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ON THE UNIVERSAL RIGIDITY OF GENERIC BAR FRAMEWORKS

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Abstract. In this paper, we present a sufficient condition for the universal rigidity of a generic bar framework $G(p)$ in terms of the Gale matrix Z corresponding to $G(p)$. We also establish a relationship between the stress matrix S and the Gale matrix Z for bar frameworks. This allows us to translate back and forth between S and Z in recently obtained results concerning universal rigidity, global rigidity and dimensional rigidity of generic bar frameworks.

1. Introduction

An r-configuration p is a finite set of points $p^1, \ldots, p^n$ in $\mathbb{R}^r$ whose affine hull is $\mathbb{R}^r$. A bar framework (or simply a framework), denoted by $G(p)$, in $\mathbb{R}^r$ is a simple graph $G = (V, E)$ on the vertices $1, \ldots, n$ together with an r-configuration p, where each vertex i of G is located at point p^i. With a slight abuse of notation, sometimes we will refer to the vertices and edges of graph G as the vertices and edges of the framework $G(p)$. To avoid trivialities, we assume that graph G is connected and not complete.

An example of two frameworks in $\mathbb{R}^2$ is given in Figure 1. The vertices of the framework are represented by little circles, while the edges (bars) are represented by straight lines.

Two frameworks $G(p)$ in $\mathbb{R}^r$ and $G(q)$ in $\mathbb{R}^s$ are said to be *equivalent*[1] if $||q^i - q^j||^2 = ||p^i - p^j||^2$ for all $(i, j) \in E$, where $||.||$ denotes the Euclidean norm. On the other hand, two frameworks $G(p)$ and $G(q)$ in $\mathbb{R}^r$ are said to be *congruent* if $||q^i - q^j||^2 = ||p^i - p^j||^2$ for all $i, j = 1, \ldots, n$; i.e., frameworks $G(p)$ and $G(q)$ in $\mathbb{R}^r$ are congruent if the r-configurations p and q can be obtained from each other by a rigid motion such as a rotation or translation in $\mathbb{R}^r$. In this paper, we do not distinguish between congruent configurations. This is particularly convenient since we use Gram matrices, or more

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[1] Many authors use the term "equivalent" only for frameworks in the same Euclidean space.

accurately projected Gram matrices, to represent r-configurations. Thus all congruent frameworks are represented by the same projected Gram matrix hence it is quite natural to identify congruent frameworks. As a result, we assume without loss of generality that the centroid of the points $p^1, \ldots, p^n$ coincides with the origin.

A framework $G(p)$ is said to be *generic* if all the coordinates of $p^1, \ldots, p^n$ are algebraically independent over the integers. That is, $G(p)$ is generic if there does not exist a nonzero polynomial $f(x_1, \ldots, x_{rn})$ with integer coefficients such that $f(p^1_1, \ldots, p^1_r, \ldots, p^n_1, \ldots, p^n_r) = 0$.

1.1. **Global Rigidity of Frameworks.** A framework $G(p)$ in $\mathbb{R}^r$ is said to be *rigid* if for some $\epsilon > 0$, there does not exist a framework $G(q)$ in $\mathbb{R}^r$ which is equivalent to $G(p)$ such that $||p^i - q^i|| < \epsilon$ for all $i = 1, \ldots, n$. Recall that in this paper we don't distinguish between congruent frameworks. A framework $G(p)$ in $\mathbb{R}^r$ is said to be *globally rigid* if there does not exist a framework $G(q)$ in the same space $\mathbb{R}^r$ which is equivalent to $G(p)$. Obviously, rigidity is a necessary, albeit not sufficient, condition for global rigidity of a framework.

The problem of global rigidity of frameworks has received a great deal of attention recently [9, 10, 12, 13, 14, 15, 16]. Hendrickson [13] proved that if a generic framework $G(p)$ in $\mathbb{R}^r$ with at least $r+1$ vertices is globally rigid, then the graph $G = (V, E)$ is $r+1$ vertex-connected and $G(p)$ is redundantly rigid. A graph G is said to be k *vertex-connected* if G remains connected after deleting fewer than k of its vertices. A framework $G(p)$ is *redundantly rigid* if it remains rigid after deleting any one edge of G. Hendrickson also conjectured that $r + 1$ vertex-connectivity of G and redundant rigidity of $G(p)$ are sufficient for global rigidity of a generic framework $G(p)$. This conjecture, which is obviously true for $r = 1$, was shown by Connelly [8] to be false for $r \geq 3$. Jackson and Jordán [15] proved that Hendrickson's conjecture is true for $r = 2$.

Theorem 1.1 (Hendrickson [13], Jackson and Jordán [15]). *Given a generic framework $G(p)$ in $\mathbb{R}^2$, then $G(p)$ is globally rigid in $\mathbb{R}^2$ if and only if G is either a complete graph on at most three vertices or G is 3-vertex-connected and $G(p)$ is redundantly rigid.*

Connelly [9] gave a sufficient condition, in terms of the stress matrix, for a generic framework $G(p)$ in $\mathbb{R}^r$, for any r, to be globally rigid; and he conjectured that this condition is also necessary. Gortler *et al.* [12] proved that Connelly's conjecture is indeed true. Thus, the following theorem characterizes generic global rigidity in any dimension.

Theorem 1.2 (Connelly [9], Gortler *et al.* [12]). *Given a generic framework $G(p)$ with n vertices in $\mathbb{R}^r$, let S be the stress matrix associated with an equilibrium stress ω for $G(p)$. Then $G(p)$ is globally rigid in $\mathbb{R}^r$ if and only if rank $S = n - 1 - r$.*

1.2. **Universal Rigidity of Frameworks.** A framework $G(p)$ in $\mathbb{R}^r$ is said to be *universally rigid* if there does not exist a framework $G(q)$ in $\mathbb{R}^s$, which is equivalent to $G(p)$, for any s, $1 \leq s \leq n-1$. It immediately follows that universal rigidity implies global rigidity but the converse is not true. The framework (b) in Figure 1 is globally rigid in $\mathbb{R}^2$ but it is not universally rigid.

The problem of framework universal rigidity has received less attention than that of global rigidity. As it turns out, the notion of universal rigidity is closely related to that of dimensional rigidity first introduced in [4]. A framework $G(p)$ in $\mathbb{R}^r$ is said to be *dimensionally rigid* if there does not exist a framework $G(q)$ in $\mathbb{R}^s$, which is equivalent to $G(p)$, for any $s \geq r+1$. If $G(p)$ is not dimensionally rigid, we say it is *dimensionally flexible.* Alfakih [4] proved that a given framework $G(p)$, not necessarily generic, is universally rigid[2] if it is both rigid and dimensionally rigid.

Theorem 1.3 (Alfakih [4])**.** *Let $G(p)$ be a given framework with n vertices in $\mathbb{R}^r$ for some $r \leq n-2$. If $G(p)$ is both rigid and dimensionally rigid then $G(p)$ is universally rigid.*

Alfakih also presented in [4] the following sufficient condition for dimensional rigidity of frameworks.

Theorem 1.4 (Alfakih [4])**.** *Let $G(p)$ be a given framework with n vertices in $\mathbb{R}^r$ for some $r \leq n-2$, and let Z be the Gale matrix corresponding to $G(p)$. Further, let $\bar{r} = n-1-r$ and let ${z^i}^T$ denote the ith row of Z. If the following condition holds*

(1) *there exists an $\bar{r} \times \bar{r}$ symmetric positive definite matrix Ψ such that ${z^i}^T \Psi z^j = 0, \forall (i,j) \notin E$,*

then $G(p)$ is dimensionally rigid.

Two remarks are in order here. First, if $G = (V, E)$ is the complete graph then condition (1) trivially holds. Second, checking the validity of condition (1) is a semi-definite programming problem which can be solved efficiently (see [4] for more details). Also note that the necessary and sufficient condition for generic global rigidity in Theorem 1.2 is given in terms of stress matrix S while the sufficient condition for dimensional rigidity in Theorem 1.4 is given in terms of Gale matrix Z.

In this paper we show that Theorem 1.3 simplifies if the given framework is generic. In particular, we show that if a generic framework $G(p)$ is dimensionally rigid then it is universally rigid. Hence, we show that condition (1) is sufficient, and we conjecture that it is also necessary, for the universal rigidity of generic frameworks.

We also establish a relationship between the stress matrix S associated with an equilibrium stress ω for a framework $G(p)$ and the Gale matrix Z

[2]The term "unique" was used instead of the term "universally rigidity" in [4] and in an earlier version of this paper.

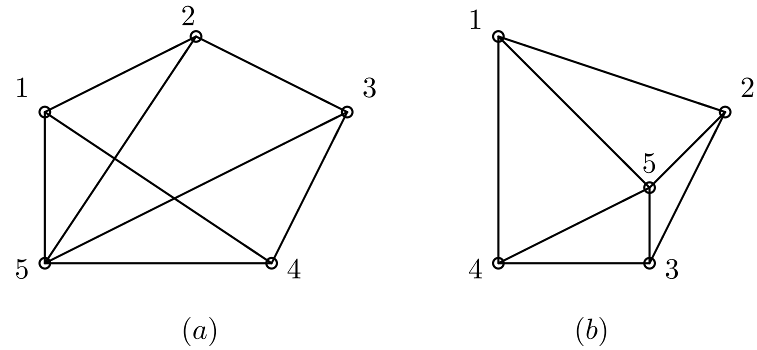

FIGURE 1. An example of two frameworks in $\mathbb{R}^2$. The framework in (a) is universally rigid (hence it is globally rigid); while the framework in (b) is globally rigid but not universally rigid.

corresponding to $G(p)$. This allows us to express the sufficient and necessary condition for generic global rigidity, and the above sufficient condition for generic universal rigidity in terms of either S or Z.

2. Gale Matrices and Stress Matrices

Let $G(p)$ be a given framework with n vertices in $\mathbb{R}^r$ and let e denote the vector of all 1's in $\mathbb{R}^n$. Consider the $(r+1) \times n$ matrix

$$\begin{bmatrix} P^T \\ e^T \end{bmatrix} = \begin{bmatrix} p^1 & p^2 & \cdots & p^n \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Recall that the affine hull of $p^1, \ldots, p^n$ has dimension r, i.e., the points $p^1, \ldots, p^n$ are not contained in a proper hyper-plane in $\mathbb{R}^r$. Then $r \leq n-1$, and the matrix $\begin{bmatrix} P^T \\ e^T \end{bmatrix}$ has full row rank. Let $\bar{r} = n - 1 - r$ and for $\bar{r} \geq 1$, let Λ be the $n \times \bar{r}$ matrix whose columns form a basis for the null space of $\begin{bmatrix} P^T \\ e^T \end{bmatrix}$. Λ is called a *Gale matrix* corresponding to $G(p)$; and the ith row of Λ, considered as a vector in $\mathbb{R}^{\bar{r}}$, is called a *Gale transform* of p^i [11]. Gale transform plays an important role in the theory of polytopes. We take advantage of the fact that Λ is not unique to define a special sparse Gale matrix Z which is also more convenient for our purposes.

Let us write Λ in block form as

$$\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix},$$

where Λ_1 is $\bar{r} \times \bar{r}$ and Λ_2 is $(r+1) \times \bar{r}$. Since Λ has full column rank, we can assume without loss of generality that Λ_1 is non-singular. Then Z is defined as

$$Z := \Lambda {\Lambda_1}^{-1} = \begin{bmatrix} I_{\bar{r}} \\ \Lambda_2 {\Lambda_1}^{-1} \end{bmatrix}. \tag{2.1}$$

Let ${z^i}^T$ denote the ith row of Z then it readily follows that $z^1, z^2, \ldots, z^{\bar{r}}$, the Gale transforms of $p^1, p^2, \ldots, p^{\bar{r}}$ respectively, are simply the standard unit vectors in $\mathbb{R}^{\bar{r}}$.

Let $G(p)$ be a framework in $\mathbb{R}^r$ where $G = (V, E)$ has n vertices and m edges. Associate with each edge (i, j) of G a scalar ω_{ij}. The vector $\omega = (\omega_{ij})$ in $\mathbb{R}^m$ such that

$$\sum_{j:(i,j)\in E} \omega_{ij}(p^i - p^j) = 0 \text{ for all } i = 1, \ldots, n, \tag{2.2}$$

is called an *equilibrium stress* for $G(p)$. Given an equilibrium stress ω, let $S = (s_{ij})$ be the $n \times n$ symmetric matrix defined by:

$$s_{ij} = \begin{cases} -\omega_{ij} & \text{if } (i,j) \in E, \\ 0 & \text{if } (i,j) \notin E, \\ \displaystyle\sum_{k:(i,k)\in E} \omega_{ik} & \text{if } i = j. \end{cases}$$

S is called the *stress matrix* associated with ω. The following lemma shows that the Gale matrix Z corresponding to $G(p)$ and the stress matrix S associated with an equilibrium stress ω of $G(p)$ are closely related.

Lemma 2.1. *Given a framework $G(p)$ with n vertices in $\mathbb{R}^r$, let Z be the Gale matrix corresponding to $G(p)$ and recall that $\bar{r} = n - 1 - r$. Further, let S be the stress matrix associated with an equilibrium stress ω for $G(p)$. Then*

$$S = Z\Psi Z^T \textit{ for some } \bar{r} \times \bar{r} \textit{ symmetric matrix } \Psi. \tag{2.3}$$

Furthermore, let ${z^i}^T$ be the ith row of Z. If Ψ' is any $\bar{r} \times \bar{r}$ symmetric matrix such that ${z^i}^T \Psi' z^j = 0$ for all $(i, j) \notin E$, then $Z\Psi' Z^T$ is a stress matrix associated with an equilibrium stress ω for $G(p)$.

Proof. Let S be the stress matrix associated with an equilibrium stress ω for $G(p)$. Then $e^T S = 0$ and $P^T S = 0$. Hence the columns of S belong to the null space of $\begin{bmatrix} P^T \\ e^T \end{bmatrix}$. Thus $S = ZA$ for some $\bar{r} \times n$ matrix A. But since S is symmetric and Z has full column rank, it follows that the columns of A^T also belong to the null space of $\begin{bmatrix} P^T \\ e^T \end{bmatrix}$. Therefore $S = Z\Psi Z^T$ for some $\bar{r} \times \bar{r}$ symmetric matrix Ψ.

On the other hand, let Ψ' be any $\bar{r} \times \bar{r}$ symmetric matrix such that ${z^i}^T \Psi' z^j = 0$ for all $(i, j) \notin E$ and let $S' = (s'_{ij}) = Z\Psi' Z^T$. Then $s'_{ij} = 0$ for all $(i, j) \notin E$ and $e^T S' = 0$ and $P^T S' = 0$. Therefore, for $i \neq j$, $\omega_{ij} = -s'_{ij}$ is an equilibrium stress for $G(p)$ and the result follows. □

Lemma 2.1 was used in [5] to establish relations between the rigidity matrix and the "dual" rigidity matrix introduced in [5]. Also, the following result obtained by Connelly follows immediately from the Lemma 2.1.

Corollary 2.2 (Connelly [7])**.** *Let S be the stress matrix associated with an equilibrium stress ω for framework $G(p)$ with n vertices in $\mathbb{R}^r$, then*

$$\text{rank } S \leq \bar{r} = n - 1 - r \tag{2.4}$$

3. Main Results

Next we show that Theorem 1.3 simplifies when the given framework is generic.

Theorem 3.1. *Let $G(p)$ be a given generic framework in $\mathbb{R}^r$ with n vertices for some $r \leq n-2$. If $G(p)$ is dimensionally rigid, then $G(p)$ is universally rigid.*

Section 4 of the paper is dedicated to a proof of this theorem. The following is an immediate corollary of Theorems 1.4 and 3.1.

Theorem 3.2. *Let $G(p)$ be a generic framework with n vertices in $\mathbb{R}^r$ for some $r \leq n-2$, and let Z be the Gale matrix corresponding to $G(p)$. Then condition (1) is sufficient for the universal rigidity of $G(p)$.*

In light of Lemma 2.1, we can express the sufficient and necessary condition for generic global rigidity and the sufficient condition for generic universal rigidity in terms of either the stress matrix S or the Gale matrix Z. Thus Theorems 1.2 and 3.2 can be equivalently stated as follows:

Theorem 3.3. *Given a generic framework $G(p)$ with n vertices in $\mathbb{R}^r$, let Z be the Gale matrix corresponding to $G(p)$. Recall that $\bar{r} = n - 1 - r$ and let ${z^i}^T$ be the ith row of Z. Then $G(p)$ is globally rigid if and only if*

(2) *there exists an $\bar{r} \times \bar{r}$ symmetric non-singular matrix Ψ such that ${z^i}^T \Psi z^j = 0, \forall (i,j) \notin E$.*

Theorem 3.4. *Let $G(p)$ be a given generic framework with n vertices in $\mathbb{R}^r$ for some $r \leq n-2$. If there exists a stress matrix S associated with an equilibrium stress ω for $G(p)$ such that S is positive semi-definite with rank $\bar{r} = n-1-r$, then $G(p)$ is universally rigid.*

Since universal rigidity implies global rigidity, it is interesting to note that whereas Ψ in the sufficient and necessary condition for generic global rigidity (condition (2)) is required to be non-singular, Ψ in the sufficient condition for generic universal rigidity (condition (1)) is required to satisfy the stronger notion of positive definiteness.

Being able to use the Gale matrix Z as an alternative to the stress matrix S in the above results offers new insights into these results. For example, in [4], the fact that the Gale matrix Z contains information on the affine dependencies among the points $p^1, \ldots, p^n$, is used to derive some interesting results concerning dimensional and universal rigidity of frameworks whose vertices are in general position. Points $p^1, \ldots, p^n$ in $\mathbb{R}^r$ are said to be in *general position* if no $r+1$ of them are affinely dependent. For example, points in the plane are in general position if no three of them lie on a straight line.

We conclude this section with the following conjecture:

Conjecture. *Let $G(p)$ be a given generic framework in $\mathbb{R}^r$ with n vertices for some $r \leq n-2$, and let Z be the Gale matrix for $G(p)$. If $G(p)$ is universally rigid then condition (1) holds.*

Example 3.1 in [4] shows that this conjecture is false if the framework $G(p)$ is not generic.

4. Proof of Theorem 3.1

Recall that e denotes the vector of all 1's in $\mathbb{R}^n$. Positive semi-definiteness of a symmetric matrix A is denoted by $A \succeq 0$. For a matrix A, $\mathrm{diag}(A)$ denotes the vector consisting of the diagonal entries of A. Finally, the $n \times n$ identity matrix will be denoted by I_n.

Let us represent an r-configuration $p^1, \ldots, p^n$ of a framework $G(p)$ in $\mathbb{R}^r$ by the $n \times r$ matrix

$$P = \begin{bmatrix} {p^1}^T \\ \vdots \\ {p^n}^T \end{bmatrix}.$$

Note that $P^T e = 0$ since we assume that the centroid of the points $p^1, \ldots, p^n$ coincides with the origin. Let B be the Gram matrix of the points $p^1, \ldots, p^n$, i.e., $B = PP^T$, and let V be an $n \times (n-1)$ matrix such that

$$V^T e = 0 \,, \quad V^T V = I_{n-1} \,. \tag{4.1}$$

For the purposes of this paper, it is convenient to represent an r-configuration of a framework $G(p)$ in $\mathbb{R}^r$ by the $(n-1) \times (n-1)$ *projected Gram matrix* X defined by

$$X := V^T B V = V^T P P^T V. \tag{4.2}$$

Clearly X, which is invariant under rigid motions, is positive semidefinite with rank r. Furthermore, since we do not distinguish between congruent frameworks, it follows that P and X uniquely determine each other (for more details see [4]). Thus, we will use $G(p)$ and $G(X)$ interchangeably.

Given a framework $G(p_1)$ in $\mathbb{R}^r$ where $G = (V, E)$ is not the complete graph. Let $\overline{m}$ denote the number of missing edges of G. For each $(i,j) \notin E$ define the matrix

$$M^{ij} := -\frac{1}{2} V^T E^{ij} V, \tag{4.3}$$

where E^{ij} is the $n \times n$ matrix with ones in the (i,j)th and (j,i)th entries and zeros elsewhere. Let X_1 be the projected Gram matrix corresponding to p_1, i.e., $X_1 = V^T P_1 P_1^T V$, and let

$$\Omega = \Big\{ y \in \mathbb{R}^{\overline{m}} : X(y) := X_1 + \sum_{(i,j) \notin E} y_{ij} M^{ij} \succeq 0 \Big\}. \tag{4.4}$$

It was shown in [1] that the set of all frameworks $G(q)$ in $\mathbb{R}^r$ that are equivalent to $G(p_1)$ is given by

$$\{G(X(y)) : y \in \Omega \text{ and rank } X(y) = r\}; \tag{4.5}$$

and that the set of all frameworks $G(q)$ in $\mathbb{R}^s$, equivalent to $G(p_1)$, for some s, $1 \le s \le n-1$, is given by

$$\{G(X(y)) : y \in \Omega\}. \tag{4.6}$$

For more details on Ω see [3].

Let $\mathcal{K}_V(.)$ be the linear map defined on the set of symmetric matrices of order $n-1$ by:

$$\mathcal{K}_V(X) := \operatorname{diag}(VXV^T)e^T + e(\operatorname{diag}(VXV^T))^T - 2VXV^T. \tag{4.7}$$

It is not difficult [1, 6] to show that the set $\{M^{ij} : (i,j) \notin E\}$ forms a basis for the null space of $H \circ \mathcal{K}_V(.)$, where H is the adjacency matrix of graph G and $H \circ \mathcal{K}_V(X)$ denotes the *Hadamard* product (or the element-wise product) of matrices H and $\mathcal{K}_V(X)$.

The following technical lemma establishes a relationship between the Gale matrix Z and the projected Gram matrix X.

Lemma 4.1 ([2])**.** *Let $G(p)$ be a given framework with n vertices in $\mathbb{R}^r$ for some $r \le n-2$, and let Z and X be, respectively, the Gale matrix and the projected Gram matrix corresponding to $G(p)$. Further, let $Q = [W\ U]$ be the orthogonal matrix whose columns are the eigenvectors of X, where the columns of U form an orthonormal basis for the null space of X. Then*

1. *$VU = ZA$ for some non-singular matrix A, i.e., VU is a Gale matrix.*
2. *$VW = PA'$ for some non-singular matrix A'.*

The next lemma is crucial for our proof of Theorem 3.1.

Lemma 4.2. *Let $G(p)$ be a framework with n vertices in $\mathbb{R}^r$ for some $r \le n-2$, and let U, W be the matrices as defined in Lemma 4.1. Then the following statements are equivalent:*

1. *There exists a nonzero $r \times r$ symmetric matrix Φ such that*
$$(p^i - p^j)^T \Phi (p^i - p^j) = 0, \forall (i,j) \in E.$$
2. *There exists a nonzero $y = (y_{ij}) \in \mathbb{R}^{\overline{m}}$ such that*
$$\sum_{(i,j) \notin E} y_{ij} M^{ij} U = 0,$$
where the matrices M^{ij} are defined in (4.3).

Proof. $(p^i - p^j)^T \Phi(p^i - p^j) = (P\Phi P^T)_{ii} + (P\Phi P^T)_{jj} - 2(P\Phi P^T)_{ij}$. Therefore, it follows from Lemma 4.1 and the remark before it that Statement 1 holds if and only if $H \circ \mathcal{K}_V(W\Phi' W^T) = 0$ for some nonzero symmetric matrix Φ' if and only if there exists a nonzero y such that $W\Phi' W^T = \sum_{(i,j)\notin E} y_{ij} M^{ij}$. But this last statement holds if and only if there exists a nonzero y such that $\sum_{(i,j)\notin E} y_{ij} M^{ij} U = 0$. □

A remark is in order here. In light of Lemma 4.1, Statement 2 of Lemma 4.2 is equivalent to: there exists a nonzero $y = (y_{ij}) \in \mathbb{R}^{\overline{m}}$ such that

$$\sum_{(i,j)\notin E} y_{ij} V^T E^{ij} Z = 0.$$

Lemma 4.3 (Connelly [9])**.** *Let $G(p)$ be a generic framework in $\mathbb{R}^r$ and let each vertex of G have degree at least r. Then there does not exist an $r \times r$ symmetric nonzero matrix Φ such that $(p^i - p^j)^T \Phi(p^i - p^j) = 0 \ \ \forall\, (i,j) \in E$.*

Proof of Theorem 3.1. Let $G(p_1)$ be a given generic framework with n vertices in $\mathbb{R}^r$ for some $r \leq n-2$, and let X_1 be the projected Gram matrix corresponding to $G(p_1)$. Let $Q = [W \ U]$ be the orthogonal matrix whose columns are the eigenvectors of X_1, where the columns of U form an orthonormal basis for the null space of X_1.

Assume that $G(p_1)$ is dimensionally rigid. Then it follows that each vertex of G has degree at least $r+1$ [4, Theorem 3.2].

Now suppose that $G(p_1)$ is not universally rigid. Then there exists a framework $G(q)$ in $\mathbb{R}^s$, which is equivalent to $G(p_1)$, for some s, $1 \leq s \leq n-1$. Therefore, there exists a nonzero $\hat{y}$ in $\mathbb{R}^{\overline{m}}$ such that $X(\hat{y}) = X_1 + \mathcal{M}(\hat{y}) \succeq 0$ where $\mathcal{M}(\hat{y}) = \sum_{(i,j)\notin E} \hat{y}_{ij} M^{ij}$. Furthermore, $X_1 + \mathcal{M}(\hat{y}) \succeq 0$ if and only if $Q^T(X_1 + \mathcal{M}(\hat{y}))Q \succeq 0$. But,

$$Q^T(X_1 + \mathcal{M}(\hat{y}))Q = \begin{bmatrix} \Lambda + W^T \mathcal{M}(\hat{y})\, W & W^T \mathcal{M}(\hat{y})\, U \\ U^T \mathcal{M}(\hat{y})\, W & U^T \mathcal{M}(\hat{y})\, U \end{bmatrix} \succeq 0,$$

where Λ is the $r \times r$ diagonal matrix consisting of the positive eigenvalues of X_1. Thus $U^T \mathcal{M}(\hat{y})\, U \succeq 0$ and the null space of $U^T \mathcal{M}(\hat{y})\, U \subseteq$ the null space of $W^T \mathcal{M}(\hat{y})\, U$. Now if $U^T \mathcal{M}(\hat{y}) U$ is nonzero then rank $X(\hat{y}) \geq r+1$. This contradicts our assumption that $G(p_1)$ is dimensionally rigid. Therefore, both matrices $U^T \mathcal{M}(\hat{y}) U$ and $W^T \mathcal{M}(\hat{y}) U$ must be zero. This implies that $\mathcal{M}(\hat{y}) U = 0 = \sum_{(i,j)\notin E} \hat{y}_{ij} M^{ij} U$ which is also a contradiction by Lemmas 4.2 and 4.3. Thus $G(p_1)$ is universally rigid. □

5. Numerical Example

In this section we present a numerical example to illustrate the main results of the paper. Consider the framework (a) in Figure 1, where P and its corresponding Gale matrix Z are

$$P = \frac{1}{5}\begin{bmatrix} -9 & 3 \\ 1 & 8 \\ 11 & 3 \\ 6 & -7 \\ -9 & -7 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -3/2 \\ 4/3 & 4/3 \\ -4/3 & -5/6 \end{bmatrix}.$$

It is easy to verify that $\Psi = \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$ satisfies condition (1). Thus this framework is dimensionally rigid and globally rigid, in fact it is also universally rigid.

Now consider the framework (b) in Figure 1, where P and its corresponding Gale matrix Z are

$$P = \frac{1}{5}\begin{bmatrix} -7 & 9 \\ 8 & 4 \\ 3 & -6 \\ -7 & -6 \\ 3 & -1 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 1/2 \\ -1 & 1/2 \\ -3 & -2 \end{bmatrix}.$$

Any 2×2 symmetric matrix Ψ that satisfies $z^{1^T}\Psi z^3 = z^{2^T}\Psi z^4 = 0$ must be of the form $\Psi = \begin{bmatrix} \alpha & -6\alpha \\ -6\alpha & -12\alpha \end{bmatrix}$, where α is a scalar. Thus condition (1) does not hold since Ψ can not be positive definite. This framework is dimensionally flexible and consequently, not universally rigid. However, for $\alpha \neq 0$, rank $\Psi = 2 = \bar{r}$. Hence the stress matrix $S = Z\Psi Z^T$ has rank $= 2$. Thus this framework is globally rigid in $\mathbb{R}^2$.

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