, \ldots, \sigma_{d-1}\rangle$ be a group satisfying (1.2). If $\langle \sigma_1, \ldots, \sigma_{d-2}\rangle$ is the even subgroup of the symmetry group of a regular $(d-1)$-polytope and the intersection condition*

$$\langle \sigma_1, \ldots, \sigma_{d-2}\rangle \cap \langle \sigma_k, \ldots, \sigma_{d-1}\rangle = \langle \sigma_k, \ldots, \sigma_{d-2}\rangle$$

holds for $k = 2, \ldots, d-1$, then Γ satisfies the intersection condition.

In particular, the intersection condition for the rotation subgroup of a regular or chiral 3-polytope is given by $\langle \sigma_1\rangle \cap \langle \sigma_2\rangle = \{\varepsilon\}$. Rank 3-polytopes are called (abstract) *polyhedra*.

The dual $\mathcal{K}^*$ of a regular or chiral polytope $\mathcal{K}$ with Schläfli type $\{p_1, \ldots, p_{d-1}\}$ is a regular or chiral polytope with Schläfli type $\{p_{d-1}, \ldots, p_1\}$.

We say that a polytope is *flat* whenever all of its facets contain every vertex. An example of this is the hemicube which consists of four vertices, six edges and three faces which are squares, implying that every vertex is contained in every face.

A regular polytope $\mathcal{K}$ is flat if and only if

$$\Gamma(\mathcal{K}) = \langle \rho_0, \ldots, \rho_{d-1}\rangle = \langle \rho_1, \ldots, \rho_{d-1}\rangle\langle \rho_0, \ldots, \rho_{d-2}\rangle$$

(see [9, Proposition 4E4]). Similarly, it can be proved that a chiral polytope $\mathcal{K}$ is flat if and only if

$$\Gamma(\mathcal{K}) = \langle \sigma_1, \ldots, \sigma_{d-1}\rangle = \langle \sigma_2, \ldots, \sigma_{d-1}\rangle\langle \sigma_1, \ldots, \sigma_{d-2}\rangle. \tag{1.7}$$

In fact, since $\mathcal{K}$ is flat, for any vertex v of $\mathcal{K}$ there is an automorphism $\phi \in \langle \sigma_1, \ldots, \sigma_{d-2}\rangle$ mapping the base vertex v_0 to v. Moreover, $\langle \sigma_2, \ldots, \sigma_{d-1}\rangle$ is the automorphism group of the vertex-figure on the base vertex v_0, and for ϕ chosen as above $\langle \sigma_2, \ldots, \sigma_{d-1} \rangle \phi$ consists of all automorphisms of $\mathcal{K}$ mapping v_0 into v, implying (1.7).

A *central involution* of a polytope $\mathcal{K}$ is an involutory (non-trivial) automorphism which commutes with every other automorphism of $\mathcal{K}$. In general, there may be more than one central involution of the same polytope. For example, there are three distinct central involutions of the regular polyhedron with Schläfli type $\{6, 6\}$ and automorphism group isomorphic to $S_4 \times (\mathbb{Z})^2$ of order 96 (see [5, Polytope $\{6, 6\}$*96], and Figure 1 for a CPR graph of it). Generally, given a regular polyhedron $\mathcal{K}$ with Schläfli type $\{p, q\}$ for some odd numbers p and q, we can construct a regular polyhedron $\mathcal{P}$ with Schläfli type $\{2p, 2q\}$ with at least three central involutions in the following way. Assuming that $\Gamma(\mathcal{K}) = \langle \rho_0, \rho_1, \rho_2 \rangle$ we adjoin to it the central elements z and w, and define $r_0 := \rho_0 z$, $r_1 := \rho_1$ and $r_2 := \rho_2 w$. It is not hard to prove that $\langle r_0, r_1, r_2 \rangle$ is the automorphism group of a regular polyhedron $\mathcal{P}$ with standard generators r_0, r_1, r_2. In fact, the elements z and w can be seen as $(r_0 r_1)^p$ and $(r_1 r_2)^q$ respectively and the intersection condition follows from the fact that $\Gamma(\mathcal{P}) \cong \Gamma(\mathcal{K}) \times (\mathbb{Z}_2)^2$. Note that the quotients of $\Gamma(\mathcal{P})$ by $\langle z \rangle$ and $\langle w \rangle$ yield string C-groups, however the quotient by $\langle zw \rangle$ no longer satisfies the intersection property.

Besides regular and chiral polyhedra we shall also consider another class of polyhedra with two orbits on the flags, denoted by 2_0 in [7] (see also [6] and [8]). The name of the class is derived from the fact that 0-adjacent flags belong to the same orbit, while 1- and 2-adjacent flags belong to distinct orbits. The generators of the automorphism group of a polyhedron in class 2_0 are the abstract reflection ρ_0 and the abstract rotation σ_2 and they satisfy the intersection condition given by

$$\langle \sigma_2 \rangle \cap \langle \rho_0, \sigma_2 \rho_0 \sigma_2^{-1} \rangle = \{\varepsilon\}. \tag{1.8}$$

Any group generated by an involution ρ_0 and an element σ_2 satisfying the intersection condition (1.8) is either the automorphism group of a polyhedron in class 2_0 or an index 2 subgroup of the automorphism group of a regular polyhedron (this is called the 2_0-admissible subgroup of the automorphism group in [11]). It is an index 2 subgroup of the automorphism group of a regular polyhedron if and only if it has a group automorphism preserving ρ_0 and mapping σ_2 into σ_2^{-1}. Although regular and chiral polyhedra can have any Schläfli type $\{p, q\}$ for $p, q \geq 3$, polyhedra in class 2_0 have faces with even number of edges (that is, its Schläfli type is $\{2p, q\}$).

The *Petrie polygons* of a polyhedron $\mathcal{K}$ are defined as polygonal paths with the property that any two consecutive edges, but not three belong to the same face of $\mathcal{K}$. In general the Petrie polygons may not be polygons in the sense that they fail to be cycles or infinite paths by repeating some vertex more than once. In any case we may consider the poset $P(\mathcal{K})$ determined by the vertex and edge sets of $\mathcal{K}$ along with the Petrie polygons of $\mathcal{K}$ and call it the *Petrial* of $\mathcal{K}$. When the Petrie polygons of a polyhedron $\mathcal{P}$ are polygons, the Petrial of $\mathcal{P}$ is a polyhedron. Let $\mathcal{K}$ be a regular polyhedron with $\Gamma(\mathcal{K}) = \langle \rho_0, \rho_1, \rho_2 \rangle$, then, if its Petrial is a polyhedron, it must be regular as well with generators $\rho_0\rho_2, \rho_1, \rho_2$. On the other hand, if $\mathcal{K}$ is a chiral polyhedron with $\Gamma(\mathcal{K}) = \langle \sigma_1, \sigma_2 \rangle$ and its Petrial is a polyhedron, then it must belong to class 2_0 and its automorphism group must have $\sigma_1\sigma_2$ and σ_2 as generators. Conversely, the Petrial, if polyhedral, of a polyhedron in class 2_0 with automorphism group $\langle \rho_0, \sigma_2 \rangle$ must be chiral with automorphism group $\langle \rho_0\sigma_2^{-1}, \sigma_2 \rangle$ (see [7] for details).

2. CPR and GPR graphs

In this section we introduce permutation representation graphs for symmetry groups of chiral and orientably regular polytopes as the natural extension of CPR graphs introduced in [12] and [13]. We also provide some basic structural results for the graphs. This idea can be extended to less symmetric polytopes. As an example we define permutation representation graphs for 2-orbit polyhedra in class 2_0.

We begin by recalling the concept of CPR graphs, referring to [13] for details.

A *CPR graph* of a regular d-polytope $\mathcal{K}$ is a permutation presentation of $\Gamma(\mathcal{K}) = \langle \rho_0, \ldots, \rho_{d-1} \rangle$ represented on a graph and is defined as follows. Let π be an embedding of $\Gamma(\mathcal{K})$ into the symmetric group S_X on the set X. The CPR graph G of $\mathcal{K}$ determined by π is a multigraph with edge labels (or colors) in the set $\{0, \ldots, d-1\}$, with vertex set $V(G) = X$, and such that two vertices u, v of the graph are joined by an edge of label j if and only if $u(\rho_j\pi) = v$. Note that since π is an embedding, these representations are faithful.

For example, Figure 1 is a CPR graph of the regular polyhedron with Schläfli type $\{6, 6\}$ described in Section 1 for some particular embedding π of the automorphism group into S_{16}. Here ρ_0, ρ_1 and ρ_2 are respectively represented by the edges colored black, red and blue.

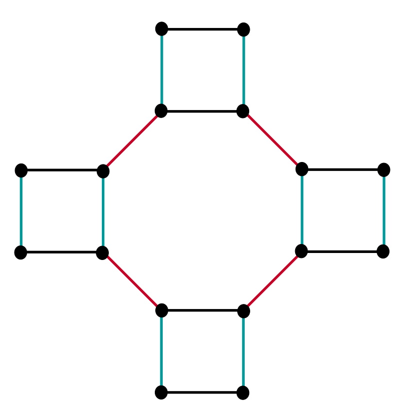

Figure 1: CPR graph of a polyhedron with Schläfli type $\{6, 6\}$

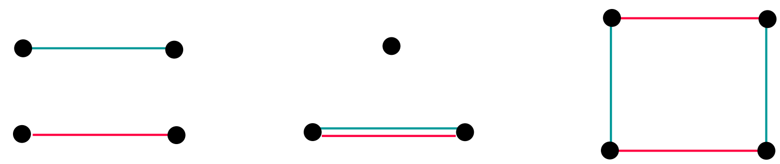

Figure 2: Connected components induced by ρ_i and ρ_j for $|i-j| \geq 2$

As a consequence of the relation $(\rho_i\rho_j)^2 = \varepsilon$ for $|i-j| \geq 2$ we have that the connected components induced by the edges with labels i and j of any CPR graph of a polytope must be among those in Figure 2.

The fact that the generators ρ_i of the automorphism group of a regular polytope are involutions allows us to represent each of them by a matching, however, to represent the abstract rotations generating the rotation subgroup of a regular or chiral polytope in a similar fashion, we need to consider oriented graphs as explained below.

Definition 2.1. *Let X be a set, $\mathcal{K}$ be a regular or chiral d-polytope and π an embedding into S_X of the rotation subgroup $\Gamma^+(\mathcal{K}) = \langle \sigma_1, \dots, \sigma_{d-1} \rangle$ of $\mathcal{K}$. The* general permutation representation graph *of $\mathcal{K}$ determined by π is the directed (multi)graph with vertex set X, and an arrow with label i from vertex u to vertex v if and only if $(u)(\sigma_i\pi) = v$.*

Although this definition can be easily generalized to (polytopes with automorphism) groups with other structures, except when specified we shall only consider directly regular and chiral polytopes and we shall refer to the general permutation representation graphs of their rotation subgroup by GPR graphs.

The label set for the arrows of any GPR graph is then $\{1, \dots, d-1\}$. We omit the loops since they give no extra information. However, occasionally it will be convenient to label by i (more than one label can be used) the vertices of a GPR graph which contain a loop of label i (see, for example, Figure 3 and Proposition 2.3).

Furthermore, when π is obvious from the context, the graph is simply referred to as a GPR graph of $\mathcal{K}$.

The arrows of each label induce disjoint directed cycles or infinite directed paths. It follows that for $j = 1, \dots, d-1$, every vertex belongs to 0 or 2 arrows of label j. Hence, each vertex has even valency in the underlying graph.

A GPR graph G of $\mathcal{K}$ obtained by the embedding of $\Gamma^+(\mathcal{K})$ into the permutation group of its j-faces will be called *j-face GPR graph* of $\mathcal{K}$. For example, the graphs in Figure 3 are the 2-face GPR graphs of the cube and of $\{4,4\}_{(2,1)}$. Here color blue corresponds to label 1 and color green to label 2. The 2-face GPR graph of $\{4,4\}_{(1,2)}$ is obtained by reversing the direction

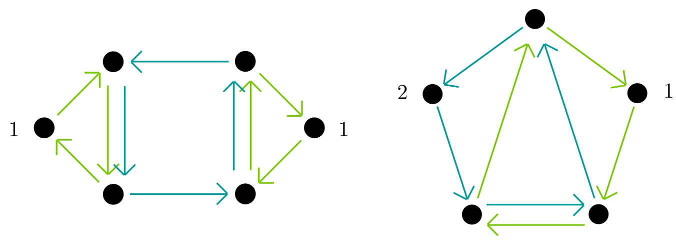

Figure 3: Cube and $\{4,4\}_{(2,1)}$ GPR graphs

of all arrows of the GPR graph of its enantiomorphic form, the polyhedron $\{4,4\}_{(2,1)}$.

The Schläfli type of a chiral or orientably regular polytope can be easily derived from any of its GPR graphs as explained in the following proposition.

Proposition 2.2. *Let G be a GPR graph of a directly regular or chiral polytope $\mathcal{K}$. The Schläfli number p_i of $\mathcal{K}$ is given by the least common multiple of the lengths of the cycles of the arrows of label i in G.*

Proof. Since p_i is the order of the element σ_i in $\Gamma(\mathcal{K})$, it suffices to calculate the order of its image under the corresponding embedding. This is the same as calculating the order of σ_i as a permutation of the vertices of G, which is the least common multiple of the lengths of all cycles in its cyclic structure. □

Each embedding π of $\Gamma(\mathcal{K})$ into a symmetric group induces a natural action of $\Gamma(\mathcal{K})$ on the vertex set of the GPR graph determined by π. Given such an action we will simply write $u\sigma_i = v$ whenever there is an arrow of label i from u to v.

Proposition 2.3. *Let $\mathcal{K}$ be a chiral or regular d-polytope and G any GPR graph of $\mathcal{K}$. Then, for any $i \in \{1, \ldots, d-2\}$, the local configurations of arrows of labels i and $i+1$ at each vertex u of G must take one of the forms shown in Figure 4.*

Proof. Let u be a vertex of a GPR graph of $\mathcal{K}$. Since any regular or chiral polyhedron satisfies the relation $(\sigma_i\sigma_{i+1})^2 = \varepsilon$, it follows that $u(\sigma_i\sigma_{i+1})^2 = u$. A simple exhaustive search will show that this is possible if and only if the arrangement of labels i and $i+1$ at u is in one of the configurations shown in Figure 4. □

The three configurations at the bottom of Figure 4 can only occur in disconnected GPR graphs or in GPR graphs of polytopes of rank 4 or higher.

Although each CPR graph corresponds to exactly one polytope, this is not the case for the permutation representation graphs defined above. In fact, if $\mathcal{K}$ is a non-orientable regular polytope and G is any GPR graph for

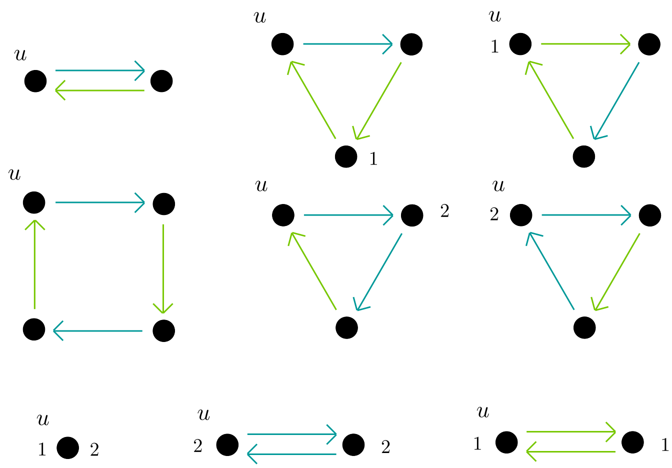

Figure 4: Local configurations

$\mathcal{K}$, then G is also a GPR graph for the orientable double cover of $\mathcal{K}$ since their rotation subgroups coincide.

We now discuss algebraic and combinatorial properties of the embedding π of connected GPR graphs.

Proposition 2.4. *Let G be a connected GPR graph of a directly regular or chiral polytope $\mathcal{K}$, let u be a vertex of G and let $\Lambda := Stab_{\Gamma(\mathcal{K})}(u)$. Then G is the GPR graph of $\mathcal{K}$ obtained by the embedding of $\Gamma(\mathcal{K})$ into the symmetric group on the right cosets of Λ. Furthermore, the arrows corresponding to the generators σ_i are determined by the natural action of $\Gamma(\mathcal{K})$ on the set of right cosets of Λ.*

Proof. To prove this proposition we give a bijection between $V(G)$ and the right cosets of Λ. Let u be any vertex of G. We note that $u\phi = u\psi$ if and only if $\phi\psi^{-1} \in \Lambda$. This allows us to identify the vertex u with Λ and, for any $\phi \in \Gamma(\mathcal{K})$, the vertex $u\phi$ with $\Lambda\phi$. It follows from the definition of GPR graphs that the adjacencies on the graph with vertex set determined by the right cosets of Λ are determined by the right action of $\Gamma(\mathcal{K})$. $\square$

Since each vertex of a GPR graph of a directly regular or chiral polytope $\mathcal{K}$ corresponds to a right coset of $\Gamma(\mathcal{K})$, it follows

Corollary 2.5. *Let G be a connected GPR graph with m vertices of a directly regular or chiral polytope $\mathcal{K}$. Then m divides the order of $\Gamma(\mathcal{K})$.*

Note that if G is not connected m needs not divide the order of $\Gamma(\mathcal{K})$. For instance, a 2-cycle and a 3-cycle of the same color, disjoint to each other represent a GPR graph of the hexagon, however 5 does not divide 12.

Corollary 2.6. *Let G be a connected GPR graph of a directly regular or chiral polytope $\mathcal{K}$. Then G is induced by the embedding of $\Gamma(\mathcal{K})$ into the symmetric group on a family $\mathcal{F} = B_1, \dots, B_s$ of sets of flags such that*

- $B_i \cap B_j = \varnothing$ *if* $i \neq j$,
- $\cup_i B_i$ *is the flag set of* $\mathcal{K}$, *and*
- *for any* $i \in \{1, \dots, s\}, k \in \{1, \dots, d-1\}$ *there exists* $j \in \{1, \dots, s\}$ *such that* σ_k *sends every flag of* B_i *into* B_j. *In other words,* $\Gamma(\mathcal{K})$ *induces an action on* $\mathcal{F}$.

Proof. Let $\Lambda = Stab_{\Gamma(\mathcal{K})}(u)$ and let $\Lambda\phi_1, \dots \Lambda\phi_k$ be the right cosets of Λ. By Proposition 2.4 we may identify the vertices of G with the right cosets of Λ.

Given a base flag f of $\mathcal{K}$ we now identify the right coset $\Lambda\phi_i$ with the set of flags given by $f\Lambda\phi_i$ obtaining the desired sets $\mathcal{F} = B_1, \dots, B_k$. □

The following proposition relates a polytope $\mathcal{K}$ with the automorphism group as a labeled graph of a given GPR graph of $\mathcal{K}$. The proof is analogous to that of [13, Proposition 3.7].

Proposition 2.7. *Let G be a GPR graph of a polytope $\mathcal{K}$. Assume that Λ is the automorphism group of G as a labeled graph, and for any vertex v of G let O_v be the orbit of v under Λ. We define the quotient graph G/Λ by*

- $V(G/\Lambda) = \{O_v \mid v \in V(G)\}$, *and*
- *there is an arrow with label i from O_v to O_w if and only if there exist $x \in O_v$ and $y \in O_w$ such that there is an arrow with label i from x to y in G.*

If G/Λ is again a GPR graph of a polytope $\mathcal{P}$, then $\mathcal{P}$ is the quotient of $\mathcal{K}$ determined by the normal subgroup

$$N = \{\phi \in \Gamma(\mathcal{K}) \mid v\phi \in O_v \text{ for every } v \in V(G)\}.$$

We now give conditions for a GPR graph G to admit central involutions in a polytope $\mathcal{K}$. The proof follows directly from the action of $\Gamma(\mathcal{K})$ on the vertex set of G.

Proposition 2.8. *Let $\mathcal{K}$ be a directly regular or chiral polytope and G be a GPR graph of $\mathcal{K}$. Any central involution ϕ in $\Gamma(\mathcal{K})$ can be seen in G as a matching M on the vertex set such that*

- *for each connected component C of G, ϕ induces a perfect matching M_C on C, or acts like ε.*
- *each connected component of the subgraph of G induced by M and the arrows of label i is either a single vertex, a single edge in M, an oriented cycle labeled i, an oriented cycle of even length labeled i with its main diagonals in M, or a pair of oriented cycles of the same length joined by edges in M as Figure 5 shows.*

Given $i \in \{1, \dots, d-1\}$, we shall also require conditions for an involution α such that $\alpha\sigma_i\alpha = \sigma_i^{-1}$. It is not hard to see that in this case, the connected

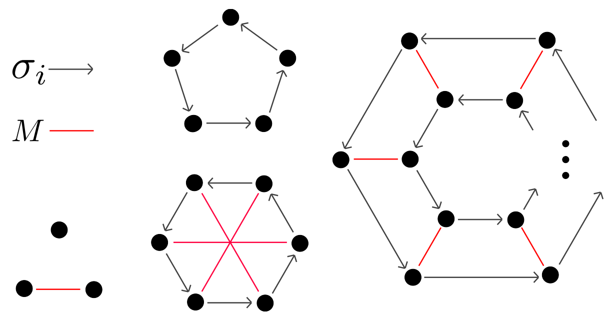

Figure 5: Central involution

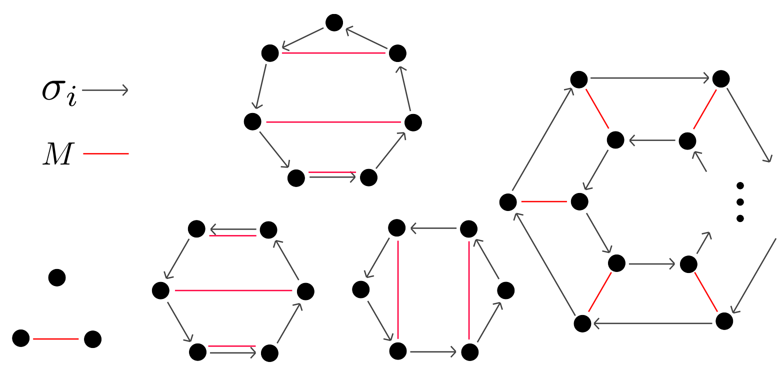

Figure 6: Matching M such that $\alpha\sigma_i\alpha = \sigma_i^{-1}$

components of the graph determined by all arrows with label i and the matching M corresponding to α must be one of the types in Figure 6.

To conclude this section we turn our attention to polyhedra. We first prove a useful criterion to determine that the intersection condition holds in certain connected (directed multi-) graphs in two colors. In order to do that we shall make use of the following lemmas.

Lemma 2.9. *The cyclic structure of the k-th power of the permutation $(a_1\ a_2\ \cdots\ a_s)$ consists of (s, k) cycles, each of them of length $s/(s, k)$.*

Lemma 2.10. *Let G be a finite connected (multi)digraph with arrows labeled with the colors $\{1, 2\}$ in such a way that the arrows of each color form disjoint cycles and every vertex belongs to one of the local configurations in Figure 4.*

Let σ_1 and σ_2 be the permutations of the vertices of G defined by $u\sigma_i = v$ if and only if there is an arrow of color i from u to v, and let $\phi \in \langle\sigma_1\rangle \cap \langle\sigma_2\rangle$.

Then the cyclic structure of ϕ as a permutation on the vertices of G consists of disjoint k-cycles for certain fixed k (this implies that no vertex is fixed if $\phi \neq \varepsilon$).

Proof. Let u be a vertex in a k-cycle in the cyclic structure of ϕ as a permutation on the vertex set of G. Assume that v is a vertex in an s-cycle of the cyclic structure of ϕ for some $s \neq k$. Since G is connected, there is a path (not necessarily oriented) $u = w_1, w_2, \ldots, w_l = v$ in the underlying graph of G. Let i be the first integer such that w_i is not in a k-cycle in the cyclic structure of ϕ. Since G is connected, there is an arrow of label j between vertices w_i and w_{i-1}. This implies that w_i and w_{i-1} are in the same connected component C of the subgraph of G induced by the arrows of label j. By Lemma 2.9, C splits into cycles of the same length when a power of σ_j (say ϕ) acts on it. This contradicts the definition of w_i and hence ϕ induces only k-cycles on the vertex set of G. □

The following proposition gives the criterion for connected digraphs to be GPR graphs of regular or chiral polyhedra. See [13, Theorem 4.3] for the analog for CPR graphs.

Proposition 2.11. *Let G be a finite connected (multi)digraph with arrows labeled with the colors $\{1, 2\}$ in such a way that the arrows of each color form disjoint cycles and every vertex belongs to one of the local configurations in Figure 4.*

If G has a connected component C with m vertices of color i and a connected component D of color j with n vertices (i may or may not be equal to j) then the group satisfies the intersection property with respect to the generators σ_1 and σ_2, and hence, G is a GPR graph of a regular or chiral polyhedron.

Proof. Let $\phi \in \langle\sigma_1\rangle \cap \langle\sigma_2\rangle$. Then $\phi = \sigma_1^{k_1} = \sigma_2^{k_2}$. Lemma 2.9 implies that ϕ acts in C like a product of disjoint cycles of the same length l_1, while it acts in D like a product of disjoint cycles of the same length l_2, with l_1 dividing m and l_2 dividing n. Lemma 2.10 imply that $l_1 = l_2$. Since $(m, n) = 1$, then $l_1 = l_2 = 1$ and, by Lemma 2.10, ϕ acts on any vertex of G like ε. □

In some cases it is easy to determine that a given GPR graph belongs to a chiral polyhedron, as the following proposition shows.

Proposition 2.12. *Let G be a GPR graph of a directly regular or chiral polyhedron $\mathcal{K}$, and let G' be the graph obtained from G by reversing the direction of all arrows. If $G \cong G'$ then $\mathcal{K}$ is regular.*

Proof. This follows directly from (1.6). □

The converse of Proposition 2.12 is false. For instance, the graph at the right of Figure 7 is obtained from the graph at the left by reversing the arrows. They are not isomorphic as graphs, but they both generate the same polyhedron, that is $\{4, 5|\,5\}$ (see [5, Polytope $\{4, 5\}$*720]).

Since the automorphism group of a 2-orbit polyhedron $\mathcal{K}$ in class 2_0 is generated by ρ_0 and σ_2 we may define a permutation representation graph for $\mathcal{K}$ in a similar way as we did for chiral and regular polytopes. In this case, ρ_0 will be represented by a non-oriented matching as was the case in

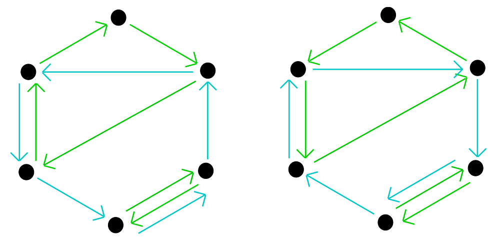

Figure 7: The graphs are not isomorphic but the polytope is regular

CPR graphs, whereas σ_2 will be represented by a set of directed cycles as was the case in GPR graphs.

Conversely, given a vertex set V together with a matching and a set of directed cycles on V, the induced graph will be a permutation representation graph of a polyhedron in class 2_0 or of an index 2 subgroup of the automorphism group of a regular polyhedron whenever the intersection condition (1.8) holds.

Note that if $\mathcal{K}$ is a polyhedron in class 2_0 and $\mathcal{P}$ is its Petrial (hence chiral), then for any permutation representation graph G of $\mathcal{K}$ we can construct a GPR graph H of $\mathcal{P}$ with the same vertex set in the following way. The vertex set of H is the vertex set of G, the arrows corresponding to σ_2 in G are the arrows corresponding to σ_2 in H, and there is an arrow corresponding to σ_1 from vertex u to vertex v in H if and only if $v = u\rho_0\sigma_2^{-1}$ in G.

3. Chiral polyhedra for alternating and symmetric groups

In this section we consider chiral polytopes with small alternating or symmetric automorphism groups. We start with some preliminary results.

Proposition 3.1. *Let G be a GPR graph with m vertices of a directly regular or chiral d-polytope $\mathcal{K}$ ($d \geq 2$) such that its automorphism group as a labeled digraph is not trivial. Then $\Gamma(\mathcal{K})$ is neither S_m nor A_m.*

Proof. We may assume that G is connected. In fact, if the connected components of G have $n_1, \dots, n_k$ vertices then $\Gamma(\mathcal{K}) \leq S_{n_1} \times \cdots \times S_{n_k}$ which is a proper subgroup of $S_{V(G)}$. Consequently, for every two vertices u, v of G there is an automorphism of $\mathcal{K}$ that maps u into v.

Let Λ be the automorphism group of G as a labeled digraph, let u, v be two vertices in the same orbit under Λ, and let $\lambda \in \Lambda$ such that $(u)\lambda = v$.

For any $\phi \in \Gamma(\mathcal{K})$ such that $(u)\phi = v$ we claim that $(v)\phi = (v)\lambda$. In fact, $(v)\phi$ is obtained from v by following the sequence of arrows used to determine that $(u)\phi = v$. Since λ is an automorphism of G, that sequence of arrows will take v to $(v)\lambda$.

Hence, for any vertex w of G such that $(v)\lambda \neq w$, the 3-cycle $(u\ v\ w)$ does not correspond to the action of any element of $\Gamma(\mathcal{K})$, and $\Gamma(\mathcal{K})$ is not isomorphic to the symmetric or alternating group on the vertex set of G. $\square$

For example, when $\mathcal{K}$ is a triangle the GPR graph determined by the natural embedding of $\Gamma(\mathcal{K})$ on the symmetric group of the flags of $\mathcal{K}$ is an alternating cycle with six vertices, three edges labeled 0 and three edges labeled 1. The automorphism group of this graph (as a labeled graph) is not trivial, and thus, $\Gamma(\mathcal{K})$ cannot be isomorphic to S_6. Note that this does not contradict that $\Gamma(\mathcal{K}) \cong S_3$, since the GPR graph determined by the embedding of $\Gamma(\mathcal{K})$ on the symmetric group on the vertices (or edges) of the triangle is an alternating path with three vertices, one edge labeled 0 and one edge labeled 1, and such a path has no non-trivial automorphisms as a labeled graph.

We shall make use of the following, rather technical lemmas, proofs of which are readily available.

Lemma 3.2. *The set of involutions* $\{(j,m) \,|\, j \in \{1,\ldots,n\} \setminus \{m\}\}$ *determines a generating set for the symmetric group* S_n*.*

Lemma 3.3. *Every automorphism of* S_n*,* $n \neq 2,6$ *is an inner automorphism.*

Figure 7 shows that Proposition 2.12 proves short to determine when the polytope $\mathcal{K}$ described by a given GPR graph G is chiral. When the group induced by G is symmetric on the vertex set of G, the regularity or chirality of $\mathcal{K}$ can be totally determined by the following criterion.

Proposition 3.4. *Let* $\mathcal{K}$ *be a regular or chiral polyhedron and let* G *be a GPR graph with at least* 7 *vertices of* $\mathcal{K}$ *such that* $\Gamma(\mathcal{K})$ *induces the symmetric group on the vertices of* G*.*

Then, $\mathcal{K}$ *is regular if and only if there exists an involutory permutation* α *of the vertices of* G *such that the mapping* h *on the arrow set of* G *given by* $(u,v)h = (u\alpha, v\alpha)$ *sends* σ_i *to* σ_i^{-1}*,* $i = 1,2$*, that is, sends* G *to* G'*, where* G' *is obtained from* G *by reversing all arrows.*

Proof. Any automorphism ψ of $\Gamma(\mathcal{K})$ that maps σ_i to σ_i^{-1}, $i = 1,2$ is an involution since

$$(\sigma_i)\psi^2 = \sigma_i^{-1}\psi = (\sigma_i\psi)^{-1} = (\sigma_i^{-1})^{-1} = \sigma_i.$$

Lemma 3.3 implies now that such a ψ exists if and only if it is the inner automorphism determined by conjugating with an involution α in S_V, where V is the vertex set of G. Now the proposition follows from (1.6). □

It follows from Propositon 3.4 that a GPR graph in two colors which induces the symmetric group on its vertex set V, corresponds to a regular polyhedron if and only if there is a perfect matching M in V such that, for each $i = 1,2$, the connected components of the subgraph induced by the edges of label i and M are like those in Figure 5. In some cases it is easy to determine if the associated polyhedron is chiral or regular using this fact.

For each $n \leq 7$ we conducted an exhaustive search for chiral polyhedra with automorphism group isomorphic to A_n and S_n. The idea is to assign a

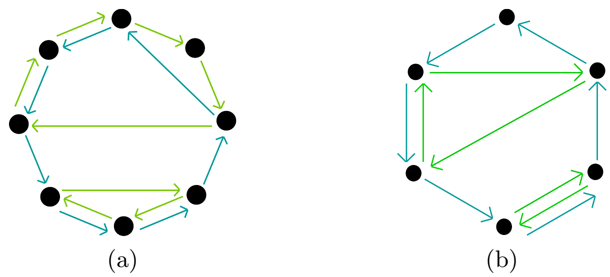

Figure 8: Automorphism groups A_8 and S_6

Schläfli type first and try to construct a GPR graph on n vertices in such a way that the graph generates A_n or S_n. For $n = 4$ we consider only the Schläfli numbers 3 and 4 since no permutation on n elements has greater order. For $n = 5, 6$ we also consider the numbers 5 and 6. For $n = 7$ we also consider the numbers $7, 10$ and 12. An easy calculation will show that those are the only possible orders of elements in S_n for each n.

We found no chiral polyhedra with automorphism group isomorphic to A_n, $n \leq 7$ (compare to [10]). The graph in Figure 8(a) corresponds to a chiral polyhedron with automorphism group isomorphic to A_8. Since the group of the polyhedron is alternating we can no longer apply Proposition 3.4. However, using a computer algebra program such as [4], it can be confirmed that the polyhedron is indeed chiral. Alternatively, one can use the fact that the automorphism group of A_n is S_n for $n \neq 2, 3, 6$.

We found no chiral polyhedra with automorphism groups isomorphic to S_4 or S_5. Figure 8(b) shows the unique chiral polyhedron (up to enantiomorphism) with automorphism group isomorphic to S_6 (see also [1, C(61.7)]). The list of all GPR graphs of chiral polyhedra (up to enantiomorphism) with automorphism group isomorphic to S_7 is in Figure 9. In Table 1 we give the Schläfli types for these polyhedra.

Table 1: Polyhedra with automorphism group S_7

Polyhedron	**Schläfli type**
(a)	{4, 6}
(b)	{4, 6}
(c)	{5, 6}
(d)	{5, 12}
(e)	{6, 6}
(f)	{6, 6}
(g)	{6, 7}

Polyhedron	**Schläfli type**
(h)	{6, 7}
(i)	{6, 7}
(j)	{6, 10}
(k)	{6, 12}
(l)	{7, 10}
(m)	{10, 10}

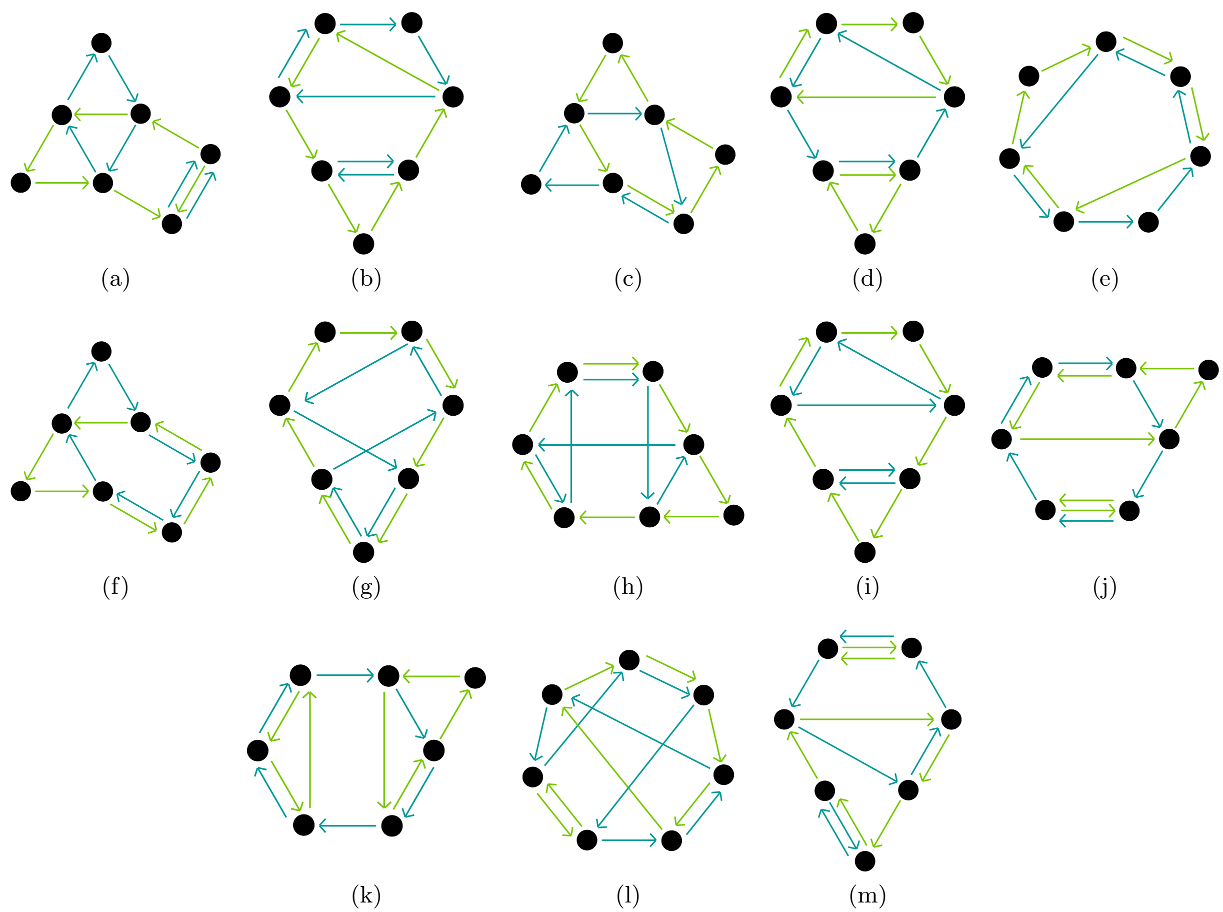

Figure 9: Automorphism group S_7

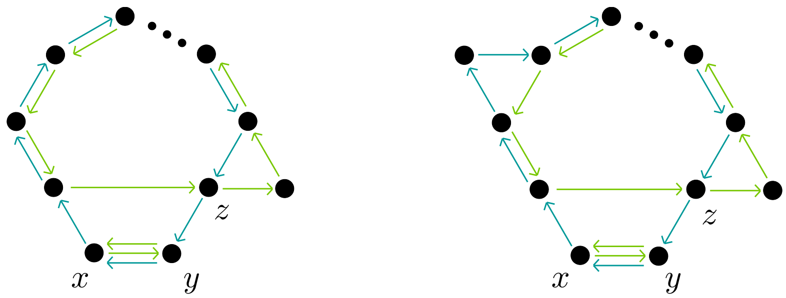

Figure 10: Automorphism group S_n

For each $n \geq 8$ there is at least one chiral polyhedron with automorphism group isomorphic to S_n. For n odd (resp. even) one such polyhedron is given by the GPR graph in Figure 10(a) (resp. (b)) (here n is the number of vertices of the graph). Proposition 2.11 can be used to prove that the intersection property holds. To see that the group generated by graph (a) is the symmetric group on its vertex set we make use of Lemma 3.2. Consider $\phi = \sigma_2^{n-2}$ for n odd and $\phi = \sigma_2^{n-3}$ for n even. Then ϕ acts on the graph like the transposition that interchanges the vertices x and y and $\psi = \sigma_1 \phi \sigma_1^{-1}$ is the transposition that interchanges the vertices y and z. Conjugating ψ with powers of σ_2 we get the remaining transpositions of type $(y\ w)$. Finally, it can be derived from the comments following Proposition 3.4 that the polytope described is chiral. A similar argument shows that graph (b) also yields a chiral polyhedron with symmetric automorphism group. Note that for Figure 10(a) with n even, and for the graph (b) in the same figure with n odd, the argument above fails to determine whether the group generated is S_n.

In order to construct other GPR graphs of perhaps different chiral polyhedra with symmetric automorphism groups it is possible to make further modifications to the graphs in Figure 10 by applying one or more times the following step. Consider two vertices $u, v \notin \{x, y\}$ of the graph such that there is a blue arrow from u to v and a green arrow from v to u. We may add a vertex w to the graph, and replace the green arrow between u and v by green arrows from v to w and from w to u. The only restrictions we need to preserve are:

- the green cycle not containing x and y must have an odd number of vertices,
- there is no matching of the graph such that satisfies the conditions of Proposition 3.4 (see comments following Proposition 3.4 and Figure 6).

The graph (a) in Figure 10 can be seen as the graph (j) in Figure 9 after replacing a double arrow with colors blue and green with a sequence of odd

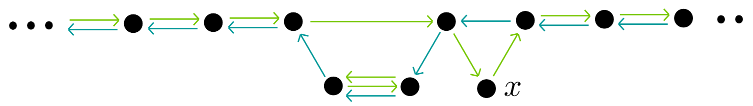

Figure 11: GPR graph of an infinite chiral polyhedron

length of such double arrows. It is possible to generate more infinite families of GPR graphs of chiral polyhedra with symmetric automorphism group by modifying other graphs in Figure 9 in the same way (for instance graphs (i), (l) and (m)).

4. Infinite chiral polyhedra

In this section we construct infinite chiral polyhedra. We provide three examples of them, one with infinite facets and vertex-figures, one with finite facets and infinite vertex-figures, and one with finite facets and vertex-figures.

Observe first that the GPR graph (a) in Figure 10 can be generalized to the infinite GPR graph in Figure 11. We claim that this is a GPR graph of an infinite chiral polyhedra. The intersection condition holds because any power of σ_1 fixes vertex x, while any non-trivial power of σ_2 moves it. It is chiral because the elements $\sigma_1^2\sigma_2^4\sigma_1^3\sigma_2$ and $\sigma_1^3\sigma_2^4\sigma_1^2\sigma_2 = (\sigma_2^{-1}(\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{-3}\sigma_2^{-1})\sigma_2)^{-1}$ have different order, proving that interchanging σ_i with σ_i^{-1} for $i = 1, 2$ does not yield an automorphism of the group determined by the graph. Note that both the facets and vertex figures of the polyhedron are infinite.

For simplicity, to construct examples with finite facets or vertex-figures we first construct (a permutation representation graph of) their Petrials, which are 2-orbit polyhedra in class 2_0. The edges corresponding to the generator ρ_0 will be colored blue whereas the arrows corresponding to the generator σ_2 will be colored green.

Figure 12 (page 96) shows a permutation representation graph of a polyhedron in class 2_0. It is not hard to see from the graph that the intersection condition (1.8) holds. Moreover, since the relation

$$\rho_0\sigma_2\rho_0\sigma_2^{-1}\rho_0\sigma_2^{-2}\rho_0\sigma_2^2$$

has order 15, whereas the relation

$$\rho_0\sigma_2^{-1}\rho_0\sigma_2\rho_0\sigma_2^2\rho_0\sigma_2^{-2}$$

has order 60, the graph does not yield a regular polyhedron because no automorphism of the induced group will preserve ρ_0 while interchanging σ_2 with σ_2^{-1}. Taking the Petrial polyhedron by defining $\sigma_1' := \rho_0\sigma_2^{-1}$ and

$\sigma_2' := \sigma_2$, we obtain a chiral infinite polyhedron with Schläfli type $\{\infty, 6\}$ and σ_1', σ_2' as canonical generators of its automorphism group. Note that the intersection condition holds here simply because one of the entries in the Schläfli type is finite while the other is infinite.

The upper graph in Figure 13 is a section of a permutation representation graph of a polyhedron in class 2_0 with schläfli type $\{12, 4\}$. The entire permutation representation graph is derived by constructing the section between vertices u_i and v_i for every integer i and by joining the vertices v_{i-1}, u_i by an edge (colored blue) corresponding to ρ_0 as shown in the upper part of Figure 13. The intersection condition (1.8) can be proved by comparing the connected components of the subgraph induced by the arrows corresponding to σ_2 with those of the subgraph induced by the matchings ρ_0 and $\sigma_2\rho_0\sigma_2^{-1}$. It is easy to see that ε is the only element in $\langle\sigma_2\rangle \cap \langle\rho_0, \sigma_2\rho_0\sigma_2^{-1}\rangle$ fixing each of such connected components. The permutation $(\rho_0\sigma_2)^3\sigma_2\rho_0\sigma_2\rho_0\sigma_2^{-1}$ has order 4 while the permutation $(\rho_0\sigma_2^{-1})^3\sigma_2^{-1}\rho_0\sigma_2^{-1}\rho_0\sigma_2$ has order 6 implying that the derived polyhedron is not regular. The (chiral) Petrial of this polyhedron has Schläfli type $\{8, 4\}$.

The lower part of Figure 13 shows two 8-cycles corresponding to σ_1 in the graph of the chiral polyhedron. Note that any 8-cycle induced by σ_1 on the vertices of the graph is a translate of one of the 8-cycles of Figure 13. The intersection condition $\langle\sigma_1\rangle \cap \langle\sigma_2\rangle = \{\varepsilon\}$ can be easily proved from the graph.

5. Higher rank chiral polytopes

In [14] the first author constructs CPR graphs of regular polytopes with preassigned facet types. In this section we give a technique for constructing a graph which, if satisfying the intersection condition, is a GPR graph of a directly regular or chiral $(d+1)$-polytope $\mathcal{P}$ with preassigned facets isomorphic to a given d-polytope $\mathcal{K}$. More precisely we construct a GPR graph G for $\mathcal{P}$ such that the graph resulting from erasing the arrows of color d from G (assigned to σ_d), correspond to a disconnected GPR graph of $\mathcal{K}$. We illustrate this technique for $d = 3, 4$.

The directed paths consisting of arrows of colors $i, i+1, \ldots, d, i, i+1, \ldots, d$ in G must be closed paths since, for every $i < d$, the relation

$$(\sigma_i \cdots \sigma_d)^2 = \varepsilon \tag{5.1}$$

must be satisfied. This is not immediately obvious from the graph and suggests we should work with $\tau_{i,j} := \sigma_i\sigma_{i+1}\cdots\sigma_j$ as generators when considering chiral $(d+1)$-polytopes. In fact, the automorphism group of any chiral polytope of rank $n \geq 4$ can easily be seen to be generated by the involutions $\tau_{i,j}$, which are represented in permutation representation graphs as (undirected) matchings.

As an example, the universal polytope

$$\{\{\{3,4\},\{4,4\}_{(2,1)}\},\{\{4,4\}_{(2,1)},\{4,6|2\}\}\}$$

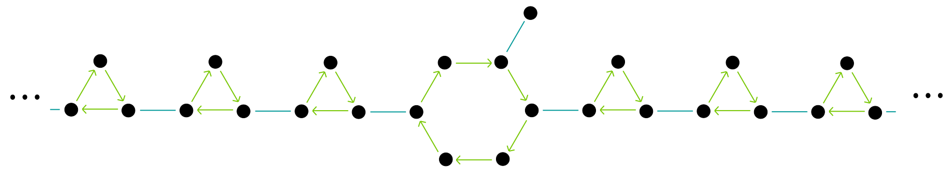

Figure 12: Permutation representation graph of an infinite 2-orbit polyhedron in class 2_0

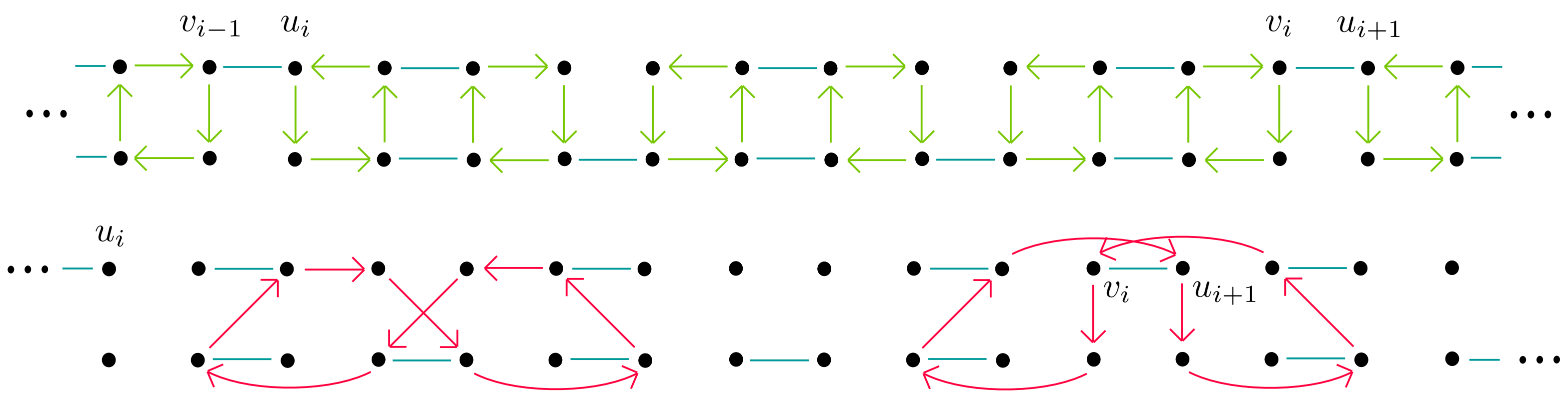

Figure 13: Permutation representation graph of an infinite 2-orbit polyhedron in class 2_0

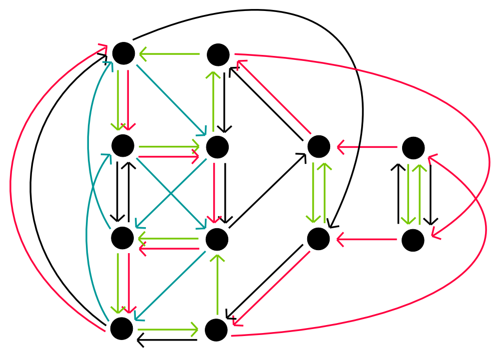

Figure 14: The polytope $\{\{\{3,4\},\{4,4\}_{(2,1)}\},\{\{4,4\}_{(2,1)},\{4,6|2\}\}\}$

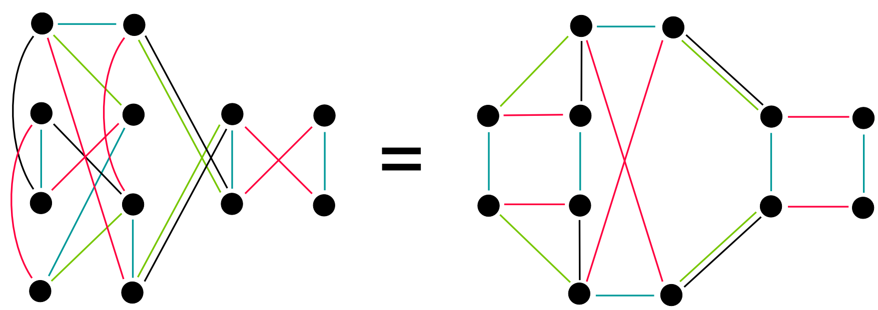

Figure 15: The polytope $\{\{\{3,4\},\{4,4\}_{(2,1)}\},\{\{4,4\}_{(2,1)},\{4,6|2\}\}\}$

described by Conder, Hubard and Pisanski in [3] allows the graphs shown in Figures 14 and 15. In Figure 14, σ_1, σ_2, σ_3 and σ_4 are represented by arrows of color blue, green, black and red, respectively. In Figure 15, $\tau_{1,2}$, $\tau_{1,3}$, $\tau_{2,3}$ and $\tau_{3,4}$ are represented by matchings of colors blue, green, black and red, respectively.

For our purposes it is convenient to consider a generating set of $\Gamma(\mathcal{K})$ consisting of some abstract rotations σ_i and some abstract half-turns $\tau_{i,j}$. The following lemma will help to determine a useful generating set for $\Gamma(\mathcal{K})$ in order to easily verify relations (5.1).

Lemma 5.1. *Let $\mathcal{P}$ be a chiral $(d+1)$-polytope with $d \geq 3$. Then*

$$(5.2) \qquad \Gamma(\mathcal{P}) = \begin{cases} \langle \sigma_1, \tau_{2,3}, \sigma_3, \tau_{4,5}, \ldots, \sigma_{d-4}, \\ \qquad \tau_{d-3,d-2}, \sigma_{d-2}, \sigma_{d-1}, \tau_{d-1,d} \rangle & \text{if } d+1 \text{ is even} \\ \langle \tau_{1,2}, \sigma_2, \tau_{3,4}, \sigma_4, \ldots, \sigma_{d-4}, \\ \qquad \tau_{d-3,d-2}, \sigma_{d-2}, \sigma_{d-1}, \tau_{d-1,d} \rangle & \text{if } d+1 \text{ is odd.} \end{cases}$$

Moreover, the set of relations $(\sigma_i \cdots \sigma_d)^2 = \varepsilon$ *for* $i < d-1$ *is equivalent to the following set of relations on the generators given in (5.2):*

$$(\sigma_{d-2}\tau_{d-1,d})^2 = \tau_{d-1,d}^2 = \varepsilon, \tag{5.3}$$

$$(\tau_{d-3,d-2}\tau_{d-1,d})^2 = \varepsilon \quad (\textit{if } d \geq 4) \tag{5.4}$$

$$\sigma_j\tau_{d-1,d} = \tau_{d-1,d}\sigma_j, \quad \textit{and} \quad (\tau_{j-1,j}\tau_{d-1,d})^2 = \varepsilon \tag{5.5}$$

for $j < d-2$*,* $d-j$ *even.*

Proof. First note that, $\sigma_{i-1} = \tau_{i-1,i}\sigma_i^{-1}$ for any i implying (5.2).

Relation (5.3) is equivalent to $(\sigma_{d-2}\sigma_{d-1}\sigma_d)^2 = (\sigma_{d-1}\sigma_d)^2 = \varepsilon$ while (5.4) is equivalent to $(\sigma_{d-3}\sigma_{d-2}\sigma_{d-1}\sigma_d)^2 = \varepsilon$.

To prove (5.5) we first assume that $(\sigma_i \cdots \sigma_d)^2 = \varepsilon$ for every $i \leq d-1$ and let $j < d-2$ with $d-j$ even. Then Corollary 1.1 implies that

$$\begin{aligned} (\tau_{j-1,j}\tau_{d-1,d})^2 &= (\sigma_{j-1}\sigma_j\sigma_{d-1}\sigma_d)^2 \\ &= \sigma_{j-1}\sigma_j(\sigma_{j+1}\cdots\sigma_{d-2})^2\sigma_{d-1}\sigma_d\sigma_{j-1}\sigma_j\sigma_{d-1}\sigma_d \\ &= \sigma_{j-1}\cdots\sigma_{d-2}\sigma_d^{-1}\sigma_{d-1}^{-1}\cdots\sigma_{j+1}^{-1}\sigma_{j-1}\sigma_j\sigma_{d-1}\sigma_d \\ &= \sigma_{j-1}\cdots\sigma_{d-2}\sigma_{d-1}\sigma_d\sigma_{d-2}^{-1}\sigma_{d-3}^{-1}\cdots\sigma_{j+1}^{-1}\sigma_j^{-1}\sigma_{j-1}^{-1}\sigma_{d-1}\sigma_d \\ &= \sigma_{j-1}\cdots\sigma_d\sigma_{j-1}\cdots\sigma_d = \varepsilon, \end{aligned}$$

and

$$\begin{aligned} \sigma_j\tau_{d-1,d} &= \sigma_j(\sigma_{j+1}\cdots\sigma_{d-2})^2\sigma_{d-1}\sigma_d = \sigma_j\cdots\sigma_{d-2}\sigma_d^{-1}\sigma_{d-1}^{-1}\cdots\sigma_{j+1}^{-1} \\ &= \sigma_j\cdots\sigma_d\sigma_{d-2}^{-1}\sigma_{d-3}^{-1}\cdots\sigma_{j+1}^{-1} = \sigma_d^{-1}\cdots\sigma_j^{-1}\sigma_{d-2}^{-1}\cdots\sigma_{j+1}^{-1} \\ &= \sigma_{d-1}\sigma_d\sigma_j\sigma_{j+1}\cdots\sigma_{d-2}\sigma_{d-2}^{-1}\cdots\sigma_{j+1}^{-1} = \sigma_{d-1}\sigma_d\sigma_j = \tau_{d-1,d}\sigma_j. \end{aligned}$$

We proceed by (backward) induction to prove that if relations (5.3), (5.4), and (5.5) hold then $(\sigma_i \cdots \sigma_d)^2 = \varepsilon$. Assume that $(\sigma_j \cdots \sigma_d)^2 = \varepsilon$ for $j > i$ then Corollary 1.1 imples that, for $d-i$ even,

$$\begin{aligned} (\sigma_i \cdots \sigma_d)^2 &= \sigma_i\cdots\sigma_{d-2}\sigma_i\sigma_{d-1}\sigma_d\sigma_{i+1}\sigma_{i+2}\cdots\sigma_d \\ &= \sigma_{i+1}\cdots\sigma_{d-2}\sigma_{d-1}\sigma_d\sigma_{i+1}\cdots\sigma_d, \end{aligned}$$

which, by the inductive hypothesis, equals ε. A similar argument follows when $d-i$ is odd. □

Lemma 5.1 suggests the generating set of $\Gamma(\mathcal{K})$ we shall employ in the reminder of this section to construct the polytope $\mathcal{P}$. For example, G must be given in terms of σ_1, σ_2 if $\mathcal{K}$ has rank 3, and in terms of $\tau_{1,2}, \sigma_2, \sigma_3$ if $\mathcal{K}$ has rank 4. Then we only need to add a matching representing $\tau_{2,3}$ if $\mathcal{K}$ has rank 3, and representing $\tau_{3,4}$ if $\mathcal{K}$ has rank 4, in such a way that relations (5.3), (5.4) and (5.5) hold. We recall that relation $(\sigma_{d-2}\tau_{d-1,d})^2 = \varepsilon$ implies that every connected component determined by arrows corresponding to σ_{d-2} and edges corresponding to $\tau_{d-1,d}$ are as Figure 6 illustrates. Relation $\sigma_j\tau_{d-1,d} = \tau_{d-1,d}\sigma_j$ implies that every connected component determined by arrows corresponding to σ_j and edges corresponding to $\tau_{d-1,d}$ are like

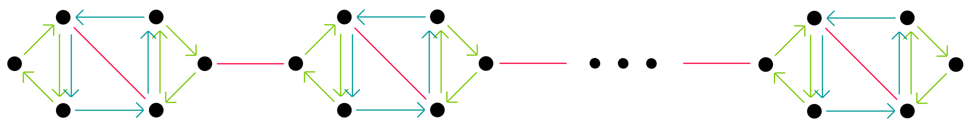

Figure 16: GPR graph of chiral 4-polytopes

those in Figure 5. Finally, as a consequence of relation $(\tau_{j-1,j}\tau_{d-1,d})^2$, each connected component determined by the edges corresponding to $\tau_{j-1,j}$ and $\tau_{d-1,d}$ must be one of those in Figure 2.

5.1. **Rank 4.** Here we construct an infinite family of chiral 4-polytopes with cubical facets obtained from disjoint GPR graphs of cubes by joining them with a matching corresponding to the involution $\tau_{2,3}$ as described above. Note that Lemma 5.1 implies that the only relation we need to preserve is

$$(\sigma_1\tau_{2,3})^2 = \varepsilon.$$

We consider q copies of the GPR graph of the cube in Figure 3 and add a (red) matching corresponding to $\tau_{2,3}$ in such a way that the connected components determined by the arrows corresponding to σ_1 and the edges corresponding to $\tau_{2,3}$ are like those in Figure 6. We show a graph G thus obtained in Figure 16. Figure 17 shows a GPR graph of the same polytope represented by a different set $\{\sigma_1, \sigma_2, \sigma_3\}$ of generators of its automorphism group.

We proceed to describe the group $\langle\sigma_2, \sigma_3\rangle$. This will be used in the proof of Proposition 5.2 for the intersection condition.

We first label the vertices as indicated in Figure 17 and let

$$A = \{a_i, d_i \,|\, i = 1, \ldots, q\}, \quad B = \{b_i, e_i \,|\, i = 1, \ldots, q\},$$
$$\text{and } C = \{c_i, f_i \,|\, i = 1, \ldots, q\}.$$

Observe that $\sigma_1, \sigma_2, \sigma_3$ have a well defined action on the set $\{A, B, C\}$ indicated in Table 2.

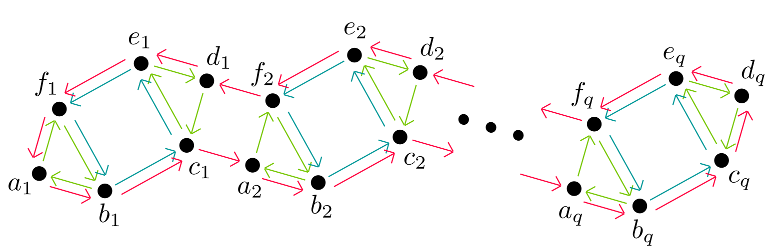

Figure 17: GPR graph of chiral 4-polytopes

Table 2: Action of $\sigma_1, \sigma_2, \sigma_3$ on $\{A, B, C\}$

Class	σ_1	σ_2	σ_3
A	A	C	B
B	C	A	C
C	B	B	A

In other words, the automorphisms $\sigma_1, \sigma_2, \sigma_3$ induce the permutations $(B, C), (A, C, B), (A, B, C)$ respectively in the set $\{A, B, C\}$.

Note that ε is the only power of σ_3 preserving the vertex set of every connected component determined by the arrows with label 2. This implies that $\langle\sigma_2, \sigma_3\rangle$ satisfies the intersection property $\langle\sigma_2\rangle \cap \langle\sigma_3\rangle = \varepsilon$ and hence it is the rotation subgroup of the automorphism group of a polyhedron. As we shall see, this polyhedron is regular.

Let Λ be the subgroup of $\langle\sigma_2, \sigma_3\rangle$ that fixes A (and therefore Λ fixes the set B and the set C). Since the action of σ_3 on $\{A, B, C\}$ coincides with the action of σ_2^{-1}, it follows that Λ is generated by the words with three letters in $\{\sigma_2^{-1}, \sigma_3\}$, that is, the set $\{\gamma_1, \dots, \gamma_8\}$ is a generating set for Λ where

$$\begin{aligned} \gamma_1 &:= \sigma_3^3, & \gamma_2 &:= \sigma_3^2\sigma_2^{-1}, \\ \gamma_3 &:= \sigma_3\sigma_2^{-1}\sigma_3, & \gamma_4 &:= \sigma_2^{-1}\sigma_3^2, \\ \gamma_5 &:= \sigma_2\sigma_3, & \gamma_6 &:= \sigma_2^{-1}\sigma_3\sigma_2^{-1}, \\ \gamma_7 &:= \sigma_3\sigma_2 & \gamma_8 &:= \sigma_2^{-3} = \varepsilon. \end{aligned}$$

In terms of permutations of the vertices of G we have

$$\begin{aligned} \gamma_1 &= (a_1, \dots, a_q, d_q, \dots, d_1)(b_1, \dots, b_q, e_q, \dots, e_1)(c_1, \dots, c_q, f_q, \dots, f_1), \\ \gamma_2 &= (a_1, d_1) \cdots (a_q, d_q)(b_1, \dots, b_q, e_q, \dots, e_1)(c_1, f_2) \cdots (c_{q-1}, f_q), \\ \gamma_3 &= (a_2, d_1) \cdots (a_q d_{q-1})(b_1, e_1) \cdots (b_q, e_q)(c_1, \dots, c_q, f_q, \dots, f_1), \\ \gamma_4 &= (a_1, \dots, a_q, d_q, \dots, d_1)(b_2, e_1) \cdots (b_q, e_{q-1})(c_1, f_1) \cdots (c_q, f_q), \\ \gamma_5 &= (b_1, e_1) \cdots (b_q, e_q)(c_1, f_2) \cdots (c_{q-1}, f_q), \\ \gamma_6 &= (a_1, d_1) \cdots (a_q, d_q)(b_2, e_1) \cdots (b_q, e_{q-1}), \\ \gamma_7 &= (a_2, d_1) \cdots (a_q d_{q-1})(c_1, f_1) \cdots (c_q, f_q). \end{aligned}$$

Note that the involution resulting from the action of γ_2 on A is not a conjugate of the involution resulting from the action of γ_2 on C, since they have a different number of transpositions in their cyclic structure. Similarly, for each $i \geq 3$, γ_i acts on two of the sets in $\{A, B, C\}$ like two involutions which are not conjugates of each other. On the other hand, γ_1 does not act as an involution in either A, B or C. Since

$$(a_1, d_1) \cdots (a_q, d_q) \cdot (a_2, d_1) \cdots (a_q d_{q-1}) = (a_1, \ldots, a_q, d_q, \ldots, d_1),$$

we see that the group Λ restricted to A acts like a dihedral group D_{2q} of order $4q$ with involutory generators $(a_1, d_1) \cdots (a_q, d_q)$ and $(a_2, d_1) \cdots (a_q d_{q-1})$. A similar argument shows that Λ restricted to B (or to C) acts like a dihedral group with analogous involutory generators. Consequently, Λ can be seen as a subgroup of D_{2q}^3 and any element λ in Λ can be written as $(\lambda_1, \lambda_2, \lambda_3)$ where λ_1, λ_2 and λ_3 respectively denote the action of λ on the sets A, B, C. With this notation,

$$\begin{aligned}\gamma_2^2 = (\varepsilon, (&b_1, b_3, \ldots b_{2q-1}, e_{2q}, e_{2q-2}, \ldots, e_2)\\ &(b_2, b_4, \ldots, b_{2q}, e_{2q-1}, e_{2q-3}, \ldots, e_1), \varepsilon),\\ \gamma_3^2 = (\varepsilon, \varepsilon, (&c_1, c_3, \ldots c_{2q-1}, f_{2q}, f_{2q-2}, \ldots, f_2)\\ &(c_2, c_4, \ldots, c_{2q}, f_{2q-1}, f_{2q-3}, \ldots, f_1))\end{aligned}$$

and

$$\begin{aligned}\gamma_4^2 = ((&a_1, a_3, \ldots a_{2q-1}, d_{2q}, d_{2q-2}, \ldots, d_2)\\ &(a_2, a_4, \ldots, a_{2q}, d_{2q-1}, d_{2q-3}, \ldots, d_1), \varepsilon, \varepsilon).\end{aligned}$$

Note here that $\gamma_i^2 \gamma_j^2 = \gamma_j^2 \gamma_i^2$ for $i, j \in \{2, 3, 4\}$. Let

$$\Delta = \langle \gamma_2^2, \gamma_3^2, \gamma_4^2 \rangle \cong \mathbb{Z}_q^3. \tag{5.6}$$

An easy calculation will show that $\gamma_i \Delta \neq \gamma_j \Delta$ whenever $i \neq j$. On the other hand, Δ is normal in Λ. Furthermore, the set $\{\gamma_i \Delta \,|\, i = 1, \ldots, 8\}$ forms a subgroup of Λ/Δ. This implies that

$$\Lambda = \{\beta \in D_{2q}^3 \;\;|\;\; \beta\Delta = \gamma_i \Delta \quad \text{for some } i \in \{1, \ldots, 8\}\}, \tag{5.7}$$

and therefore Δ has index 8 in Λ with $\Lambda/\Delta \cong \mathbb{Z}_2^3$. Note here that if $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \gamma_i \Delta$ and $\gamma_i = (\gamma_{i1}, \gamma_{i2}, \gamma_{i3})$ then the cyclic structure of any involutory λ_j coincides with the cyclic structure of γ_{ij} for $j = 1, 2, 3$.

Since Λ is normal in $\langle \sigma_2, \sigma_3 \rangle$ we observe that

$$\langle \sigma_2, \sigma_3 \rangle \cong \Lambda \rtimes \langle \sigma_2 \rangle.$$

Furthermore, an easy calculation will show that $\Lambda = \langle \gamma_2, \gamma_3, \gamma_4 \rangle$ and that

$$\begin{aligned}\gamma_2 \sigma_2 &= \sigma_2 \gamma_4,\\ \gamma_3 \sigma_2 &= \sigma_2 \gamma_2^{-1},\\ \gamma_4 \sigma_2 &= \sigma_2 \gamma_3^{-1}.\end{aligned}$$

From the structure of the group it is not hard to derive now that the relations

$$\begin{aligned}&\sigma_2^3 = \sigma_3^{6q} = (\sigma_2 \sigma_3)^2 = \varepsilon,\\ &\gamma_2 := \sigma_3^2 \sigma_2^{-1},\end{aligned}$$

$$\begin{aligned}
\gamma_3 &:= \sigma_3\sigma_2^{-1}\sigma_3, \gamma_4 := \sigma_2^{-1}\sigma_3^2, \\
\gamma_2^{2q} &= \gamma_3^{2q} = \gamma_4^{2q} = \varepsilon, \\
\gamma_i^2\gamma_j^2 &= \gamma_j^2\gamma_i^2 && \text{(for } i,j = 2,3,4), \\
\gamma_3\gamma_2 &= \gamma_2\gamma_3\gamma_2^2\gamma_3^{-2}\gamma_4^{-2}, \\
\gamma_4\gamma_2 &= \gamma_2\gamma_4\gamma_2^2\gamma_3^{-2}\gamma_4^{-2}, \\
\gamma_4\gamma_3 &= \gamma_3\gamma_4\gamma_2^{-2}\gamma_3^2\gamma_4^{-2}, \\
\gamma_i^2\gamma_j &= \gamma_j\gamma_i^{-2} && \text{(for } i \neq j),
\end{aligned}$$

constitute a set of defining relations for $\langle\sigma_2, \sigma_3\rangle$. An easy calculation now implies that $\langle\sigma_2, \sigma_3\rangle$ has order $24q^3$, and the corresponding analog to (1.6) shows that it is the rotation subgroup of a regular polyhedron.

We now return our attention to the graph in Figure 17.

Proposition 5.2. *The graph in Figure 17 is a GPR graph of a chiral polytope $\mathcal{K}$ with Schläfli type $\{4, 3, 6q\}$, $q \geq 2$, where q is the number of GPR graphs of the cube (connected components determined by blue and green arrows) in G.*

Proof. We have to prove that the group $\langle\sigma_1, \sigma_2, \sigma_3\rangle$ satisfies the intersection condition, and that the induced polytope is not regular.

Lemma 1.2 implies that to prove the intersection condition it suffices to prove

$$\langle\sigma_1, \sigma_2\rangle \cap \langle\sigma_3\rangle = \varepsilon, \text{ and} \tag{5.8}$$

$$\langle\sigma_1, \sigma_2\rangle \cap \langle\sigma_2, \sigma_3\rangle = \langle\sigma_2\rangle. \tag{5.9}$$

Note that ε is the only element in $\langle\sigma_3\rangle$ that fixes the set of vertices of every connected component determined by the arrows labeled 1 and 2. On the other hand, these sets of vertices are preserved by every element in $\langle\sigma_1, \sigma_2\rangle$, implying (5.8).

To prove (5.9), let $\alpha \in \langle\sigma_1, \sigma_2\rangle \cap \langle\sigma_2, \sigma_3\rangle$. We may assume that α fixes the set A defined above, otherwise we substitute α by $\alpha\sigma_2^k$ for some suitable k. This implies that $\alpha \in \Lambda$.

Since $\langle\sigma_1, \sigma_2\rangle$ is the rotation subgroup of the automorphism group of the cube, it follows that the only elements in $\langle\sigma_1, \sigma_2\rangle$ fixing A are $\{\sigma_1^k, \sigma_1^k\sigma_2^{-1}\sigma_1\sigma_2^{-1} \mid k = 0, 1, 2, 3\}$, however, the action on the set $\{A, B, C\}$ of σ_1, σ_1^3, $\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ and of $\sigma_1^2\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ is the transposition (B, C), which cannot be obtained as the action of an element in $\langle\sigma_2, \sigma_3\rangle$. Therefore, α must be either ε, σ_1^2, $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$, or $\sigma_1^3\sigma_2^{-1}\sigma_1\sigma_2^{-1}$. However, σ_1^2 induces the involution $(b_1, e_1)\cdots(b_q, e_q)$ on B and the involution $(c_1, f_1)\cdots(c_q, f_q)$ on C, both having the same cyclic structure. The discussion above implies that there is no index $i \in \{1, \dots, 7\}$ such that $\sigma_1^2\Delta = \gamma_i\Delta$ with Δ defined as in (5.6), and consequently $\sigma_1^2 \notin \Lambda$. Similarly, if $\beta = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ or $\beta = \sigma_1^3\sigma_2^{-1}\sigma_1\sigma_2^{-1}$, its cyclic structure as a permutation on the vertices in A, B and C induces two involutions with the same cyclic structure implying that $\beta \notin \Lambda$. It follows from (5.7) that $\alpha = \varepsilon$ and the intersection condition holds.

To prove that the $\mathcal{K}$ is chiral, note that the order of $\sigma_1^{-1}\sigma_3$ is 4, while the order of $\sigma_1\sigma_3$ is $4q > 4$ if $q \geq 2$. Since $\mathcal{K}$ is regular if and only if $\Gamma(\mathcal{K})$ admits an automorphism interchanging σ_1 and σ_2 with σ_1^{-1} and $\sigma_1^2\sigma_2^{-1}$, while fixing σ_3, we conclude that $\mathcal{K}$ cannot be regular for $q \geq 2$. □

The polytope constructed above is flat since $\Gamma(\mathcal{K}) = \langle\sigma_2,\sigma_3\rangle\langle\sigma_1,\sigma_2\rangle$. To see this we take one representative β in each of the eight right cosets of $\langle\sigma_2\rangle$ in $\langle\sigma_1,\sigma_2\rangle$ and show that $\beta\sigma_3 \in \langle\sigma_2,\sigma_3\rangle\sigma_1\sigma_2$. Choosing as representatives $\{\varepsilon, \sigma_1, \sigma_1^2, \sigma_1^{-1}, \sigma_1^{-1}\sigma_2, \sigma_1^2\sigma_2, \sigma_1^2\sigma_2^{-1}, \sigma_1^{-1}\sigma_2\sigma_1^2\}$ we have

$$\begin{array}{rcl}
\varepsilon &=& \varepsilon,\\
\sigma_1\sigma_3 &=& \sigma_3^2\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2\sigma_1^2,\\
\sigma_1^2\sigma_3 &=& \sigma_2\sigma_3^2\sigma_2^{-1}\sigma_3\sigma_1^2\sigma_2^{-1},\\
\sigma_1^{-1}\sigma_3 &=& \sigma_2^2\sigma_3\sigma_1^{-1},\\
\sigma_1^{-1}\sigma_2\sigma_3 &=& \sigma_2\sigma_3\sigma_1,\\
\sigma_1^2\sigma_2\sigma_3 &=& \sigma_2\sigma_3\sigma_1^2,\\
\sigma_1^2\sigma_2^{-1}\sigma_3 &=& \sigma_3^2\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2\sigma_1^{-1},\\
\sigma_1^{-1}\sigma_2\sigma_1^2\sigma_3 &=& \sigma_2^{-1}\sigma_3^2\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2.
\end{array}$$

This implies the existence of a recursive procedure to rewrite any word on the generators $\sigma_1, \sigma_2, \sigma_3$ as a product $\mu\nu$ with $\mu \in \langle\sigma_2,\sigma_3\rangle$ and $\nu \in \langle\sigma_1,\sigma_2\rangle$.

Since $\langle\sigma_1,\sigma_2\rangle \cap \langle\sigma_2,\sigma_3\rangle = \langle\sigma_2\rangle$, the automorphism group of the polytope $\mathcal{K}$ just constructed has order $|\langle\sigma_2,\sigma_3\rangle| \cdot |\langle\sigma_1,\sigma_2\rangle|/|\langle\sigma_2\rangle| = 192q^3$, the polytope has only 8 vertices, each of them incident to each of the $8q^3$ cubical facets of $\mathcal{K}$.

5.2. **Rank 5.** For chiral 5-polytopes, the relation

$$(\sigma_1\sigma_2\sigma_3\sigma_4)^2 = \varepsilon$$

becomes

$$(\tau_{1,2}\tau_{3,4})^2 = \varepsilon.$$

This implies that to construct chiral 5-polytopes from a regular or chiral 4-polytope $\mathcal{K}$ we may consider a disconnected GPR graph of $\mathcal{K}$ in terms of $\tau_{1,2}$, σ_2 and σ_3, and add edges of a new color corresponding to $\tau_{3,4}$ in such a way that each connected component of the subgraph induced by $\tau_{1,2}$ and $\tau_{3,4}$ is one of those in Figure 2, and that each connected component of the subgraph induced by σ_2 and $\tau_{3,4}$ is among those in Figure 6.

As an example, the connected component of colors blue (corresponding to $\tau_{1,2}$), green (corresponding to σ_2) and red (corresponding to σ_3) with 16 vertices in Figure 18 is a GPR graph of the cross polytope of rank 4. Attaching the connected component with the same colors with 4 vertices in the bottom does not modify the group generated by $\tau_{1,2}, \sigma_2$ and σ_3 and, thus, the graph generated by the blue edges and by the green and red arrows is a

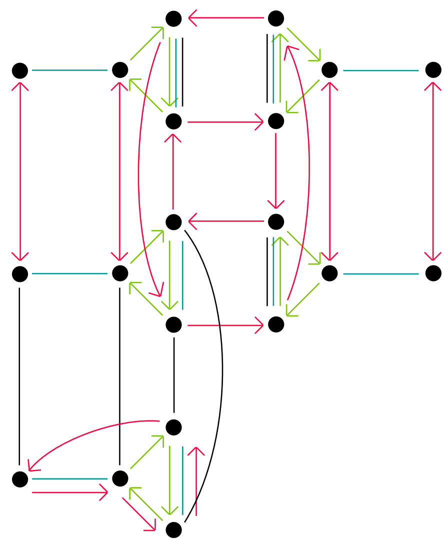

Figure 18: GPR graph of a chiral 5-polytope

disconnected GPR graph of the 4-dimentional cross-polytope $\{3,3,4\}$. Now we join the connected components with a (black) matching corresponding to $\tau_{3,4}$, satisfying the properties described above. Note that the order of $\sigma_2\sigma_4$ is 30 while the order of $\sigma_2^{-1}\sigma_4$ is 168, implying that the vertex figure, and thus the whole polytope, is chiral. GAP [4] can be used to show that the intersection conditions in Lemma (1.2) hold. The Schläfli type is $\{3,3,4,18\}$.

We conclude the paper with an interesting remark. Lemma 5.1 implies that in the procedure to obtain a $(d+1)$-polytope $\mathcal{P}$ from a d-polytope $\mathcal{K}$ we need to define an involution $\tau_{d-1,d}$ that commutes with $\sigma_1, \sigma_2, \ldots, \sigma_{d-4}$ and $\tau_{d-3,d-2}$ while satisfies the relation $(\sigma_{d-2}\tau_{d-1,d})^2 = \varepsilon$, this is, $\tau_{d-1,d}$ commutes with all the generators σ_i of the facet of $\mathcal{K}$ except σ_{d-2} and σ_{d-3} while $\tau_{d-1,d}\sigma_{d-2}\tau_{d-1,d} = \sigma_{d-2}^{-1}$ and $\tau_{d-1,d}\sigma_{d-3}\tau_{d-1,d} = \sigma_{d-3}\sigma_{d-2}^2$, but that is equivalent to (1.5). This is an alternative proof to that in [18] that the $(d-2)$-faces of any chiral d-polytope must be regular.

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