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FACE MODULE FOR REALIZABLE $\mathbb{Z}$-MATROIDS

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Abstract. In this work, we define the face module for a realizable matroid over $\mathbb{Z}$. Its Hilbert series is, indeed, the expected specialization of the Grothendieck–Tutte polynomial defined by Fink and Moci in [10].

A *matroid* M is a simplicial complex $\mathcal{I}$ on a *ground set* $[n] = \{1, \dots, n\}$, such that

$$A, B \in \mathcal{I}, \#A > \#B \Rightarrow \exists a \in A \setminus B : B \cup \{a\} \in \mathcal{I}.$$

The latter is called *independent set exchange property* and $\mathcal{I}$ is often called *independent sets* family. Matroids encapsulate the combinatorics that underline the arrangements of hyperplanes in affine or projective space.

There are two classical objects one associates to a matroid: the Stanley–Reisner ring $\mathbf{k}[M]$, that is the face ring of $\mathcal{I}$, and the Tutte polynomial T_M. They are related by the following result:

Theorem. *Let M be a matroid of rank r with ground set $[n]$ and call M^* its dual matroid. Then:*

$$\mathrm{Hilb}(\mathbf{k}[M], t) = \frac{t^r}{(1-t)^r} T_{M^*}(1, {}^1\!/\!_t).$$

where $\mathrm{Hilb}(\mathbf{k}[M], t)$ *is the Hilbert series of* $\mathbf{k}[M]$.

Fink and Moci [10] generalize the concept of matroid to a larger setting: a matroid $\mathcal{M}$ over a commutative ring R on the ground set $[n]$ is an assignment of an R-module $\mathcal{M}(A)$ for every subset A of $[n]$. This assignment has to respect a certain local patching condition.

One of the reasons behind this generalization is to deal with arrangements of hypersurfaces. Steps toward a Rota cryptomorphism are already done over a Dedekind domain [10], over valuation ring [11] and, more relevant for this manuscript, over $\mathbb{Z}$ [8].

For the goal of this paper, it is worth recalling that a realizable matroid over $\mathbb{Z}$ relates to a generalized toric arrangement [17, 4].

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In this paper, we introduce a candidate for the role of the independent set complex for realizable matroids over $\mathbb{Z}$, that we call a *partially order set (poset) of torsions* of $\mathcal{M}$. This new poset provides a combinatorial tool for (generalized) toric arrangements to compute their integral cohomology [2], to construct their wonderful models [18] and their projective wonderful models [3]. Moreover, this paves the way to show one of Rota's cryptomorphisms for matroids over $\mathbb{Z}$.

Given a matroid $\mathcal{M}$ over $\mathbb{Z}$, $\mathcal{M}(A)$ is an abelian group and so

$$\mathcal{M}(A) = \mathbb{Z}^{d(A)} \times G_A,$$

where G_A is a torsion group and its cardinality, $\#G_A$, is often referred as the multiplicity of A, $m(A)$. Call $d = d(\emptyset)$, the dimension of the matroid $\mathcal{M}$ and C_A, the dual group of G_A.

Definition. *We denote by* $\operatorname{Gr}\mathcal{M}$ *the set of torsions of* $\mathcal{M}$*. This is the set of all pairs* (A, l) *with* $d - d(A) = \#A$ *and* $l \in C_A$*.*

First, observe that $(\emptyset, e)$ always belongs to $\operatorname{Gr}\mathcal{M}$. We are going to give a partial order to $\operatorname{Gr}\mathcal{M}$ by defining certain covering relations inspired by the poset of layers of a toric arrangement [17, 4]. Similar ad hoc constructions appear also in [2]. This order depends on the realization of the matroids $\mathcal{M}$. Indeed, every realization provides a surjective map $\pi : C_{A\cup b} \twoheadrightarrow C_A$. (See Section 2 for further details.)

Definition. *Let* $\mathcal{M}$ *be a realized* $\mathbb{Z}$*-matroid. Let* $(A \cup \{b\}, h)$ *and* (A, l) *be two elements of* $\operatorname{Gr}\mathcal{M}$*. We say that* $(A \cup \{b\}, h)$ *covers* (A, l) *if and only if* $\pi(h) = l$*.*

This poset is not a simplicial complex, but it is the union of identical simplicial posets.

Theorem A. *If* $\mathcal{M}$ *is a realized* $\mathbb{Z}$*-matroid, then* $\operatorname{Gr}\mathcal{M}$ *is a disjoint union of* $m(\emptyset)$ *simplicial posets isomorphic to the link of* $(\emptyset, e)$ *in the poset* $\operatorname{Gr}\mathcal{M}$*.*

As a byproduct, one can reproduce many of the results of Sections 5 and 6 of [17], but at the cost of losing part of the geometrical intuition.

From the poset $\operatorname{Gr}\mathcal{M}$ we define a *face module* $\mathbf{k}[\mathcal{M}]$ associated to $\mathcal{M}$. For this, we use Stanley's construction [19] of the face ring for simplicial posets. As proof that $\operatorname{Gr}\mathcal{M}$ is the correct combinatorial object to study, we also show that the Hilbert series of its face *module* is the specialization of the Grothendieck–Tutte polynomial, as in the classical case:

Theorem B. *If* $\mathcal{M}$ *is a realizable* $\mathbb{Z}$*-matroid of rank* r*, then*

$$\operatorname{Hilb}(\mathbf{k}[\mathcal{M}], t) = \frac{t^r}{(1-t)^r} T_{\mathcal{M}^*}(1, {}^1\!/_t).$$

The Grothendieck–Tutte polynomial for matroid over a ring has been defined by Fink and Moci [10] as a function of $\mathcal{M}$ in a certain Grothendieck ring of matroids, but, in our setting, $T_{\mathcal{M}}$ is more concretely the arithmetic Tutte polynomial, see [4]. Precisely,

$$T_{\mathcal{M}}(x,y) = \sum_{A\subseteq[n]} m(A)(x-1)^{r-\mathrm{cork}(A)}(y-1)^{\#A-\mathrm{cork}(A)},$$

where $\mathrm{cork}(A) = d(\emptyset) - d(A)$.

It is worth mentioning that it is not clear if the poset of torsion is uniquely defined for nonrealizable matroids. On the other hand, the proof of Theorem B holds easily for any simplicial poset with the correct $\mathbf{f}$-vector. Therefore we conjecture that Theorem B holds for every $\mathbb{Z}$-matroid. We say more about this in Remark 4.2.

The paper is organized as follows: in Section 1 we recall all the basic notions needed for a full comprehension of the results. In Section 2 we define the poset of torsions and in Section 3 we prove Theorem A. Finally, in Section 4 we show Theorem B.

1. Basic notions

1.1. **Simplicial posets.** Let $(P,<)$ be a finite partially ordered set (poset). A poset with a unique initial element, denoted by $\hat{0}$, is said to be *simplicial* if for each $\sigma \in P$ the segment $[\hat{0},\sigma] = \{x \in P : \hat{0} \leq x \leq \sigma\}$ is a boolean lattice. We say that the *rank* of $[\hat{0},\sigma]$ is the length of its maximal chain; therefore $(P,<)$ has a natural rank function rk induced by the rank of the segments $[\hat{0},\sigma]$. We denote by r the rank of P, the maximal rank among all its segments.

For any σ and τ in P, $\sigma \wedge \tau$ is the set of their greatest lower bounds (*meets*) and $\sigma \vee \tau$ is the set of their least common upper bounds (*joins*). For a simplicial poset, $\sigma \wedge \tau$ is a singleton and by an abuse of notation we identify the $\sigma \wedge \tau$ with the unique greatest lower bound of σ and τ.

Example 1.1. Consider the set given by $P_1 = \{\hat{0}, a, b, 1, 2\}$, where every number is greater or equal to every letter and every element is greater or equal to $\hat{0}$, see Figure 1.a). This is a simplicial poset. It is not the face poset of any simplicial complex, but it is the face poset of a digon, a CW-complex shown in Figure 1.b). Its order complex is a triangulation of the one dimensional sphere, see Figure 1.c). We compute few examples of meets and joins that are useful in future computations: $1 \wedge a = \{a\}$, $a \wedge b = \{\hat{0}\}$, $a \vee b = \{1, 2\}$, and $1 \vee 2 = \emptyset$.

Example 1.2. Consider the set $P_2 = \{\hat{0}, a, b, c, 1\}$ with the same order law given for P_1, see Figure 1.d). This is not a simplicial poset, because $[\hat{0}, 1]$ is not boolean.

1.2. **Face ring.** Given a field $\mathbf{k}$, we set the polynomial ring $R_P = \mathbf{k}[x_\sigma : \sigma \in P]$ where x_σ has degree $\mathrm{rk}\,\sigma$. In this recap section we are going to follow the notation in [19].

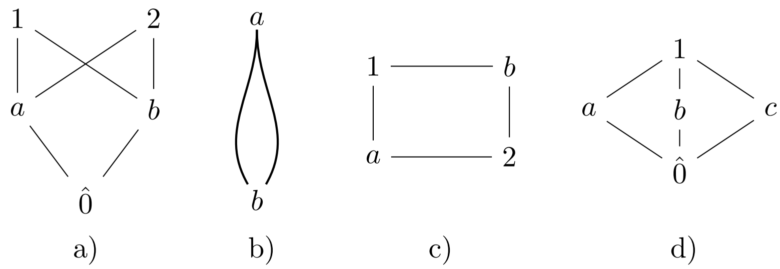


FIGURE 1. a) The poset P_1 in Example 1.1; b) The CW-complex with face poset P_1; c) The order complex of P_1; d) The poset P_2 in Example 1.2.

Definition 1.3. *The* face ideal *of a simplicial poset P is the ideal of R_P defined as*

$$I_P = \left(x_{\hat{0}} - 1,\, x_\sigma x_\tau - x_{\sigma\wedge\tau}\left(\sum_{\gamma\in\sigma\vee\tau} x_\gamma\right) \text{ for any } \sigma,\tau \in P \right).$$

As a notation, the sum $\sum_{\gamma\in\sigma\vee\tau} x_\gamma$ is zero if $\sigma \vee \tau = \emptyset$. Moreover, the face ring *of a simplicial poset P is the quotient*

$$\mathbf{k}[P] = \frac{\mathbf{k}[x_\sigma : \sigma \in P]}{I_P}.$$

Example 1.4. Consider the simplicial poset P_1 defined in Example 1.1, R_{P_1} is the polynomial ring $\mathbf{k}[x_{\hat{0}}, x_a, x_b, x_1, x_2]$.

The ideal $I_{P_1} = \langle x_{\hat{0}} - 1, x_a x_b - (x_1 + x_2), x_1 x_2 \rangle$. Indeed, the basis generators are obtained by substituting respectively (a, b) and $(1, 2)$ to the pair the (σ, τ) into $x_\sigma x_\tau - x_{\sigma\wedge\tau}\left(\sum_{\gamma\in\sigma\vee\tau} x_\gamma\right)$. For all other values of (σ, τ), the previous relation is trivial. The face ring of P_1 is therefore defined as the quotient

$$\mathbf{k}[P_1] = \frac{\mathbf{k}[x_{\hat{0}}, x_a, x_b, x_1, x_2]}{\left(x_{\hat{0}} - 1, x_a x_b - (x_1 + x_2), x_1 x_2\right)}.$$

This definition generalizes the *Stanley–Reisner ring* of a simplicial complex. Given an abstract simplicial complex Δ on n vertices its *Stanley–Reisner ring* $\mathbf{k}[\Delta]$ the following quotient ring $\mathbf{k}[\Delta] = {}^{\mathbf{k}[x_1,\ldots,x_n]}/_{I_\Delta}$, where $I_\Delta = \langle x_{i_1} \ldots x_{i_r} : \{i_1, \ldots, i_r\} \notin \Delta \rangle$. One can easily check that the two face ring definitions coincide in the case of an abstract simplicial complex.

Example 1.5. Consider the simplicial poset P_2 defined in Figure 2.a). This is the face poset of a simplicial complex. Precisely, the path graph on 3 elements. Its face ring, $\mathbf{k}[P_2]$, is isomorphic to ${}^{\mathbf{k}[x,y,z]}/_{(xz)}$.

From now on, we assume that P is a simplicial poset, that Δ is an abstract simplicial complex and r denotes their ranks.

1.3. **The Hilbert series of the face ring.** Let N be a finitely generated $\mathbb{N}$-graded A-module where A is a finitely generated $\mathbb{N}$-graded commutative algebra over $\mathbf{k}$. Denote by N_i the homogeneous part of degree i. The Hilbert series of N is the following generating function:

$$\mathrm{Hilb}(N,t) = \sum_{i\geq 0} \dim_{\mathbf{k}}(N_i)t^i,$$

where $\dim_{\mathbf{k}}(N_i)$ is the dimension of N_i as a $\mathbf{k}$-vector space. We consider $\mathrm{Hilb}(A,t)$ as the Hilbert series of A seen as a module over its self.

The ring $\mathbf{k}[P]$ is graded and its Hilbert series encodes many combinatorial objects, like the $\mathbf{f}$-*vector* and the $\mathbf{h}$-*vector*. Here, we briefly recall their definitions. The $\mathbf{f}$-vector, $\mathbf{f}(P)$, of a simplicial poset P is the vector $(f_{-1}, f_0, \dots, f_{r-1})$ where f_i is the number of elements of rank $i+1$ in P; by notation $f_{-1} = 1$ counts the empty set as a dimension -1 object. The $\mathbf{h}$-vector of P is the vector $\mathbf{h}(P) = (h_0, h_1, \dots, h_r)$ defined recursively from the $\mathbf{f}$-vector by using $\sum_{i=0}^{r} f_{i-1}(t-1)^{r-i} = \sum_{i=0}^{r} h_i t^{r-i}$.

Example 1.6. We compute $\mathbf{f}(P_1)$ and $\mathbf{h}(P_1)$ for the simplicial poset in Example 1.1. Trivially, $\mathbf{f}(P_1) = (1,2,2)$. Expanding $\sum_{i=0}^{2} f_{i-1}(t-1)^{2-i}$ one gets t^2+1 and therefore $\mathbf{h}(P_1) = (1,0,1)$.

Example 1.7. Let us make similar computation for P_2 in Example 1.5. Clearly $\mathbf{f}(P_2) = (1,3,2)$ and by expanding $\sum_{i=0}^{2} f_{i-1}(t-1)^{2-i} = t^2+t$ and therefore $\mathbf{h}(P_2) = (1,1,0)$.

As said, one can read the $\mathbf{f}$-*vector* and the $\mathbf{h}$-*vector* from the face ring $\mathbf{k}[P]$.

Theorem 1.8 ([19, Proposition 3.8])**.** *Let P be a simplicial poset of rank r and let $\mathbf{k}[P]$ be its face ring. Then*

$$\mathrm{Hilb}(\mathbf{k}[P],t) = \frac{h_0 + h_1 t + \cdots + h_r t^r}{(1-t)^r}.$$

Example 1.9. Let us verify the previous theorem for our toy simplicial poset P_1. Its face ring is computed in Example 1.4 and its $\mathbf{f}$-vector and $\mathbf{h}$-vector are shown in Example 1.6.

By a dirty hands computation or by using `Macaulay2` [13] we see that this Hilbert series simplifies to

$$\mathrm{Hilb}(\mathbf{k}[P_1],t) = \frac{1-t^2-t^4+t^6}{(1-t^2)^2(1-t^2)} = \frac{1+t^2}{(1-t)^2}$$

and this is indeed the expected result.

Example 1.10. In the case of the face poset P_2, it is easy to verify what we have just stated. Indeed, in Example 1.5, the face ring and in Example 1.6 we computed the face ring, the $\mathbf{f}$-vector and $\mathbf{h}$-vector. It is trivial to observe the following:

$$\mathrm{Hilb}\left(\frac{\mathbf{k}[x,y,z]}{(xz)}, t\right) = \frac{1-t^2}{(1-t)^3} = \frac{1+t}{(1-t)^2}.$$

1.4. **Matroid over a ring R.** In [10], Fink and Moci generalize the concept of matroid to matroid over a commutative ring R. In this section we give the general definition then deal with the case $R = \mathbb{Z}$.

1.4.1. *R-matroids.* Let $2^{[n]}$ be the set of all subsets of $[n]$ and let R–mod the category of finitely generated R-modules.

Definition 1.11. *A matroid over the ring R is the function*

$$\mathcal{M} : 2^{[n]} \to R\text{–mod}$$

such that for any subset A of $[n]$ and any elements b and c of $[n] \setminus A$ there exist $x_{b,c}$ and $y_{b,c}$ in $\mathcal{M}(A)$, such that:

$$\begin{aligned} \mathcal{M}(A \cup \{b\}) &\simeq \mathcal{M}(A)/(x_{b,c}) \\ \mathcal{M}(A \cup \{c\}) &\simeq \mathcal{M}(A)/(y_{b,c}) \\ \mathcal{M}(A \cup \{b,c\}) &\simeq \mathcal{M}(A)/(x_{b,c}, y_{b,c}) \end{aligned}$$

The choice of $\mathcal{M}$ is relevant only up to isomorphism. Moreover, we are going to assume that $\mathcal{M}$ is *essential*, that is no nontrivial projective module is a direct summand of $\mathcal{M}([n])$.

In Proposition 2.6 of [10], Fink and Moci had shown that an essential matroid over a field $\mathbf{k}$ is a matroid in the classical case. For this reason, from now on, we are going to call these $\mathbf{k}$-matroids.

1.4.2. *Realizable $\mathbb{Z}$-Matroids.* For this paper, we are going to set $R = \mathbb{Z}$. We define a corank function $\operatorname{cork}(A)$ of $\mathcal{M}$ as the corank function of the $\mathbb{Q}$-matroid $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$\operatorname{cork}(A) = \operatorname{cork}_{\mathbb{Q}} \mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Q}(A).$$

For any subset A of $[n]$, $\mathcal{M}(A)$ is an abelian group,

$$\mathcal{M}(A) = \mathbb{Z}^{d(A)} \times G_A,$$

where G_A is the torsion part. We call $m(A) = \#G_A$ the *multiplicity* of A. Clearly, $\operatorname{cork}(A) = d(\emptyset) - d(A)$.

A $\mathbb{Z}$-matroid $\mathcal{M}$ on $[n]$ is realizable if there is a list of elements $z_1, \dots, z_n \in \mathcal{M}(\emptyset)$ such that $\mathcal{M}(A) = \mathcal{M}(\emptyset)/(z_i : i \in A)$.

The definition (see Section 2) of the poset $\operatorname{Gr} \mathcal{M}$ depends on the realization of the matroid $\mathcal{M}$. For this, when we need to use explicitly a realization of a matroid $\mathcal{M}$ we are going to talk about *realized matroid* $\mathcal{M}$.

Example 1.12. Let $R = \mathbb{Z}$ and $n = 2$. Set $\mathcal{M}(\emptyset) = \mathbb{Z}^2$, $\mathcal{M}(\{1\}) = \mathbb{Z}^2/(2,0)$, $\mathcal{M}(\{2\}) = \mathbb{Z}^2/(0,1)$ and $\mathcal{M}(\{1,2\}) = \mathbb{Z}^2/((2,0),(0,1))$.

Example 1.13. Let $A = \mathbb{Z}$ and $n = 2$. Set $\mathcal{M}(\emptyset) = \mathbb{Z}^2$, $\mathcal{M}(\{1\}) = \mathbb{Z}^2/(1,1)$, $\mathcal{M}(\{2\}) = \mathbb{Z}^2/(1,-1)$ and $\mathcal{M}(\{1,2\}) = \mathbb{Z}^2/((1,1),(1,-1))$.

Remark 1.14. The definition and results in the next sections hold also for *arithmetic matroids*. We do not need any of the arithmetic matroids tools. In Section 6.1 of [10, Section 6.1] it is shown that $\mathbb{Z}$-matroids with an extra *molecule* condition are arithmetic matroids. (Not all arithmetic matroids can be seen as a $\mathbb{Z}$-matroid.)

1.4.3. *Grothendieck–Tutte polynomial.* In the rest of this section, we are going to define the Grothendieck–Tutte polynomial for a matroid over $\mathbb{Z}$.

Let $\mathrm{L}_0(\mathrm{Ab})$ be the Grothendieck type ring of abelian groups, that is the free group generated by the isomorphic classes $[G]$ for any finitely generated abelian group G. There is a ring multiplication given by $[G][G'] = [G \times G']$. This object is very useful; for instance it appears in [6, 7, 15, 16]. In Section 7.1 of [10], it is proved the Grothendieck–Tutte *class* is well defined to be the following element of $\mathrm{L}_0(\mathrm{Ab}) \otimes \mathrm{L}_0(\mathrm{Ab})$:

$$GT_{\mathcal{M}} = \sum_{A \subseteq E} [\mathcal{M}(A)][\mathcal{M}^*(E \setminus A)]$$

where $\mathcal{M}^*$ is the matroid dual to $\mathcal{M}$ and E is their common ground set. (We use E to avoid confusion between $[n]$ and $[\mathcal{M}(-)]$.) For a precise definition of the dual matroid $\mathcal{M}^*$ we refer to Section 7 of [4].

Let G be a group and consider its class $[G]$ in $\mathrm{L}_0(\mathrm{Ab})$: since $G \simeq \mathbb{Z}^d \times G_t$, one has that $[G] = [\mathbb{Z}^d][G_t] \in \mathrm{L}_0(\mathrm{Ab})$. Now, fix the following evaluations for $\mathrm{L}_0(\mathrm{Ab})$: $v_x([G]) = \#G_t(x-1)^d$ and similarly, $v_y([G]) = \#G_t(y-1)^d$ and consider the image of $GT_{\mathcal{M}}$ with respect to the map

$$v_x \otimes v_y : \mathrm{L}_0(\mathrm{Ab}) \otimes_{\mathbb{Z}} \mathrm{L}_0(\mathrm{Ab}) \to \mathbb{Z}[x, y].$$

The Grothendieck–Tutte polynomial for $\mathcal{M}$ is $T_{\mathcal{M}}(x, y) = (v_x \otimes v_y)(GT_{\mathcal{M}})$. Then,

$$T_{\mathcal{M}}(x, y) = \sum_{A \subseteq [n]} m(A)(x-1)^{r-d(A)}(y-1)^{\#A-d(A)}. \tag{1}$$

This polynomial was first introduced by Moci in [17] and it is often called the arithmetic Tutte polynomial. It easy to observe that

$$T_{\mathcal{M}}(x, y) = T_{\mathcal{M}^*}(y, x). \tag{2}$$

Example 1.15. Let us compute $T_{\mathcal{M}}$ for the $\mathbb{Z}$-matroid given in Example 1.12. In (1) the contribution of the empty set is $(x-1)^2$; the contribution of the singleton $\{1\}$ is $2(x-1)$; the contribution of the singleton $\{2\}$ is $(x-1)$; finally, the contribution of the full ground set $[2]$ is 2. Thus, $T_{\mathcal{M}}(x, y) = x^2 + x$.

Example 1.16. We compute $T_{\mathcal{M}}$ for the matroid in Example 1.13. In (1) the contribution of the empty set is $(x-1)^2$; the contribution of each singleton is $(x-1)$; finally, the contribution of the full ground set $[2]$ is 2. Thus, $T_{\mathcal{M}}(x, y) = x^2 + 1$.

As remarked in the introduction, the face ring and the Grothendieck–Tutte polynomial of a **k**-matroid are related.

Theorem 1.17. *Let $\mathcal{M}$ be a* $\mathbf{k}$*-matroid of rank* r*. Let* $\mathbf{k}[\mathcal{M}]$ *be its face ring. Then,*

$$\mathrm{Hilb}(\mathbf{k}[\mathcal{M}], t) = \frac{t^r}{(1-t)^r} T_{\mathcal{M}^*}(1, 1/t).$$

Proof. To the author's best knowledge, this result appears first in the above form in the Appendix (section A.3) by Björner in the work of De Concini and Procesi [5]. □

The goal of the paper is to extend this result to realizable $\mathbb{Z}$-matroids.

2. The poset of torsions

The aim of this section is to define a new poset taking the role of the independent complex in the case of $\mathbf{k}$-matroids.

Throughout this section we assume that A is a subset of $[n]$ and b is in $[n] \setminus A$. Let $b = c$ in Definition 1.11: it requires the existence of a quotient homomorphism by $x_{b,b} \in \mathcal{M}(A)$:

$$\pi_{(A,b)} : \mathcal{M}(A) \to \mathcal{M}(A \cup \{b\}). \qquad (3)$$

Call $\pi_{(A,b)}$ the *canonical projection* associated to A and b. While the homomorphism $\pi_{(A,b)}$ is unique, the choice of $x_{b,b}$ is not. In the case of realizable $\mathbb{Z}$-matroids, $x_{b,b}$ is unique and we denote it by x_b.

For any subset A of $[n]$, $\mathcal{M}(A) \simeq \mathbb{Z}^{d(A)} \times G_A$, where $d(A)$ is the rank of $\mathcal{M}(A)$ and G_A is the torsion part of $\mathcal{M}(A)$. Call C_A, the dual group of G_A.

Definition 2.1. *We call* $\mathrm{Gr}\,\mathcal{M}$ *the set of torsions of* $\mathcal{M}$*. This is the set of all pairs* (A, l) *with* $d(\emptyset) - d(A) = \#A$ *and* $l \in C_A$*.*

Inspired by Section 5 of [17], we are going to view such a set as a bunch of tori with the right dimension and cardinality, prescribed by the $\mathbb{Z}$-matroid. This is the reason that lead us to work with the dual group C_A instead of G_A, even if they are isomorphic. Moreover, consider A and $A \cup \{b\}$ such that $d(\emptyset) - d(A) = \#A$ and $d(\emptyset) - d(A \cup \{b\}) = \#A + 1$. Then, the map $\pi_{(A,b)}$ restricted to G_A is injective and its dual $\pi^{(A,b)} : C_{A \cup \{b\}} \twoheadrightarrow C_A$ is surjective.

Definition 2.2. *Let* $(A \cup \{b\}, h)$ *and* (A, l) *be two elements of* $\mathrm{Gr}\,\mathcal{M}$*. We say that* $(A \cup \{b\}, h)$ *covers* (A, l)*, and we write* $(A \cup \{b\}, h) \rhd (A, l)$*, if and only if* $\overline{h} \subseteq \underline{l}$*.*

Example 2.3. Let us compute the poset of torsions of the matroid given in Example 1.12. We show the poset in Figure 2.a). Clearly there are six elements $(\emptyset, e)$, $(\{1\}, e)$, $(\{1\}, \zeta)$, $(\{2\}, e)$, $([2], e)$, and $([2], \zeta)$.

Now observe that trivially $(\{1\}, e)$, $(\{1\}, \zeta)$, $(\{2\}, e)$ cover $(\emptyset, e)$ and $([2], x)$ covers $(\{2\}, e)$ because $C_{[2]}$ surjects to $C_{\{2\}} = \{e\}$. Moreover $([2], x)$ covers $(\{1\}, x)$, because $C_{[2]} \simeq C_{\{1\}}$, thus $\pi^{(\{1\},2)}(x) = x$. This also shows that $([2], x)$ does not cover $(\{1\}, y)$ if $x \neq y \in \mathbb{Z}/2\mathbb{Z}$.

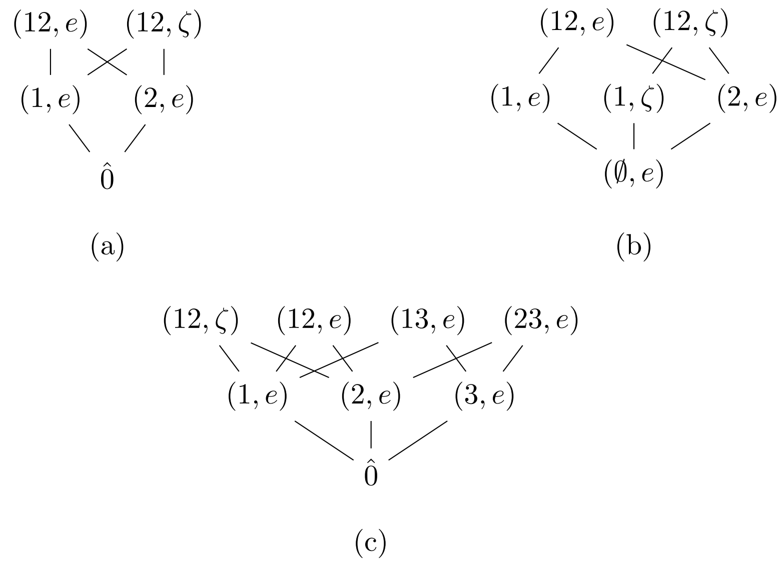


FIGURE 2. a) The poset of torsions of the matroid given in Example 1.12; b) The poset of torsions P_1 of the matroid given in Example 1.13 c) The poset of torsions of the matroid in Example 4.3.

Example 2.4. The poset of torsions of the matroid given in Example 1.13 is actually the poset P_1 defined in Example 1.1 and discussed throughout Section 1. This poset is in Figure 2.b). We leave to the reader to verify the covering relations, which are straightforward.

3. Gr $\mathcal{M}$ IS A UNION OF SIMPLICIAL POSETS

In this section we are going to prove Theorem A, that is $\operatorname{Gr}\mathcal{M}$ is a union of simplicial posets. We start by proving two properties of $\operatorname{Gr}\mathcal{M}$.

Proposition 3.1. *Let $\mathcal{M}$ be a realizable matroid over $\mathbb{Z}$. Let $(A \cup \{b\}, h)$, (A, l_1), (A, l_2) be in $\operatorname{Gr}\mathcal{M}$. If $(A \cup \{b\}, h) \triangleright (A, l_1)$ and $(A \cup \{b\}, h) \triangleright (A, l_2)$ then $l_1 = l_2 \in \mathcal{M}(A)$.*

Proof. By Definition 2.2 if $(A \cup \{b\}, h) \triangleright (A, l_1)$ and $(A \cup \{b\}, h) \triangleright (A, l_2)$ then $\pi^{(A,b)}(h) = l_1$ and $\pi^{(A,b)}(h) = l_2$, thus $l_1 = l_2 \in C_A$. □

Proposition 3.2. *Let $\mathcal{M}$ be a realizable matroid over $\mathbb{Z}$. Let $(A \cup \{b\}, h)$ be in $\operatorname{Gr}\mathcal{M}$. Then, there exist $l \in G_A$ such that $(A \cup \{b\}, h) \triangleright (A, l)$.*

Proof. Consider the $\mathbb{Q}$-matroid $\mathcal{M} \otimes \mathbb{Q}$ and observe that $\mathcal{M} \otimes \mathbb{Q}(A) = \mathbb{Q}^{d(A)}$. Thus, $(A \cup \{b\}, h) \in \operatorname{Gr}\mathcal{M}$ implies that $A \cup \{b\}$ belongs to the independent set complex of $\mathcal{M} \otimes \mathbb{Q}$. This is a simplicial complex and so if $A \subset A \cup \{b\}$ then A also belongs to the independent set complex and so, by definition, one has that $d(\emptyset)-d(A)=\#A$. Hence, remark that the map $\pi^{(A,b)}: C_{A\cup\{b\}} \to C_A$ is well defined and pick $l=\pi^{(A,b)}(h)$. Such $l \in C_A$ satisfies the statement. □

Theorem 3.3. *For every representable matroid $\mathcal{M}$ over $\mathbb{Z}$ with $\mathcal{M}(\emptyset)=\mathbb{Z}^d$, $\operatorname{Gr}\mathcal{M}$ is a simplicial poset.*

Proof. The element $(\emptyset, e)$ is the bottom element. The only thing to check is that the interval $I=[(\emptyset,e),(A\cup\{b\},h)]$ is boolean for every independent set $A\cup\{b\}$ and every $h\in G_{A\cup\{b\}}$. Recursively using Proposition 3.2, for every subset E of $A\cup\{b\}$ there exists $l_e\in G_E$ such that (E,l_e) belongs to I. Moreover, because of Proposition 3.1, such subset E appears only once in the interval I. Thus, I is isomorphic as a poset to the boolean lattice $[\emptyset, A\cup\{b\}]$. □

Many of the facts shown in Sections 5 and 6 of [17] can be proved as an application of the previous theorem. It is worth mentioning, for instance, that as a corollary of Theorem 3.3, one gets Lemma 6.1 of [17].

Lemma 3.4 ([17, Lemma 6.1])**.** *Let $\mathcal{M}$ be a realizable matroid over $\mathbb{Z}$. We call $E_{\mathcal{M}}(y)=\sum_{A\subseteq[n]}(y-1)^{\#A-\operatorname{cork}(A)}$ the polynomial of the external activity of $\mathcal{M}$. Denote by C_0 the pair $(A,l)\in\operatorname{Gr}\mathcal{M}$ such that $d(A)=0$. Finally call M_A the restriction of the matroid $\mathcal{M}$ to (A,l).*

Then,

$$T_{\mathcal{M}}(1,y)=\sum_{(A,l)\in C_0} E_{M_A}(y).$$

Indeed, the fact that each interval $[(\emptyset,e),(A\cup\{b\},h)]$ is isomorphic to the boolean lattice $[\emptyset, A\cup\{b\}]$ implies that in the realizable arithmetic case, the toric arrangement associated looks locally as a hyperplane arrangement.

By applying Theorem 3.3, one can extend the main result to any realizable $\mathbb{Z}$-matroid. To do this, we need the following technical definition. Given an element σ of a poset P we denote the *link* of σ by $\operatorname{link}_P(\sigma)$:

$$\operatorname{link}_P(\sigma)=\{\tau\in P:\sigma\le\tau\}\subseteq P.$$

Theorem A. *If $\mathcal{M}$ is a realizable $\mathbb{Z}$-matroid, then $\operatorname{Gr}\mathcal{M}$ is a disjoint union of $m(\emptyset)$ $(=\#G_\emptyset)$ simplicial posets isomorphic to $\operatorname{link}_{\operatorname{Gr}\mathcal{M}}(\emptyset,e)$.*

Proof. If $\mathcal{M}(\emptyset)$ is a free group, we have already proved that the statement is true in Theorem 3.3. If $\mathcal{M}(\emptyset)$ is not free, pick $c\in C_\emptyset$. Each pair $(\emptyset,c)$ is minimal in $\operatorname{Gr}\mathcal{M}$. Moreover, there is a natural poset isomorphism from the elements of $\operatorname{link}_{\operatorname{Gr}\mathcal{M}}(\emptyset,e)$ to the elements of $\operatorname{link}_{\operatorname{Gr}\mathcal{M}}(\emptyset,c)$. The isomorphism sends $(E,l)\in\operatorname{link}_{\operatorname{Gr}\mathcal{M}}(\emptyset,e)$ to $(E,cl)\in\operatorname{link}_{\operatorname{Gr}\mathcal{M}}(\emptyset,c)$.

Finally, define for every $A\subseteq[n]$, $\mathcal{M}'(A)={}^{\mathcal{M}(A)}/_{G_\emptyset}$. This is a realizable $\mathbb{Z}$-matroid and $\mathcal{M}'(\emptyset)$ is free. Moreover, $\operatorname{Gr}(\mathcal{M}')=\operatorname{link}_{\operatorname{Gr}\mathcal{M}}(\emptyset,e)$. □

4. The Hilbert series of the face module

In this section we show that the face module and the Grothendieck–Tutte polynomial of a realizable $\mathbb{Z}$-matroid are related as in the classical case.

Recall that the face ring of a simplicial poset P has been defined in Section 1 and it is denoted by $\mathbf{k}[P]$. Theorem 3.3 shows that if $\mathcal{M}$ is representable matroid over $\mathbb{Z}$ with $\mathcal{M}(\emptyset) = \mathbb{Z}^d$, then $\operatorname{Gr}\mathcal{M}$ is a simplicial poset. Therefore, we might define the face ring of such matroid $\mathcal{M}$ as

$$\mathbf{k}[\mathcal{M}] = \mathbf{k}[\operatorname{Gr}(\mathcal{M})].$$

In the general realizable case $\mathcal{M}(\emptyset) = \mathbb{Z}^{d(\emptyset)} \times G_\emptyset$, Theorem A ensures that $\operatorname{Gr}\mathcal{M}$ is a union of $m(\emptyset) = \#G_\emptyset$ a simplicial posets. The correct algebraic structure is no longer a ring, but a module. Combining these facts, the reader can make sense of the following definition.

Definition 4.1. *The face module* $\mathbf{k}[\mathcal{M}]$ *of* $\mathcal{M}$ *is*

$$\mathbf{k}[\mathcal{M}] = \mathbf{k}[\operatorname{Gr}(\mathcal{M}')]^{m(\emptyset)},$$

where $\mathcal{M}'$ *is the matroid defined for every* $A \subseteq [n]$ *by* $\mathcal{M}'(A) = {}^{\mathcal{M}(A)}/_{G_\emptyset}$.

Note that $\mathbf{k}[\mathcal{M}]$ is a free module over the ring $\mathbf{k}[\operatorname{link}_{\operatorname{Gr}\mathcal{M}}(\emptyset, e)]$. In other words, $\mathbf{k}[\mathcal{M}] = \mathbf{k}[\operatorname{link}_{\operatorname{Gr}\mathcal{M}}(\emptyset, e)]^{m(\emptyset)}$. If $\mathcal{M}(\emptyset)$ is free then the face module has a ring structure, i.e. $\mathbf{k}[\mathcal{M}] = \mathbf{k}[\operatorname{Gr}\mathcal{M}]$. Finally, recall that the dual of a realizable $\mathbb{Z}$-matroid is still realizable (see Section 2 of [4]).

Theorem B. *If* $\mathcal{M}$ *is a realizable* $\mathbb{Z}$*-matroid of rank* r*, then*

$$\operatorname{Hilb}(\mathbf{k}[\mathcal{M}], t) = \frac{t^r}{(1-t)^r} T_{\mathcal{M}^*}(1, {}^1/_t).$$

Proof. For the additivity property of the Hilbert series, it is enough to show that the theorem is true in the case $m(\emptyset) = 1$. $\operatorname{Gr}\mathcal{M}$ is a simplicial poset because of Theorem 3.3. One defines its $\mathbf{h}$-vector as

$$\sum_{i=0}^{r} f_{i-1}(\operatorname{Gr}\mathcal{M})(t-1)^{r-i} = \sum_{i=0}^{r} h_i(\operatorname{Gr}\mathcal{M})t^{r-i}.$$

We observe that

$$f_{i-1}(\operatorname{Gr}\mathcal{M}) = \sum_{\#A=i} m(A),$$

where $m(A)$ is the order of the torsion part of $\mathcal{M}(A)$. Hence

$$\sum_{i=0}^{r} h_i(\operatorname{Gr}\mathcal{M})t^{r-i} = \sum_{A\in[n]} m(A)(t-1)^{r-d(A)} = T_{\mathcal{M}}(t, 1).$$

Therefore, $t^r T_{\mathcal{M}}({}^1/_t, 1) = \sum_{i=0}^{r} h_i(\operatorname{Gr}\mathcal{M})t^i$. We now apply Theorem 1.8 together with (2) to get the result. ☐

Remark 4.2. The proof of the above theorem works for every simplicial partial order of the set in Definition 2.1. We conjecture Theorem B is true for every matroid over $\mathbb{Z}$ and the only obstacle to this result is hidden in the nature of the *canonical projections*. Indeed, for a nonrealizable $\mathbb{Z}$-matroid, it is not clear if there is a unique simplicial order of the set in Definition 2.1, that respects Definition 2.2.

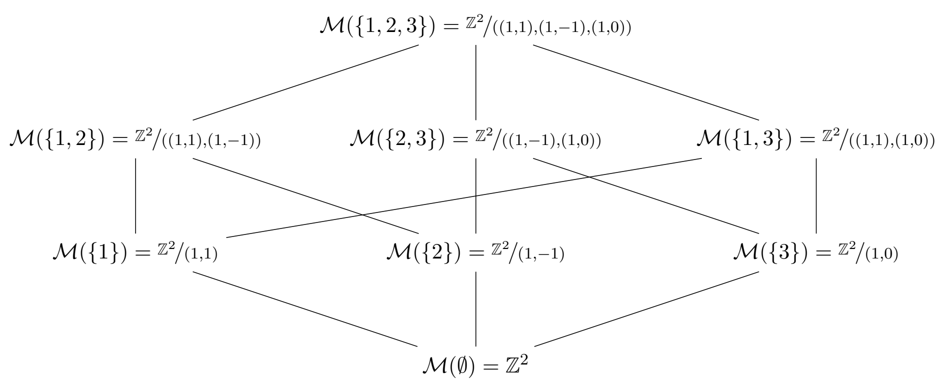
Throughout the paper we have played with two toy examples: $\mathcal{M}_1$ defined in Example 1.12 and $\mathcal{M}_2$ defined in Example 1.13. In Table 1 we summarize where to find the related calculations: the computations of the poset, the Tutte polynomial, the face ring, etc.

	$\mathcal{M}$	$T_{\mathcal{M}}$	$\operatorname{Gr}\mathcal{M}$	$\mathbf{h}$, $\mathbf{f}$	$\mathbf{k}[\mathcal{M}]$	Hilb	Fig.
$\mathcal{M}_1$	Ex. 1.12	Ex. 1.15	Ex. 2.3	Ex. 1.7	Ex. 1.5	Ex. 1.10	Fig. 2.a)
$\mathcal{M}_2$	Ex. 1.13	Ex. 1.16	Ex. 2.3	Ex. 1.6	Ex. 1.4	Ex. 1.9	Fig. 2.b)

TABLE 1. The computations of the toy examples $\mathcal{M}_1$ and $\mathcal{M}_2$.

We now provide a more substantial example to verify the recently proved Theorem B.

Example 4.3. Let $n = 3$ and we define $\mathcal{M}$ as follows:



Let us compute $T_{\mathcal{M}}$. We list the contribution in (1) for each subset:

$\emptyset$	$(x-1)^2$
$\{1\},\{2\},\{3\}$	$(x-1)$
$\{1,2\}$	2
$\{1,3\}$	1
$\{2,3\}$	1
$\{1,2,3\}$	$(y-1)$

Thus, we get $T_{\mathcal{M}}(x,y) = x^2 + x + y + 1$.

Now we want to construct $\operatorname{Gr}\mathcal{M}$. We start by observing that the matroid in Example 1.13 is a submatroid of $\mathcal{M}$. We have already studied this submatroid and therefore we do not need to explain the covering relation among $(\emptyset, e)$, $(\{1\}, e)$, $(\{1\}, \zeta)$, $(\{2\}, e)$, $([2], e)$, and $([2], \zeta)$.

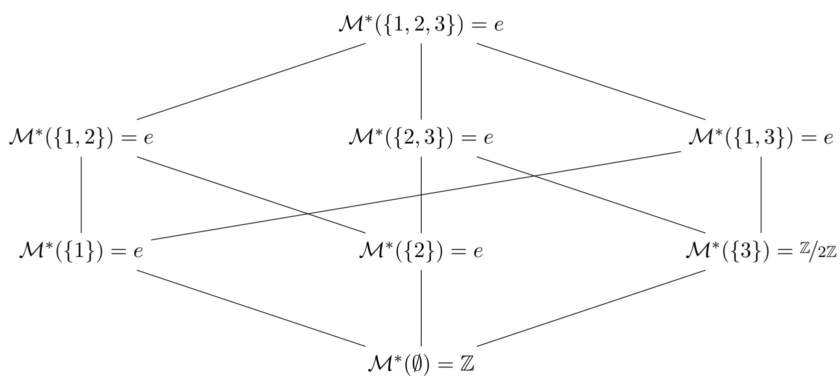
In $\operatorname{Gr}\mathcal{M}$, we also find $(\{3\}, e)$, $(\{1,3\}, e)$, and $(\{2,3\}, e)$. We remark that the subset $[3]$ does not appear in the poset, because $2 = \operatorname{cork}([3]) \neq \#[3] = 3$. Thus, it remains to study which elements are covered by the subsets containing 3. Readily, $(\{3\}, e)$ covers $(\emptyset, e)$.

Since $C_{\{2,3\}}$, $C_{\{1,3\}}$ are trivial groups, then $(\{1,3\},e)$ covers $(\{1\},e)$ and $(\{3\},e)$, and similarly $(\{2,3\},e)$ covers $(\{2\},e)$ and $(\{3\},e)$. Figure 2.c) shows $\operatorname{Gr}\mathcal{M}$.

Using Macaulay2 [13], we compute the Hilbert series of the face ring:

$$\operatorname{Hilb}(\mathbf{k}[\mathcal{M}],t) = \frac{1+t+2t^2}{(1-t)^2}.$$

Let us focus on the dual matroid; one can easily compute that



By duality $T_{\mathcal{M}^*}(x,y) = y^2+y+x+1$ and by trivial computation,

$$\operatorname{Hilb}(\mathbf{k}[\mathcal{M}^*],t) = \frac{1+3t}{(1-t)}.$$

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