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ARRANGEMENTS OF HOMOTHETS OF A CONVEX BODY II

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Abstract. A family of homothets of an o-symmetric convex body K in d-dimensional Euclidean space is called a Minkowski arrangement if no homothet contains the center of any other homothet in its interior. We show that any pairwise intersecting Minkowski arrangement of a d-dimensional convex body has at most $2 \cdot 3^d$ members. This improves a result of Polyanskii (Discrete Mathematics **340** (2017), 1950–1956).

Using similar ideas, we also give a proof the following result of Polyanskii: Let $K_1, \dots, K_n$ be a sequence of homothets of the o-symmetric convex body K, such that for any $i < j$, the center of K_j lies on the boundary of K_i. Then $n = O(3^d d)$.

1. Introduction

We use the notation $[n] = \{1, 2, \dots, n\}$. A *convex body* K in the d-dimensional Euclidean space $\mathbb{R}^d$ is a compact convex set with nonempty interior, and is *o-symmetric* if $K = -K$. A (positive) *homothet* of K is a set of the form $\lambda K + v := \{\lambda k + v : k \in K\}$, where $\lambda > 0$ is the homothety ratio, and $v \in \mathbb{R}^d$ is a translation vector. If K is o-symmetric, we also call v the *center* of the homothet $\lambda K + v$. An *arrangement of homothets of* K is a collection $\{\lambda_i K + v_i : i \in [n]\}$. A *Minkowski arrangement* of an o-symmetric convex body K is a family $\{v_i + \lambda_i K\}$ of homothets of K such that none of the homothets contains the center of any other homothet in its interior. This notion was introduced by L. Fejes Tóth [3] in the context of Minkowski's fundamental theorem on the minimal determinant of a packing lattice for a symmetric convex body, and was further studied by him in [4, 5], by Böröczky and Szabó in [2], and in connection with the Besicovitch covering theorem by Füredi and Loeb [6]. Recently, Minkowski arrangements have been used to study a problem arising in the design of wireless networks [9]. In [10] it was shown that the largest cardinality of a pairwise intersecting

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Minkowski arrangement of homothets of an o-symmetric convex body in $\mathbb{R}^d$ is $O(3^d d \log d)$. This was improved to 3^{d+1} by Polyanskii [11]. We make the following slight improvement.

Theorem 1.1. *For any o-symmetric convex body K in $\mathbb{R}^d$, a pairwise intersecting Minkowski arrangement has at most $2 \cdot 3^d$ members.*

Note that the d-cube has 3^d pairwise intersecting translates that form a Minkowski arrangement. The proof uses ideas from [7] and [8].

In [10], bounds on pairwise intersecting Minkowski arrangements were used to give an upper bound of $O(6^d d^2 \log d)$ on the length of a sequence of homothets $v_i + \lambda_i K$ of an o-symmetric convex body K such that $v_j \in \mathrm{bd}(v_i + \lambda_i K)$ whenever $j > i$. This bound was improved to $O(3^d d)$ by Polyanskii [11]. We use some similar ideas to the proof of Theorem 1.1 to give a short proof of this result of Polyanskii.

Theorem 1.2 (Polyanskii [11])**.** *Let K be an o-symmetric convex body, and $v_1, v_2, \dots, v_n \in \mathbb{R}^d$. Let $\lambda_1, \lambda_2, \dots, \lambda_{n-1} > 0$, and assume that for any $1 \le i < j \le n$ we have $v_j \in \mathrm{bd}(v_i + \lambda_i K)$. Then $n = O(3^d d)$.*

Clearly, when K is the cube, $n = 2^d$ is attained. It would be interesting to find better bounds for the maximum size of a family satisfying the conditions of Theorem 1.2.

The interest in this result is that it gives the upper bound $k^{O(3^d d)}$ to the cardinality of a set in a d-dimensional normed space in which only k nonzero distances occur between pairs of points. This is currently the best known upper bound if $k = \Omega(3^d d)$ (see [12] for a survey of this problem).

2. Proof of Theorem 1.1

Theorem 2.1. *Let $d \ge 1$. Suppose that there exists an o-symmetric convex body K in $\mathbb{R}^d$ which has a pairwise intersecting Minkowski arrangement of n homothets. Then there exists a set $\{x_1, \dots, x_n\}$ of n points in $\mathbb{R}^{d+1}$ such that $o \notin \mathrm{conv}\{x_1, \dots, x_n\}$, and for any distinct $i, j \in [n]$, $i < j$, there exists a nonzero linear functional $f_{ij} \colon \mathbb{R}^{d+1} \to \mathbb{R}$ with*

$$|f_{ij}(x_k)| \le |f_{ij}(x_i) - f_{ij}(x_j)| \quad \text{for all } k \in [n]. \tag{2.1}$$

We remark that the converse of the above theorem does not hold. We describe a simple counterexample for $d = 1$. On the one hand, clearly, a pairwise intersecting Minkowski arrangement of intervals in $\mathbb{R}$ has at most two members. On the other hand, there is a set of 5 points on the plane satisfying the conclusion of Theorem 2.1. Indeed, let $\{x_1, \dots, x_5\}$ be the vertex set of a regular pentagon, with o just outside the pentagon, close to the midpoint of an edge. It is easy to see that for any pair x_i, x_j of vertices, there is a line through o such that the projections $\pi(x_k)$ of the vertices onto the line are all within distance $|\pi(x_i) - \pi(x_j)|$ of o.

The above remark is to be contrasted with the equivalence in the following result, which generalizes part of Theorem 1.4 of [7].

Theorem 2.2. *Given $\lambda \geq 1$, and $D \in \mathbb{Z}, D \geq 1$. Then the following statements are equivalent.*

(i) There exists a set $\{x_1, \dots, x_n\}$ of n points in $\mathbb{R}^D$, such that $o \notin \operatorname{conv}\{x_1, \dots, x_n\}$, and for any distinct $i, j \in [n], i < j$ there exists a nonzero linear functional $f_{ij} : \mathbb{R}^D \to \mathbb{R}$ with

$$|f_{ij}(x_k)| \leq \frac{\lambda}{2} |f_{ij}(x_i) - f_{ij}(x_j)| \quad \text{for all } k \in [n]. \tag{2.2}$$

(ii) There is an o-symmetric convex set L in $\mathbb{R}^D$ that has n nonoverlapping translates $L + t_1, \dots, L + t_n$, each intersecting $(\lambda - 1)L$, with $o \notin \operatorname{conv}\{t_1, \dots, t_n\}$.

We note that the equivalence between (ii) and (iv) of Theorem 1.4 in [7] is exactly the above theorem in the case $\lambda = 1$.

Theorem 2.3. *Let K be an o-symmetric convex set in $\mathbb{R}^D$ with $D \geq 2$, and let $\alpha K + t_1, \dots, \alpha K + t_n$ be n nonoverlapping translates of αK with $\alpha > 0$ such that each translate intersects K, and $o \notin \operatorname{int}(\operatorname{conv}\{t_1, \dots, t_n\})$. Then*

$$n \leq \frac{(1 + 2\alpha)^{D-1}(1 + 3\alpha)}{2\alpha^D}. \tag{2.3}$$

This theorem is a slight modification of Theorem 1.5 of [7]. There the translates of αK touch K, whereas here they may overlap with K. Theorem 2.3 is sharp for $\alpha = 1$. Indeed, let K be the cube $[-1, 1]^D$, and consider the $2 \cdot 3^{D-1}$ translation vectors $\{t \in \{-2, 0, 2\}^D \ : \ t^{(1)} \geq t^{(2)}\}$.

Combining Theorems 2.1, 2.2, and 2.3 (with $\lambda = 2$, $K = (\lambda - 1)L = L$, $\alpha = 1/(\lambda - 1) = 1$), we immediately obtain Theorem 1.1.

3. Proof of Theorem 2.1

Let the Minkowski arrangement be $\{v_i + \lambda_i K : i \in [n]\}$, where $\lambda_i > 0$ and $v_i \in \mathbb{R}^d$ for each $i \in [n]$. Let $x_i = (\lambda_i^{-1} v_i, \lambda_i^{-1}) \in \mathbb{R}^d \times \mathbb{R}$, $i \in [n]$. Fix distinct $i, j \in \{1, \dots, n\}$. We will find a linear $f \colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ that satisfies (2.1). Let $\varphi \colon \mathbb{R}^d \to \mathbb{R}$ be a linear functional such that $\varphi(x) \leq \|x\|_K$ for all $x \in \mathbb{R}^d$ and $\varphi(v_j - v_i) = \|v_j - v_i\|_K$. (Thus, $\varphi^{-1}(1)$ is a hyperplane that supports K at $\|v_j - v_i\|_K^{-1} (v_j - v_i)$.)

Since any two homothets $v_k + \lambda_k K$ and $v_\ell + \lambda_\ell K$ intersect, any two of the compact intervals $\varphi(v_k + \lambda_k K)$ and $\varphi(v_\ell + \lambda_\ell K)$ intersect in $\mathbb{R}$. By Helly's Theorem in $\mathbb{R}$, there exists $\alpha \in \bigcap_{t=1}^n \varphi(v_t + \lambda_t K)$. Since $\varphi(v_i + \lambda_i K) = [\varphi(v_i) - \lambda_i, \varphi(v_i) + \lambda_i]$ and $\varphi(v_j + \lambda_j K) = [\varphi(v_j) - \lambda_j, \varphi(v_j) + \lambda_j]$, we have

$$\varphi(v_j) - \lambda_j \leq \alpha \leq \varphi(v_i) + \lambda_i.$$

By the Minkowski property,

$$\varphi(v_j - v_i) = \|v_j - v_i\|_K \geq \max\{\lambda_i, \lambda_j\}.$$

It follows that

$$\varphi(v_i) \leq \alpha \leq \varphi(v_j). \tag{3.1}$$

We set $f = (\varphi, -\alpha) \in (\mathbb{R}^d \times \mathbb{R})^*$, that is, define $f(x) = \varphi(v) - \alpha\mu$, where $x = (v, \mu) \in \mathbb{R}^d \times \mathbb{R}$. We show that $f(x_j - x_i) \geq 1$, and $|f(x_k)| \leq 1$ for all $k \in \{1, \dots, n\}$. This will show that (2.1) is satisfied, which will finish the proof.

$$\begin{aligned} f(x_j - x_i) &= \varphi(\lambda_j^{-1} v_j - \lambda_i^{-1} v_i) - \alpha(\lambda_j^{-1} - \lambda_i^{-1}) \\ &= \frac{\varphi(v_j) - \alpha}{\lambda_j} + \frac{\alpha - \varphi(v_i)}{\lambda_i} \\ &\overset{(3.1)}{\geq} \frac{\varphi(v_j) - \alpha + \alpha - \varphi(v_i)}{\max\{\lambda_i, \lambda_j\}} \\ &= \frac{\|v_j - v_i\|_K}{\max\{\lambda_i, \lambda_j\}} \geq 1. \end{aligned}$$

Since $\alpha \in \varphi(v_k + \lambda_k K)$, there exists $x \in K$ such that $\varphi(v_k + \lambda_k x) = \alpha$. Therefore,

$$|f(x_k)| = \left|\varphi(\lambda_k^{-1} v_k) - \alpha\lambda_k^{-1}\right| = |\varphi(x)| \leq \|x\|_K \leq 1.$$

□

4. Proof of Theorem 1.2

The following proof is very similar to the proof of Theorem 2.1.

Without loss of generality, $\min_i \lambda_i = 1$. Denote the unit ball of $\|\cdot\|$ by K. Let $x_i = (\lambda_i^{-1} v_i, \lambda_i^{-1}) \in \mathbb{R}^d \times \mathbb{R}$, $i = 1, \dots, n-1$. Let $N \geq 1$, to be fixed later. For each $m = 0, \dots, N$, let

$$X_m = \{x_i : i \in [n-1], \lfloor N \log_2 \lambda_i \rfloor \equiv m \pmod{N+1}\}.$$

Then $X_0, \dots, X_N$ partition $\{x_1, \dots, x_{n-1}\}$ into $N+1$ parts. Fix $x_i, x_j \in X_m$ such that $1 \leq i < j < n$. We will find a linear $f\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ such that (2.2) is satisfied for all $x_k \in X_m$ and $\lambda = 2 - 2^{1/N}$. Let $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ be a linear functional such that $\varphi(v) \leq \|v\|$ for all $v \in \mathbb{R}^d$ and

$$\varphi(v_j - v_i) = \|v_j - v_i\| = \lambda_i. \tag{4.1}$$

(Thus, $\varphi^{-1}(1)$ is a hyperplane that supports K at $\|v_j - v_i\|_K^{-1}(v_j - v_i)$.)

Since any two homothets $v_k + \lambda_k K$ and $v_\ell + \lambda_\ell K$ intersect in their interiors, any two of the open intervals $\varphi(v_k + \lambda_k \operatorname{int} K)$ and $\varphi(v_\ell + \lambda_\ell \operatorname{int} K)$ intersect in $\mathbb{R}$. By Helly's Theorem in $\mathbb{R}$, there exists $\alpha \in \bigcap_{t=1}^n \varphi(v_t + \lambda_t \operatorname{int} K)$. Since $\varphi(v_i + \lambda_i \operatorname{int} K) = (\varphi(v_i) - \lambda_i, \varphi(v_i) + \lambda_i)$ and $\varphi(v_j + \lambda_j \operatorname{int} K) = (\varphi(v_j) - \lambda_j, \varphi(v_j) + \lambda_j)$, we have

$$\varphi(v_j) - \lambda_j < \alpha < \varphi(v_i) + \lambda_i.$$

By (4.1), we can rewrite this as

$$-\lambda_i < \varphi(v_i) - \alpha < \lambda_j - \lambda_i. \tag{4.2}$$

We set $f = (\varphi, -\alpha) \in (\mathbb{R}^d \times \mathbb{R})^*$, that is, for $x = (v, \mu) \in \mathbb{R}^d \times \mathbb{R}$, we let $f(x) = \varphi(v) - \alpha\mu$. It remains to show that $f(x_j - x_i) > 2 - 2^{1/N}$, and $|f(x_k)| \leq 1$ for all $k \in \{0, \dots, n\}$, since this will show that (2.2) is satisfied with $\lambda = 2 - 2^{1/N}$. By applying Theorems 2.2 and 2.3 with $\lambda = 2/(2 - 2^{1/N}) = 2 + \frac{\log 4}{N} + O(N^{-2})$, $K = (\lambda - 1)L$ and $\alpha = 1/(\lambda - 1) = 2^{1-1/N} - 1$, we obtain $|X_m| \le (1 + \lambda/2)(1 + \lambda)^d$, and it follows that

$$n - 1 \le (N + 1)(1 + \lambda/2)(1 + \lambda)^d.$$

If we choose $N = d$, we obtain $\lambda = 2 + \frac{\log 4}{d} + O(d^{-2})$ and $n = 3^d O(d)$, which would finish the proof.

By definition of X_m,

$$\lfloor N \log_2 \lambda_j \rfloor - \lfloor N \log_2 \lambda_i \rfloor = kN \quad \text{for some } k \in \mathbb{Z}.$$

If $k \ge 1$, then $N \log_2 \lambda_j - N \log_2 \lambda_i > N$, hence $\lambda_j/\lambda_i > 2$. However, we also have

$$\lambda_i = \|v_i - v_j\| \ge \|v_j - v_n\| - \|v_n - v_i\| = \lambda_j - \lambda_i,$$

a contradiction. Therefore, $k \le 0$, that is, $\lfloor N \log_2 \lambda_j \rfloor - \lfloor N \log_2 \lambda_i \rfloor \le 0$. This gives $N \log_2 \lambda_j - N \log_2 \lambda_i < 1$ and

$$\frac{\lambda_j}{\lambda_i} < 2^{1/N}. \tag{4.3}$$

It follows that

$$\begin{aligned}
f(x_j - x_i) &= \varphi(\lambda_j^{-1} v_j - \lambda_i^{-1} v_i) - \alpha(\lambda_j^{-1} - \lambda_i^{-1}) \\
&= \frac{\varphi(v_j) - \alpha}{\lambda_j} + \frac{\alpha - \varphi(v_i)}{\lambda_i} \\
&= \frac{\varphi(v_i) + \lambda_i - \alpha}{\lambda_j} + \frac{\alpha - \varphi(v_i)}{\lambda_i} \\
&\overset{(4.2),(4.3)}{>} \frac{2^{-1/N}(\varphi(v_i) + \lambda_i - \alpha) + \alpha - \varphi(v_i)}{\lambda_i} \\
&= 2^{-1/N} + \frac{(1 - 2^{-1/N})(\alpha - \varphi(v_i))}{\lambda_i} \\
&\overset{(4.2)}{>} 2^{-1/N} + \frac{(1 - 2^{-1/N})(\lambda_i - \lambda_j)}{\lambda_i} \\
&= 1 - (1 - 2^{-1/N}) \frac{\lambda_j}{\lambda_i} \\
&\overset{(4.2)}{>} 1 - (1 - 2^{-1/N}) 2^{1/N} \\
&= 2 - 2^{1/N}.
\end{aligned}$$

Since $\alpha \in \varphi(v_k + \lambda_k K)$, there exists $x \in K$ such that $\varphi(v_k + \lambda_k x) = \alpha$. Therefore,

$$|f(x_k)| = \left|\varphi(\lambda_k^{-1} v_k) - \alpha \lambda_k^{-1}\right| = |\varphi(x)| \le \|x\|_K \le 1.$$

□

5. Proof of Theorem 2.2

Assume that (i) holds. Let $C := \bigcap_{i\neq j} S_{ij}$ be the intersection of the o-symmetric slabs $S_{ij} := \left\{p \in \mathbb{R}^D : |f_{ij}(p)| \leq \frac{\lambda}{2} |f_{ij}(x_i) - f_{ij}(x_j)|\right\}$. By assumption, $C \supseteq \{x_1, \ldots, x_n\}$. For each $i \in [n]$, let $C_i := \frac{\lambda x_i + C}{\lambda+1}$ be the homothetic copy of C with center of homothety x_i, and of ratio $\frac{1}{\lambda+1}$. It is an easy exercise that the C_is are nonoverlapping. Moreover, by the symmetry of C, we have $\frac{\lambda-1}{\lambda+1}x_i \in C_i \cap \frac{\lambda-1}{\lambda+1}C$. Thus, for $L := \frac{1}{\lambda+1}C$, and $t_i := \frac{\lambda}{\lambda+1}x_i$, (ii) holds as promised.

Next, assume that (ii) holds. Fix $i, j \in [n], i \neq j$. Since $L + t_i$ and $L + t_j$ are nonoverlapping, there is a linear functional f such that the two real intervals $s_i := f(L + t_i)$ and $s_j := f(L + t_i)$ do not overlap. These two intervals are of equal length, which we denote by w. Thus, we have

$$w \leq |f(t_i) - f(t_j)|\,. \tag{5.1}$$

On the other hand, $s_k := f(L + t_k)$ is also a real interval of length w for any $k \in [n]$; and $s_0 := f((\lambda - 1)L)$ is a 0-symmetric real interval of length $(\lambda - 1)w$, which intersects each s_k. Thus, for the center $f(t_k)$ of s_k, we have $|f(t_k)| \leq \frac{(\lambda-1)w}{2} + \frac{w}{2} = \frac{\lambda w}{2}$. Now, (5.1) yields $|f(t_k)| \leq \frac{\lambda}{2} |f(t_i) - f(t_j)|$. Thus, we may set $f_{ij} := f$. This argument is valid for any i and j, thus, with $x_i := t_i$, we obtain (i).

6. Proof of Theorem 2.3

The proof is an almost verbatim copy of the proof of Theorem 1.5 of [7]. There are two points of difference, which we will note.

We recall Lemma 3.1 of [7], which is a slightly more general version of the Lemma of [1].

Lemma 6.1. *Let f be a function on $[0,1]$ with the properties $f(0) \geq 0$, f is positive and monotone increasing on $(0,1]$, and $f(x) = (g(x))^k$ for some concave function g and $k > 0$. Then*

$$F(y) := \frac{1}{f(y)} \int_0^y f(x)\,\mathrm{d}x$$

is strictly increasing on $(0,1]$.

Proof of Theorem 2.3. Clearly, we may assume that K is bounded, otherwise, by a projection, we can reduce the dimension. Let $\alpha K + t_1$, $\alpha K + t_2, \ldots, \alpha K + t_n$ be pairwise nonoverlapping translates of αK that intersect K. By the assumptions of the theorem, there is a nonzero vector $v \in \mathbb{R}^D$ such that $a_i := \langle t_i, v\rangle \geq 0$ for $i \in [n]$. Set $h(x) := \{p \in \mathbb{R}^D : \langle p, v\rangle = x\}$. Without loss of generality, we may assume that $h(-1)$ and $h(1)$ are supporting hyperplanes of K.

Clearly, $\alpha K + t_i$ is between $h(-\alpha)$ and $h(1 + 2\alpha)$, and it is contained in $(1 + 2\alpha)K$, for $i \in [n]$.

$$\int_{-\alpha}^{1+2\alpha} \mathrm{V}_{D-1}\left(\left(\bigcup_{i=1}^{n} \alpha K + t_i\right) \cap h(x)\right) \mathrm{d}x = n\alpha^D \mathrm{V}_D(K). \tag{6.1}$$

$$\int_{0}^{1+2\alpha} \mathrm{V}_{D-1}\left(\left(\bigcup_{i=1}^{n} \alpha K + t_i\right) \cap h(x)\right) \mathrm{d}x \tag{6.2}$$

$$\leq \int_{0}^{1+2\alpha} \mathrm{V}_{D-1}\left((1+2\alpha)K \cap h(x)\right) \mathrm{d}x = \frac{(1+2\alpha)^d}{2} \mathrm{V}_D(K).$$

We note that this was the first point of difference from the proof in [7]: here, we do not subtract the contribution of K in the total volume on the right hand side of the inequality.

Set $f(x) := \mathrm{V}_{D-1}\left(\alpha K \cap h(x-\alpha)\right)$, and observe that the conditions of Lemma 6.1 are satisfied by f (with $k = D-1$, by the Brunn–Minkowski inequality). We may assume that $a_1, \ldots, a_m \leq \alpha < a_{m+1}, \ldots, a_n$. By Lemma 6.1,

$$\int_{-\alpha}^{0} \mathrm{V}_{D-1}\left(\left(\bigcup_{i=1}^{n} (\alpha K + t_i)\right) \cap h(x)\right) \mathrm{d}x = \sum_{i=1}^{m} \int_{0}^{\alpha - a_i} f(x)\,\mathrm{d}x$$

$$\leq \sum_{i=1}^{m} \int_{0}^{\alpha} f(x)\,\mathrm{d}x \frac{f(\alpha - a_i)}{f(\alpha)} = \frac{\alpha^d \mathrm{V}_D(K)}{2f(\alpha)} \sum_{i=1}^{m} \mathrm{V}_{D-1}\left((\alpha K + t_i) \cap h(0)\right)$$

$$= \frac{\alpha^d \mathrm{V}_D(K)}{2f(\alpha)} \mathrm{V}_{D-1}\left(\left(\bigcup_{i=1}^{m} (\alpha K + t_i)\right) \cap h(0)\right)$$

$$\leq \frac{\alpha^d \mathrm{V}_D(K)}{2f(\alpha)} \left[\mathrm{V}_{D-1}\left((1+2\alpha)K \cap h(0)\right)\right] = \frac{\alpha(1+2\alpha)^{D-1}}{2} \mathrm{V}_D(K).$$

We note that this was the second point of difference from the proof in [7]: again, the contribution of K to the volume is not subtracted.

This inequality, combined with (6.1) and (6.2), yields (2.3). □

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