FACE MODULE FOR REALIZABLE $\mathbb{Z}$-MATROIDS

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Abstract. In this work, we define the face module for a realizable matroid over $\mathbb{Z}$. Its Hilbert series is, indeed, the expected specialization of the Grothendieck–Tutte polynomial defined by Fink and Moci in [10].

A matroid $M$ is a simplicial complex $\mathcal{I}$ on a ground set $[n] = \{1, \ldots, n\}$, such that

$$A, B \in \mathcal{I}, \#A > \#B \Rightarrow \exists a \in A \setminus B : B \cup \{a\} \in \mathcal{I}.$$ 

The latter is called independent set exchange property and $\mathcal{I}$ is often called independent sets family. Matroids encapsulate the combinatorics that underlie the arrangements of hyperplanes in affine or projective space.

There are two classical objects one associates to a matroid: the Stanley–Reisner ring $k[M]$, that is the face ring of $\mathcal{I}$, and the Tutte polynomial $T_M$. They are related by the following result:

**Theorem.** Let $M$ be a matroid of rank $r$ with ground set $[n]$ and call $M^*$ its dual matroid. Then:

$$\text{Hilb}(k[M], t) = \frac{t^r}{(1-t)^r} T_{M^*}(1, 1/t).$$

where $\text{Hilb}(k[M], t)$ is the Hilbert series of $k[M]$.

Fink and Moci [10] generalize the concept of matroid to a larger setting: a matroid $\mathcal{M}$ over a commutative ring $R$ on the ground set $[n]$ is an assignment of an $R$-module $\mathcal{M}(A)$ for every subset $A$ of $[n]$. This assignment has to respect a certain local patching condition.

One of the reasons behind this generalization is to deal with arrangements of hypersurfaces. Steps toward a Rota cryptomorphism are already done over a Dedekind domain [10], over valuation ring [11] and, more relevant for this manuscript, over $\mathbb{Z}$ [8].

For the goal of this paper, it is worth recalling that a realizable matroid over $\mathbb{Z}$ relates to a generalized toric arrangement [17, 4].
In this paper, we introduce a candidate for the role of the independent set complex for realizable matroids over \( \mathbb{Z} \), that we call a partially order set (poset) of torsions of \( M \). This new poset provides a combinatorial tool for (generalized) toric arrangements to compute their integral cohomology [2], to construct their wonderful models [18] and their projective wonderful models [3]. Moreover, this paves the way to show one of Rota’s cryptomorphisms for matroids over \( \mathbb{Z} \).

Given a matroid \( M \) over \( \mathbb{Z} \), \( M(A) \) is an abelian group and so \( M(A) = \mathbb{Z}^{d(A)} \times G_A \), where \( G_A \) is a torsion group and its cardinality, \( \#G_A \), is often referred as the multiplicity of \( A \), \( m(A) \). Call \( d = d(\emptyset) \), the dimension of the matroid \( M \) and \( C_A \), the dual group of \( G_A \).

**Definition.** We denote by \( \text{Gr}_M \) the set of torsions of \( M \). This is the set of all pairs \( (A,l) \) with \( d - d(A) = \#A \) and \( l \in C_A \).

First, observe that \( (\emptyset, e) \) always belongs to \( \text{Gr}_M \). We are going to give a partial order to \( \text{Gr}_M \) by defining certain covering relations inspired by the poset of layers of a toric arrangement [17, 4]. Similar ad hoc constructions appear also in [2]. This order depends on the realization of the matroids \( M \). Indeed, every realization provides a surjective map \( \pi : C_{A \cup b} \twoheadrightarrow C_A \). (See Section 2 for further details.)

**Definition.** Let \( M \) be a realized \( \mathbb{Z} \)-matroid. Let \( (A \cup \{b\}, h) \) and \( (A, l) \) be two elements of \( \text{Gr}_M \). We say that \( (A \cup \{b\}, h) \) covers \( (A, l) \) if and only if \( \pi(h) = l \).

This poset is not a simplicial complex, but it is the union of identical simplicial posets.

**Theorem A.** If \( M \) is a realized \( \mathbb{Z} \)-matroid, then \( \text{Gr}_M \) is a disjoint union of \( m(\emptyset) \) simplicial posets isomorphic to the link of \( (\emptyset, e) \) in the poset \( \text{Gr}_M \).

As a byproduct, one can reproduce many of the results of Sections 5 and 6 of [17], but at the cost of losing part of the geometrical intuition.

From the poset \( \text{Gr}_M \) we define a face module \( k[M] \) associated to \( M \). For this, we use Stanley’s construction [19] of the face ring for simplicial posets. As proof that \( \text{Gr}_M \) is the correct combinatorial object to study, we also show that the Hilbert series of its face module is the specialization of the Grothendieck–Tutte polynomial, as in the classical case:

**Theorem B.** If \( M \) is a realizable \( \mathbb{Z} \)-matroid of rank \( r \), then

\[
\text{Hilb}(k[M], t) = \frac{t^r}{(1-t)^r} T_{M^*}(1, 1/t).
\]

The Grothendieck–Tutte polynomial for matroid over a ring has been defined by Fink and Moci [10] as a function of \( M \) in a certain Grothendieck
ring of matroids, but, in our setting, $T_M$ is more concretely the arithmetic Tutte polynomial, see [4]. Precisely,

$$T_M(x, y) = \sum_{A \subseteq [n]} m(A)(x - 1)^{r - \text{cork}(A)}(y - 1)^{\#A - \text{cork}(A)},$$

where $\text{cork}(A) = d(\emptyset) - d(A)$.

It is worth mentioning that it is not clear if the poset of torsion is uniquely defined for nonrealizable matroids. On the other hand, the proof of Theorem B holds easily for any simplicial poset with the correct $f$-vector. Therefore we conjecture that Theorem B holds for every $\mathbb{Z}$-matroid. We say more about this in Remark 4.2.

The paper is organized as follows: in Section 1 we recall all the basic notions needed for a full comprehension of the results. In Section 2 we define the poset of torsions and in Section 3 we prove Theorem A. Finally, in Section 4 we show Theorem B.

1. Basic notions

1.1. Simplicial posets. Let $(P, <)$ be a finite partially ordered set (poset). A poset with a unique initial element, denoted by $\hat{0}$, is said to be simplicial if for each $\sigma \in P$ the segment $[\hat{0}, \sigma] = \{x \in P : \hat{0} \leq x \leq \sigma\}$ is a boolean lattice. We say that the rank of $[\hat{0}, \sigma]$ is the length of its maximal chain; therefore $(P, <)$ has a natural rank function $\text{rk}$ induced by the rank of the segments $[\hat{0}, \sigma]$. We denote by $r$ the rank of $P$, the maximal rank among all its segments.

For any $\sigma$ and $\tau$ in $P$, $\sigma \land \tau$ is the set of their greatest lower bounds (meets) and $\sigma \lor \tau$ is the set of their least common upper bounds (joins). For a simplicial poset, $\sigma \land \tau$ is a singleton and by an abuse of notation we identify the $\sigma \land \tau$ with the unique greatest lower bound of $\sigma$ and $\tau$.

**Example 1.1.** Consider the set given by $P_1 = \{\hat{0}, a, b, 1, 2\}$, where every number is greater or equal to every letter and every element is greater or equal to $\hat{0}$, see Figure 1.a). This is a simplicial poset. It is not the face poset of any simplicial complex, but it is the face poset of a digon, a CW-complex shown in Figure 1.b). Its order complex is a triangulation of the one dimensional sphere, see Figure 1.c). We compute few examples of meets and joins that are useful in future computations: $1 \land a = \{a\}$, $a \land b = \{\hat{0}\}$, $a \lor b = \{1, 2\}$, and $1 \lor 2 = \emptyset$.

**Example 1.2.** Consider the set $P_2 = \{\hat{0}, a, b, c, 1\}$ with the same order law given for $P_1$, see Figure 1.d). This is not a simplicial poset, because $[\hat{0}, 1]$ is not boolean.

1.2. Face ring. Given a field $k$, we set the polynomial ring $R_P = k[x_\sigma : \sigma \in P]$ where $x_\sigma$ has degree $\text{rk} \sigma$. In this recap section we are going to follow the notation in [19].
\textbf{Definition 1.3.} The face ideal of a simplicial poset $P$ is the ideal of $R_P$ defined as

$$I_P = \left( x_0 - 1, x_\sigma x_\tau - x_\sigma \land x_\tau \left( \sum_{\gamma \in \sigma \lor \tau} x_\gamma \right) \right) \text{ for any } \sigma, \tau \in P.$$

As a notation, the sum $\sum_{\gamma \in \sigma \lor \tau} x_\gamma$ is zero if $\sigma \lor \tau = \emptyset$. Moreover, the face ring of a simplicial poset $P$ is the quotient

$$k[P] = \frac{k[x_\sigma : \sigma \in P]}{I_P}.$$

\textbf{Example 1.4.} Consider the simplicial poset $P_1$ defined in Example 1.1, $R_{P_1}$ is the polynomial ring $k[x_0, x_a, x_b, x_1, x_2]$.

The ideal $I_{P_1} = \langle x_0 - 1, x_a x_b - (x_1 + x_2), x_1 x_2 \rangle$. Indeed, the basis generators are obtained by substituting respectively $(a, b)$ and $(1, 2)$ to the pair the $(\sigma, \tau)$ into $x_\sigma x_\tau - x_\sigma \land x_\tau \left( \sum_{\gamma \in \sigma \lor \tau} x_\gamma \right)$. For all other values of $(\sigma, \tau)$, the previous relation is trivial. The face ring of $P_1$ is therefore defined as the quotient

$$k[P_1] = \frac{k[x_0, x_a, x_b, x_1, x_2]}{(x_0 - 1, x_a x_b - (x_1 + x_2), x_1 x_2)}.$$

This definition generalizes the Stanley–Reisner ring of a simplicial complex. Given an abstract simplicial complex $\Delta$ on $n$ vertices its Stanley–Reisner ring $k[\Delta]$ the following quotient ring $k[\Delta] = k[x_1, \ldots, x_n]/I_\Delta$, where $I_\Delta = \langle x_{i_1}, \ldots, x_{i_r} : \{i_1, \ldots, i_r\} \notin \Delta \rangle$. One can easily check that the two face ring definitions coincide in the case of an abstract simplicial complex.

\textbf{Example 1.5.} Consider the simplicial poset $P_2$ defined in Figure 2.a). This is the face poset of a simplicial complex. Precisely, the path graph on 3 elements. Its face ring, $k[P_2]$, is isomorphic to $k[x,y,z]/(xz)$.

From now on, we assume that $P$ is a simplicial poset, that $\Delta$ is an abstract simplicial complex and $r$ denotes their ranks.
1.3. The Hilbert series of the face ring. Let $N$ be a finitely generated $\mathbb{N}$-graded $A$-module where $A$ is a finitely generated $\mathbb{N}$-graded commutative algebra over $k$. Denote by $N_i$ the homogeneous part of degree $i$. The Hilbert series of $N$ is the following generating function:

$$\text{Hilb}(N,t) = \sum_{i \geq 0} \dim_k(N_i)t^i,$$

where $\dim_k(N_i)$ is the dimension of $N_i$ as a $k$-vector space. We consider $\text{Hilb}(A,t)$ as the Hilbert series of $A$ seen as a module over its self.

The ring $k[P]$ is graded and its Hilbert series encodes many combinatorial objects, like the $f$-vector and the $h$-vector. Here, we briefly recall their definitions. The $f$-vector, $f(P)$, of a simplicial poset $P$ is the vector $(f_{-1}, f_0, \ldots, f_{r-1})$ where $f_i$ is the number of elements of rank $i + 1$ in $P$; by notation $f_{-1} = 1$ counts the empty set as a dimension $-1$ object. The $h$-vector of $P$ is the vector $h(P) = (h_0, h_1, \ldots, h_r)$ defined recursively from the $f$-vector by using

$$\sum_{i=0}^r f_{i-1}(t-1)^{r-i} = \sum_{i=0}^r h_it^{r-i}.$$ 

**Example 1.6.** We compute $f(P_1)$ and $h(P_1)$ for the simplicial poset in Example 1.1. Trivially, $f(P_1) = (1, 2, 2)$. Expanding $\sum_{i=0}^2 f_{i-1}(t-1)^{2-i}$ one gets $t^2 + 1$ and therefore $h(P_1) = (1, 0, 1)$.

**Example 1.7.** Let us make similar computation for $P_2$ in Example 1.5. Clearly $f(P_2) = (1, 3, 2)$ and by expanding $\sum_{i=0}^2 f_{i-1}(t-1)^{2-i} = t^2 + t$ and therefore $h(P_2) = (1, 1, 0)$.

As said, one can read the $f$-vector and the $h$-vector from the face ring $k[P]$.

**Theorem 1.8 ([19, Proposition 3.8]).** Let $P$ be a simplicial poset of rank $r$ and let $k[P]$ be its face ring. Then

$$\text{Hilb}(k[P],t) = \frac{h_0 + h_1t + \cdots + h_rt^r}{(1-t)^r}.$$ 

**Example 1.9.** Let us verify the previous theorem for our toy simplicial poset $P_1$. Its face ring is computed in Example 1.4 and its $f$-vector and $h$-vector are shown in Example 1.6.

By a dirty hands computation or by using Macaulay2 [13] we see that this Hilbert series simplifies to

$$\text{Hilb}(k[P_1],t) = \frac{1 - t^2 - t^4 + t^6}{(1-t^2)^2(1-t^2)} = \frac{1 + t^2}{(1-t)^2}$$

and this is indeed the expected result.

**Example 1.10.** In the case of the face poset $P_2$, it is easy to verify what we have just stated. Indeed, in Example 1.5, the face ring and in Example 1.6 we computed the face ring, the $f$-vector and $h$-vector. It is trivial to observe the following:

$$\text{Hilb}\left(\frac{k[x,y,z]}{(xz)},t\right) = \frac{1 - t^2}{(1-t)^3} = \frac{1 + t}{(1-t)^2}.$$
1.4. Matroid over a ring $R$. In [10], Fink and Moci generalize the concept of matroid to matroid over a commutative ring $R$. In this section we give the general definition then deal with the case $R = \mathbb{Z}$.

1.4.1. $R$-matroids. Let $2^{[n]}$ be the set of all subsets of $[n]$ and let $R$-mod the category of finitely generated $R$-modules.

Definition 1.11. A matroid over the ring $R$ is the function

$$\mathcal{M} : 2^{[n]} \to R\text{-mod}$$

such that for any subset $A$ of $[n]$ and any elements $b$ and $c$ of $[n] \setminus A$ there exist $x_{b,c}$ and $y_{b,c}$ in $\mathcal{M}(A)$, such that:

$$\mathcal{M}(A \cup \{b\}) \cong \mathcal{M}(A)/\langle x_{b,c} \rangle$$

$$\mathcal{M}(A \cup \{c\}) \cong \mathcal{M}(A)/\langle y_{b,c} \rangle$$

$$\mathcal{M}(A \cup \{b,c\}) \cong \mathcal{M}(A)/\langle x_{b,c}, y_{b,c} \rangle$$

The choice of $\mathcal{M}$ is relevant only up to isomorphism. Moreover, we are going to assume that $\mathcal{M}$ is essential, that is no nontrivial projective module is a direct summand of $\mathcal{M}([n])$.

In Proposition 2.6 of [10], Fink and Moci had shown that an essential matroid over a field $k$ is a matroid in the classical case. For this reason, from now on, we are going to call these $k$-matroids.

1.4.2. Realizable $\mathbb{Z}$-Matroids. For this paper, we are going to set $R = \mathbb{Z}$. We define a corank function $\text{cork}(A)$ of $\mathcal{M}$ as the corank function of the $\mathbb{Q}$-matroid $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$\text{cork}(A) = \text{cork}_{\mathbb{Q}} \mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Q}(A).$$

For any subset $A$ of $[n]$, $\mathcal{M}(A)$ is an abelian group,

$$\mathcal{M}(A) = \mathbb{Z}^{d(A)} \times G_A,$$

where $G_A$ is the torsion part. We call $m(A) = \#G_A$ the multiplicity of $A$. Clearly, $\text{cork}(A) = d(\emptyset) - d(A)$.

A $\mathbb{Z}$-matroid $\mathcal{M}$ on $[n]$ is realizable if there is a list of elements $z_1, \ldots, z_n \in \mathcal{M}(\emptyset)$ such that $\mathcal{M}(A) = \mathcal{M}(\emptyset)/\langle z_i : i \in A \rangle$.

The definition (see Section 2) of the poset $\text{Gr} \mathcal{M}$ depends on the realization of the matroid $\mathcal{M}$. For this, when we need to use explicitly a realization of a matroid $\mathcal{M}$ we are going to talk about realized matroid $\mathcal{M}$.

Example 1.12. Let $R = \mathbb{Z}$ and $n = 2$. Set $\mathcal{M}(\emptyset) = \mathbb{Z}^2$, $\mathcal{M}(\{1\}) = \mathbb{Z}^2/(2,0)$, $\mathcal{M}(\{2\}) = \mathbb{Z}^2/(0,1)$ and $\mathcal{M}(\{1,2\}) = \mathbb{Z}^2/((2,0),(0,1))$.

Example 1.13. Let $A = \mathbb{Z}$ and $n = 2$. Set $\mathcal{M}(\emptyset) = \mathbb{Z}^2$, $\mathcal{M}(\{1\}) = \mathbb{Z}^2/(1,1)$, $\mathcal{M}(\{2\}) = \mathbb{Z}^2/(1,-1)$ and $\mathcal{M}(\{1,2\}) = \mathbb{Z}^2/((1,1),(1,-1))$.

Remark 1.14. The definition and results in the next sections hold also for arithmetic matroids. We do not need any of the arithmetic matroids tools. In Section 6.1 of [10, Section 6.1] it is shown that $\mathbb{Z}$-matroids with an extra
1.4.3. Grothendieck–Tutte polynomial. In the rest of this section, we are going to define the Grothendieck–Tutte polynomial for a matroid over \( \mathbb{Z} \).

Let \( L_0(\text{Ab}) \) be the Grothendieck type ring of abelian groups, that is the free group generated by the isomorphic classes \([G]\) for any finitely generated abelian group \( G \). There is a ring multiplication given by \([G][G'] = [G \times G']\).

This object is very useful; for instance it appears in [6, 7, 15, 16]. In Section 7.1 of [10], it is proved the Grothendieck–Tutte class is well defined to be the following element of \( L_0(\text{Ab}) \) ⊗ \( L_0(\text{Ab}) \):

\[
GT_M = \sum_{A \subseteq E} [M(A)][M^*(E \setminus A)]
\]

where \( M^* \) is the matroid dual to \( M \) and \( E \) is their common ground set. (We use \( E \) to avoid confusion between \([n]\) and \([M(\cdot)]\).) For a precise definition of the dual matroid \( M^* \) we refer to Section 7 of [4].

Let \( G \) be a group and consider its class \([G]\) in \( L_0(\text{Ab}) \): since \( G \cong \mathbb{Z}^d \times G_t \), one has that \([G] = [\mathbb{Z}^d][G_t] \in L_0(\text{Ab})\). Now, fix the following evaluations for \( L_0(\text{Ab}) \):

\[
v_x([G]) = \#G_t(x - 1)^d \quad \text{and} \quad v_y([G]) = \#G_t(y - 1)^d
\]

and consider the image of \( GT_M \) with respect to the map

\[
v_x \otimes v_y : L_0(\text{Ab}) \otimes \mathbb{Z} L_0(\text{Ab}) \to \mathbb{Z}[x, y].
\]

The Grothendieck–Tutte polynomial for \( M \) is 

\[
T_M(x, y) = (v_x \otimes v_y)(GT_M).
\]

Then,

\[
T_M(x, y) = \sum_{A \subseteq [n]} m(A)(x - 1)^{r-d(A)}(y - 1)^{#A-d(A)}.
\]

This polynomial was first introduced by Moci in [17] and it is often called the arithmetic Tutte polynomial. It easy to observe that

\[
T_M(x, y) = T_{M^*}(y, x).
\]

**Example 1.15.** Let us compute \( T_M \) for the \( \mathbb{Z} \)-matroid given in Example 1.12. In (1) the contribution of the empty set is \( (x - 1)^2 \); the contribution of the singleton \( \{1\} \) is \( 2(x - 1) \); the contribution of the singleton \( \{2\} \) is \( (x - 1) \); finally, the contribution of the full ground set \( [2] \) is \( 2 \). Thus, \( T_M(x, y) = x^2 + x \).

**Example 1.16.** We compute \( T_M \) for the matroid in Example 1.13. In (1) the contribution of the empty set is \( (x - 1)^2 \); the contribution of each singleton is \( (x - 1) \); finally, the contribution of the full ground set \( [2] \) is \( 2 \). Thus, \( T_M(x, y) = x^2 + 1 \).

As remarked in the introduction, the face ring and the Grothendieck–Tutte polynomial of a \( k \)-matroid are related.
Theorem 1.17. Let $\mathcal{M}$ be a $k$-matroid of rank $r$. Let $k[\mathcal{M}]$ be its face ring. Then,
\begin{equation}
\text{Hilb}(k[\mathcal{M}], t) = \frac{t^r}{(1-t)^r} T_{M^*}(1, 1/t).
\end{equation}

Proof. To the author's best knowledge, this result appears first in the above form in the Appendix (section A.3) by Björner in the work of De Concini and Procesi [5]. 

The goal of the paper is to extend this result to realizable $\mathbb{Z}$-matroids.

2. The Po Set of Torsions

The aim of this section is to define a new poset taking the role of the independent complex in the case of $k$-matroids.

Throughout this section we assume that $A$ is a subset of $[n]$ and $b$ is in $[n] \setminus A$. Let $b = c$ in Definition 1.11: it requires the existence of a quotient homomorphism by $x_{b,b} \in M(A)$:
\begin{equation}
\pi_{(A,b)} : M(A) \rightarrow M(A \cup \{b\}).
\end{equation}

Call $\pi_{(A,b)}$ the canonical projection associated to $A$ and $b$. While the homomorphism $\pi_{(A,b)}$ is unique, the choice of $x_{b,b}$ is not. In the case of realizable $\mathbb{Z}$-matroids, $x_{b,b}$ is unique and we denote it by $x_b$.

For any subset $A$ of $[n]$, $M(A) \simeq \mathbb{Z}^{d(A)} \times G_A$, where $d(A)$ is the rank of $M(A)$ and $G_A$ is the torsion part of $M(A)$. Call $C_A$, the dual group of $G_A$.

Definition 2.1. We call $\text{Gr} M$ the set of torsions of $M$. This is the set of all pairs $(A,l)$ with $d(\emptyset) - d(A) = \#A$ and $l \in C_A$.

Inspired by Section 5 of [17], we are going to view such a set as a bunch of tori with the right dimension and cardinality, prescribed by the $\mathbb{Z}$-matroid. This is the reason that lead us to work with the dual group $C_A$ instead of $G_A$, even if they are isomorphic. Moreover, consider $A$ and $A \cup \{b\}$ such that $d(\emptyset) - d(A) = \#A$ and $d(\emptyset) - d(A \cup \{b\}) = \#A + 1$. Then, the map $\pi_{(A,b)}$ restricted to $G_A$ is injective and its dual $\pi^{(A,b)} : C_{A \cup \{b\}} \rightarrow C_A$ is surjective.

Definition 2.2. Let $(A \cup \{b\}, h)$ and $(A, l)$ be two elements of $\text{Gr} M$. We say that $(A \cup \{b\}, h)$ covers $(A, l)$, and we write $(A \cup \{b\}, h) \triangleright (A, l)$, if and only if $h \subseteq l$.

Example 2.3. Let us compute the poset of torsions of the matroid given in Example 1.12. We show the poset in Figure 2.a). Clearly there are six elements $(\emptyset, e), ((1), e), ((1), \zeta), ((2), e), ([2], e)$, and $([2], \zeta)$.

Now observe that trivially $(\{1\}, e), (\{1\}, \zeta), ([2], e)$ cover $(\emptyset, e)$ and $([2], x)$ covers $([2], e)$ because $C_{[2]}$ surjects to $C_{\{2\}} = \{e\}$. Moreover $([2], x)$ covers $([1], x)$, because $C_{[2]} \simeq C_{\{1\}}$, thus $\pi^{\{1\}, [2]}(x) = x$. This also shows that $([2], x)$ does not cover $([1], y)$ if $x \neq y \in \mathbb{Z}/2\mathbb{Z}$.
Example 2.4. The poset of torsions of the matroid given in Example 1.13 is actually the poset $P_1$ defined in Example 1.1 and discussed throughout Section 1. This poset is in Figure 2.b). We leave it to the reader to verify the covering relations, which are straightforward.

3. $\text{Gr}\mathcal{M}$ is a union of simplicial posets

In this section we are going to prove Theorem A, that is $\text{Gr}\mathcal{M}$ is a union of simplicial posets. We start by proving two properties of $\text{Gr}\mathcal{M}$.

**Proposition 3.1.** Let $\mathcal{M}$ be a realizable matroid over $\mathbb{Z}$. Let $(A \cup \{b\}, h)$, $(A, l_1)$, $(A, l_2)$ be in $\text{Gr}\mathcal{M}$. If $(A \cup \{b\}, h) \triangleright (A, l_1)$ and $(A \cup \{b\}, h) \triangleright (A, l_2)$ then $l_1 = l_2 \in \mathcal{M}(A)$.

**Proof.** By Definition 2.2 if $(A \cup \{b\}, h) \triangleright (A, l_1)$ and $(A \cup \{b\}, h) \triangleright (A, l_2)$ then $\pi^{(A, b)}(h) = l_1$ and $\pi^{(A, b)}(h) = l_2$, thus $l_1 = l_2 \in C_A$. \hfill $\square$

**Proposition 3.2.** Let $\mathcal{M}$ be a realizable matroid over $\mathbb{Z}$. Let $(A \cup \{b\}, h)$ be in $\text{Gr}\mathcal{M}$. Then, there exist $l \in G_A$ such that $(A \cup \{b\}, h) \triangleright (A, l)$.

**Proof.** Consider the $\mathbb{Q}$-matroid $\mathcal{M} \otimes \mathbb{Q}$ and observe that $\mathcal{M} \otimes \mathbb{Q}(A) = \mathbb{Q}^{d(A)}$. Thus, $(A \cup \{b\}, h) \in \text{Gr}\mathcal{M}$ implies that $A \cup \{b\}$ belongs to the independent set complex of $\mathcal{M} \otimes \mathbb{Q}$. This is a simplicial complex and so if $A \subset A \cup \{b\}$ then $A$ also belongs to the independent set complex and so, by definition, one has...
that $d(\emptyset) - d(A) = \# A$. Hence, remark that the map $\pi^{(A,b)} : C_{A \cup \{b\}} \to C_A$ is well defined and pick $l = \pi^{(A,b)}(h)$. Such $l \in C_A$ satisfies the statement. \qed

**Theorem 3.3.** For every representable matroid $\mathcal{M}$ over $\mathbb{Z}$ with $\mathcal{M}(\emptyset) = \mathbb{Z}^d$, $\text{Gr} \mathcal{M}$ is a simplicial poset.

**Proof.** The element $(\emptyset, e)$ is the bottom element. The only thing to check is that the interval $I = [(\emptyset, e), (A \cup \{b\}, h)]$ is boolean for every independent set $A \cup \{b\}$ and every $h \in G_{A \cup \{b\}}$. Recursively using Proposition 3.2, for every subset $E$ of $A \cup \{b\}$ there exists $l_e \in G_E$ such that $(E, l_e)$ belongs to $I$. Moreover, because of Proposition 3.1, such subset $E$ appears only once in the interval $I$. Thus, $I$ is isomorphic as a poset to the boolean lattice $[\emptyset, A \cup \{b\}]$. \qed

Many of the facts shown in Sections 5 and 6 of [17] can be proved as an application of the previous theorem. It is worth mentioning, for instance, that as a corollary of Theorem 3.3, one gets Lemma 6.1 of [17].

**Lemma 3.4 ([17, Lemma 6.1]).** Let $\mathcal{M}$ be a realizable matroid over $\mathbb{Z}$. We call $E_M(y) = \sum_{A \subseteq [n]} (y - 1)^{\# A - \text{rk}(A)}$ the polynomial of the external activity of $\mathcal{M}$. Denote by $C_0$ the pair $(A, l) \in \text{Gr} \mathcal{M}$ such that $d(A) = 0$. Finally call $M_A$ the restriction of the matroid $\mathcal{M}$ to $(A, l)$.

Then,

$$T_M(1, y) = \sum_{(A, l) \in C_0} E_{MA}(y).$$

Indeed, the fact that each interval $[\emptyset, e], (A \cup \{b\}, h)]$ is isomorphic to the boolean lattice $[\emptyset, A \cup \{b\}]$ implies that in the realizable arithmetic case, the toric arrangement associated looks locally as a hyperplane arrangement.

By applying Theorem 3.3, one can extend the main result to any realizable $\mathbb{Z}$-matroid. To do this, we need the following technical definition. Given an element $\sigma$ of a poset $P$ we denote the link of $\sigma$ by link$_P(\sigma)$:

$$\text{link}_P(\sigma) = \{ \tau \in P : \sigma \leq \tau \} \subseteq P.$$

**Theorem A.** If $\mathcal{M}$ is a realizable $\mathbb{Z}$-matroid, then $\text{Gr} \mathcal{M}$ is a disjoint union of $m(\emptyset) (= \# G_\emptyset)$ simplicial posets isomorphic to $\text{link}_{\text{Gr} \mathcal{M}}(\emptyset, e)$.

**Proof.** If $\mathcal{M}(\emptyset)$ is a free group, we have already proved that the statement is true in Theorem 3.3. If $\mathcal{M}(\emptyset)$ is not free, pick $c \in C_\emptyset$. Each pair $(\emptyset, c)$ is minimal in $\text{Gr} \mathcal{M}$. Moreover, there is a natural poset isomorphism from the elements of $\text{link}_{\text{Gr} \mathcal{M}}(\emptyset, c)$ to the elements of $\text{link}_{\text{Gr} \mathcal{M}}((0, c))$. The isomorphism sends $(E, l) \in \text{link}_{\text{Gr} \mathcal{M}}(\emptyset, c)$ to $(E, cl) \in \text{link}_{\text{Gr} \mathcal{M}}((0, c))$.

Finally, define for every $A \subseteq [n]$, $\mathcal{M}'(A) = \mathcal{M}(A)/G_A$. This is a realizable $\mathbb{Z}$-matroid and $\mathcal{M}'(\emptyset)$ is free. Moreover, $\text{Gr} \mathcal{M}' = \text{link}_{\text{Gr} \mathcal{M}}(\emptyset, e)$. \qed

4. **The Hilbert series of the face module**

In this section we show that the face module and the Grothendieck–Tutte polynomial of a realizable $\mathbb{Z}$-matroid are related as in the classical case.
Recall that the face ring of a simplicial poset $P$ has been defined in Section 1 and it is denoted by $k[P]$. Theorem 3.3 shows that if $\mathcal{M}$ is representable matroid over $\mathbb{Z}$ with $\mathcal{M}(\emptyset) = \mathbb{Z}^d$, then $\text{Gr} \mathcal{M}$ is a simplicial poset. Therefore, we might define the face ring of such matroid $\mathcal{M}$ as $k[\mathcal{M}] = k[\text{Gr}(\mathcal{M})]$. 

In the general realizable case $\mathcal{M}(\emptyset) = \mathbb{Z}^d(\emptyset) \times G_{\emptyset}$, Theorem A ensures that $\text{Gr} \mathcal{M}$ is a union of $m(\emptyset) = \#G_{\emptyset}$ a simplicial posets. The correct algebraic structure is no longer a ring, but a module. Combining these facts, the reader can make sense of the following definition.

**Definition 4.1.** The face module $k[\mathcal{M}]$ of $\mathcal{M}$ is $k[\mathcal{M}] = k[\text{Gr}(\mathcal{M}')]_{m(\emptyset)}$, where $\mathcal{M}'$ is the matroid defined for every $A \subseteq [n]$ by $\mathcal{M}'(A) = \mathcal{M}(A)/G_{\emptyset}$.

Note that $k[\mathcal{M}]$ is a free module over the ring $k[\text{link}_G \mathcal{M}(\emptyset,e)]$. In other words, $k[\mathcal{M}] = k[\text{link}_G \mathcal{M}(\emptyset,e)]^{m(\emptyset)}$. If $\mathcal{M}(\emptyset)$ is free then the face module has a ring structure, i.e. $k[\mathcal{M}] = k[\text{Gr} \mathcal{M}]$. Finally, recall that the dual of a realizable $\mathbb{Z}$-matroid is still realizable (see Section 2 of [4]).

**Theorem B.** If $\mathcal{M}$ is a realizable $\mathbb{Z}$-matroid of rank $r$, then

$$\text{Hilb}(k[\mathcal{M}], t) = \frac{t^r}{(1-t)^r} T_{\mathcal{M}'}(1, 1/t).$$

**Proof.** For the additivity property of the Hilbert series, it is enough to show that the theorem is true in the case $m(\emptyset) = 1$. $\text{Gr} \mathcal{M}$ is a simplicial poset because of Theorem 3.3. One defines its $h$-vector as

$$\sum_{i=0}^r f_{i-1}(\text{Gr} \mathcal{M})(t - 1)^{r-i} = \sum_{i=0}^r h_i(\text{Gr} \mathcal{M}) t^{r-i}.$$ 

We observe that

$$f_{i-1}(\text{Gr} \mathcal{M}) = \sum_{\#A = i} m(A),$$

where $m(A)$ is the order of the torsion part of $\mathcal{M}(A)$. Hence

$$\sum_{i=0}^r h_i(\text{Gr} \mathcal{M}) t^{r-i} = \sum_{A \in [n]} m(A)(t - 1)^{r-d(A)} = T_{\mathcal{M}}(t, 1).$$

Therefore, $t^r T_{\mathcal{M}}(1/t, 1) = \sum_{i=0}^r h_i(\text{Gr} \mathcal{M}) t^i$. We now apply Theorem 1.8 together with (2) to get the result. $\square$

**Remark 4.2.** The proof of the above theorem works for every simplicial partial order of the set in Definition 2.1. We conjecture Theorem B is true for every matroid over $\mathbb{Z}$ and the only obstacle to this result is hidden in the nature of the canonical projections. Indeed, for a nonrealizable $\mathbb{Z}$-matroid, it is not clear if there is a unique simplicial order of the set in Definition 2.1, that respects Definition 2.2.
Throughout the paper we have played with two toy examples: $\mathcal{M}_1$ defined in Example 1.12 and $\mathcal{M}_2$ defined in Example 1.13. In Table 1 we summarize where to find the related calculations: the computations of the poset, the Tutte polynomial, the face ring, etc.

$$
\begin{array}{ccccccc}
\text{Ex.} & \text{Ex.} & \text{Ex.} & \text{Ex.} & \text{Ex.} & \text{Ex.} & \text{Fig.} \\
1.12 & 1.15 & 2.3 & 1.7 & 1.5 & 1.10 & 2.a) \\
1.13 & 1.16 & 2.3 & 1.6 & 1.4 & 1.9 & 2.b) \\
\end{array}
$$

Table 1. The computations of the toy examples $\mathcal{M}_1$ and $\mathcal{M}_2$.

We now provide a more substantial example to verify the recently proved Theorem B.

**Example 4.3.** Let $n = 3$ and we define $\mathcal{M}$ as follows:

Let us compute $T_{\mathcal{M}}$. We list the contribution in (1) for each subset:

<table>
<thead>
<tr>
<th>Subset</th>
<th>Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>$(x - 1)^2$</td>
</tr>
<tr>
<td>${1}$, ${2}$, ${3}$</td>
<td>$(x - 1)$</td>
</tr>
<tr>
<td>${1, 2}$</td>
<td>2</td>
</tr>
<tr>
<td>${1, 3}$</td>
<td>1</td>
</tr>
<tr>
<td>${2, 3}$</td>
<td>1</td>
</tr>
<tr>
<td>${1, 2, 3}$</td>
<td>$(y - 1)$</td>
</tr>
</tbody>
</table>

Thus, we get $T_{\mathcal{M}}(x, y) = x^2 + x + y + 1$.

Now we want to construct $\text{Gr} \mathcal{M}$. We start by observing that the matroid in Example 1.13 is a submatroid of $\mathcal{M}$. We have already studied this submatroid and therefore we do not need to explain the covering relation among $(\emptyset, e)$, $(\{1\}, e)$, $(\{1\}, \zeta)$, $(\{2\}, e)$, $(\{2\}, e)$, and $(\{2\}, \zeta)$.

In $\text{Gr} \mathcal{M}$, we also find $(\{3\}, e)$, $(\{1, 3\}, e)$, and $(\{2, 3\}, e)$. We remark that the subset $[3]$ does not appear in the poset, because $2 = \text{cork}([3]) \neq \# [3] = 3$. Thus, it remains to study which elements are covered by the subsets containing 3. Readily, $(\{3\}, e)$ covers $(\emptyset, e)$. 
Since \( C_{\{2,3\}}, C_{\{1,3\}} \) are trivial groups, then \((\{1,3\}, e)\) covers \((\{1\}, e)\) and \((\{3\}, e)\), and similarly \((\{2,3\}, e)\) covers \((\{2\}, e)\) and \((\{3\}, e)\). Figure 2.c shows \(\text{Gr} \, \mathcal{M}\).

Using Macaulay2 [13], we compute the Hilbert series of the face ring:

\[
\text{Hilb}(k[\mathcal{M}], t) = \frac{1 + t + 2t^2}{(1 - t)^2}.
\]

Let us focus on the dual matroid; one can easily compute that

\[
\begin{align*}
\mathcal{M}^*(&\{1,2,3\}) = e \\
\mathcal{M}^*(&\{1,2\}) = e \\
\mathcal{M}^*(&\{1\}) = e \\
\mathcal{M}^*(&\{2\}) = e \\
\mathcal{M}^*(&\{3\}) = \mathbb{Z}/2\mathbb{Z} \\
\mathcal{M}^*(\emptyset) = \mathbb{Z}
\end{align*}
\]

By duality \( T_{\mathcal{M}^*}(x, y) = y^2 + y + x + 1 \) and by trivial computation,

\[
\text{Hilb}(k[\mathcal{M}^*], t) = \frac{1 + 3t}{(1 - t)}.
\]

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