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ON THE ORDER OF APPEARANCE OF PRODUCTS OF FIBONACCI NUMBERS

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Abstract. Let F_n be the nth Fibonacci number. For each positive integer m, the order of appearance of m, denoted by $z(m)$, is the smallest positive integer k such that m divides F_k. Recently, D. Marques has obtained a formula for $z(F_nF_{n+1})$, $z(F_nF_{n+1}F_{n+2})$, and $z(F_nF_{n+1}F_{n+2}F_{n+3})$. In this paper, we extend Marques' result to the case $z(F_nF_{n+1}\cdots F_{n+k})$, for $4 \leq k \leq 6$.

1. Introduction

Throughout this article, we write $(a_1, a_2, \ldots, a_k)$ and $[a_1, a_2, \ldots, a_k]$ for the greatest common divisor and the least common multiple of $a_1, a_2, \ldots, a_k$, respectively.

The Fibonacci sequence $(F_n)_{n\geq 1}$ is defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. For each $m \in \mathbb{N}$, the order of appearance of m in the Fibonacci sequence, denoted by $z(m)$, is the smallest positive integer k such that m divides F_k. The divisibility property of Fibonacci numbers and the behavior of the order of appearance have been a popular area of research, see [1, 2, 5, 6, 8, 15, 18, 19, 24, 26, 27, 28, 29] and references therein for additional details and history. Recently, D. Marques [10, 11, 12, 13, 14] has obtained formulas for $z(m)$ for various types of m. In particular, he [13] obtains formulas for $z(F_nF_{n+1})$, $z(F_nF_{n+1}F_{n+2})$, and $z(F_nF_{n+1}F_{n+2}F_{n+3})$. In this article, we extend his results to the case $z(F_nF_{n+1}\cdots F_{n+k})$, for $4 \leq k \leq 6$. Our method is simpler and gives a general idea on how to obtain formulas for $z(F_nF_{n+1}\cdots F_{n+k})$, for every $k \geq 1$.

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2. Auxiliary Results

In this section, we give some lemmas that will be used in the proof of the main theorems. First we recall the following well-known results [4, 6, 8, 27] which will be applied throughout this article:

$$\text{For } n \geq 3,\ m \geq 1,\ F_n \mid F_m \text{ if and only if } n \mid m. \tag{2.1}$$

$$\text{For } m, n \geq 1,\ (F_m, F_n) = F_{(m,n)}. \tag{2.2}$$

We will need to calculate 2-adic and 3-adic orders of Fibonacci numbers; the next lemma will be useful.

Lemma 2.1 (Lengyel [9]). *For each $n \geq 1$, let $v_p(n)$ be the p-adic order of n. Then*

$$v_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 3; \\ 1, & \text{if } n \equiv 3 \pmod 6; \\ v_2(n) + 2, & \text{if } n \equiv 0 \pmod 6, \end{cases}$$

$v_5(F_n) = v_5(n)$, and if p is a prime, $p \neq 2$, and $p \neq 5$, then

$$v_p(F_n) = \begin{cases} v_p(n) + v_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{if } n \not\equiv 0 \pmod{z(p)}. \end{cases}$$

In particular,

$$v_3(F_n) = \begin{cases} v_3(n) + 1, & \text{if } n \equiv 0 \pmod 4; \\ 0, & \text{if } n \not\equiv 0 \pmod 4. \end{cases}$$

We will also need to calculate the least common multiple of consecutive integers such as $[n, n+1, n+2, n+3, n+4]$. It is not difficult to compute directly the formula for $[n, n+1, \ldots, n+k]$ in terms of $n, n+1, \ldots, n+k$ for $1 \leq k \leq 6$. But it is more convenient to apply the result of Farhi and Kane [3] on the recursive relation of the function $g_k : \mathbb{N} \to \mathbb{N}$ given by

$$g_k(n) = \frac{n(n+1)\cdots(n+k)}{[n, n+1, \ldots, n+k]}. \tag{2.3}$$

Lemma 2.2 (Farhi and Kane [3]). *For each $k \in \mathbb{N} \cup \{0\}$, let g_k be the function defined by (2.3). Then $g_0(n) = g_1(n) = 1$ for every $n \in \mathbb{N}$ and g_k satisfies the recursive relation*

$$g_k(n) = (k!, (n+k)g_{k-1}(n)) \text{ for all } k, n \in \mathbb{N}.$$

Let a, b, c be positive integers. Recall the basic results in elementary number theory that if $(a, b) = 1$, then $(c, ab) = (c, a)(c, b)$, and $(a, bc) = (a, c)$. In addition, $((a, b), c) = (a, b, c)$, $(a, b) = (b, a)$, $(ca, cb) = c(a, b)$, and if $a \equiv b \pmod c$, then $(a, c) = (b, c)$. Combining these and Lemma 2.2, we obtain the following result.

Lemma 2.3. *For each $k, n \in \mathbb{N}$, let $L_k(n) = [n, n+1, \ldots, n+k]$. Then the following statements hold.*

$$\begin{aligned}
L_1(n) &= n(n+1),\\
L_2(n) &= \frac{n(n+1)(n+2)}{(2,n)},\\
L_3(n) &= \frac{n(n+1)(n+2)(n+3)}{2(3,n)},\\
L_4(n) &= \frac{n(n+1)(n+2)(n+3)(n+4)}{2(4,n)(3,n(n+1))},\\
L_5(n) &= \frac{n(n+1)(n+2)(n+3)(n+4)(n+5)}{6(5,n)(4,n(n+1))},\\
L_6(n) &= \frac{n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}{12(3,n)(5,n(n+1))\left(4,(n+2)\left(2,\frac{n(n+1)}{2}\right)\right)}.
\end{aligned}$$

Proof. By the definition of the function $g_k(n)$, we obtain

$$[n, n+1, \ldots, n+k] = \frac{n(n+1)\cdots(n+k)}{g_k(n)}.$$

So we only need to find $g_k(n)$ for $k = 1,2,3,4,5,6$. Since each case is similar, we will only give the proof in the cases $k = 5,6$ assuming that cases $k = 1,2,3,4$ are already obtained.

Case 1: $k = 5$.

From the case $k = 4$, we have $g_4(n) = 2(4,n)(3,n(n+1))$ and we obtain by Lemma 2.2 that

$$\begin{aligned}
g_5(n) &= (5!, (n+5)g_4(n))\\
&= (5!, 2(n+5)(4,n)(3,n(n+1)))\\
&= 2(5\cdot 4\cdot 3, (n+5)(4,n)(3,n(n+1)))\\
&= 2(5,n+5)(4,(n+5)(4,n))(3,(n+5)(3,n(n+1)))\\
&= 2(5,n)(4,(n+1)(4,n))(3,3(n+5),n(n+1)(n+5))\\
&= 2(5,n)(4,4(n+1),n(n+1))(3,n(n+1)(n+5))\\
&= 2(5,n)(4,n(n+1))3\\
&= 6(5,n)(4,n(n+1)).
\end{aligned}$$

Case 2: $k = 6$.

We have

$$\begin{aligned}
g_6(n) &= (6!, (n+6)g_5(n)) \\
&= (6!, 6(n+6)(5,n)(4,n(n+1))) \\
&= 6(8 \cdot 5 \cdot 3, (n+6)(5,n)(4,n(n+1))) \\
&= 6(8,(n+6)(4,n(n+1)))(5,(n+6)(5,n))(3,n+6) \\
&= 6(8,(n+6)(4,n(n+1)))(5,(n+1)(5,n))(3,n) \\
&= 6(8,(n+6)(4,n(n+1)))(5,5(n+1),n(n+1))(3,n) \\
&= 12\left(4,(n+6)\left(2,\frac{n(n+1)}{2}\right)\right)(5,n(n+1))(3,n) \\
&= 12\left(4,(n+2)\left(2,\frac{n(n+1)}{2}\right)\right)(5,n(n+1))(3,n).
\end{aligned}$$

This completes the proof. □

Next we calculate the least common multiple of consecutive Fibonacci numbers.

Lemma 2.4. *For each $k, n \in \mathbb{N}$, let $LF_k(n) = [F_n, F_{n+1}, \dots, F_{n+k}]$. Then the following statements hold.*

(i) $LF_1(n) = F_nF_{n+1}$.
(ii) $LF_2(n) = F_nF_{n+1}F_{n+2}$.
(iii) $LF_3(n) = \frac{F_nF_{n+1}F_{n+2}F_{n+3}}{F_{(n,3)}}$.
(iv) $LF_4(n) = \begin{cases} \frac{F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}}{F_{(n,4)}}, & \text{if } n \equiv 1 \pmod 3; \\ \frac{F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}}{2F_{(n,4)}}, & \text{if } n \equiv 0, 2 \pmod 3). \end{cases}$
(v) $LF_5(n) = \begin{cases} \frac{F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}F_{n+5}}{2F_{(n,5)}}, & \text{if } n \equiv 1, 2 \pmod 4; \\ \frac{F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}F_{n+5}}{6F_{(n,5)}}, & \text{if } n \equiv 0, 3 \pmod 4. \end{cases}$
(vi) $LF_6(n) = \begin{cases} \frac{F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}F_{n+5}F_{n+6}}{2F_{(n(n+1),5)}F_{(n,6)}}, & \text{if } n \equiv 1 \pmod 4; \\ \frac{F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}F_{n+5}F_{n+6}}{6F_{(n(n+1),5)}F_{(n,6)}}, & \text{if } n \equiv 0, 2, 3 \pmod 4. \end{cases}$

Proof. By (2.2), it is easy to check that F_n, F_{n+1}, F_{n+2} are pairwise relatively prime. So (i) and (ii) follow immediately. Since (iii), (iv), (v), and (vi) follow from the same idea, we will only show the proof for (iii), (v), and (vi).

Recall that $[a_1, a_2, \dots, a_k] = [[a_1, a_2, \dots, a_{k-1}], a_k]$ and $[a, b] = ab/(a, b)$. For convenience, we let $P_k = F_nF_{n+1}\cdots F_{n+k}$. Then (iii) follows from (ii) by

$$\begin{aligned}[F_n, F_{n+1}, F_{n+2}, F_{n+3}] &= [[F_n, F_{n+1}, F_{n+2}], F_{n+3}] \\ &= \frac{[F_n, F_{n+1}, F_{n+2}]F_{n+3}}{([F_n, F_{n+1}, F_{n+2}], F_{n+3})} = \frac{F_nF_{n+1}F_{n+2}F_{n+3}}{(F_nF_{n+1}F_{n+2}, F_{n+3})} \\ &= \frac{P_3}{(F_n, F_{n+3})(F_{n+1}, F_{n+3})(F_{n+2}, F_{n+3})} \\ &= \frac{P_3}{F_{(n,n+3)}} = \frac{P_3}{F_{(n,3)}}.\end{aligned}$$

Assuming (iv), we can obtain (v) in the following similar way. Since F_{n+3}, F_{n+4}, F_{n+5} are pairwise relatively prime, we see that

$$\begin{aligned}(P_4, F_{n+5}) &= (F_n, F_{n+5})(F_{n+1}, F_{n+5})(F_{n+2}, F_{n+5}) \\ &= F_{(n,n+5)}F_{(n+1,n+5)}F_{(n+2,n+5)} \\ &= F_{(n,5)}F_{(n+1,4)}F_{(n+2,3)}. \qquad (2.4)\end{aligned}$$

Case 1: $n \equiv 1 \pmod 3$.

Then

$$\begin{aligned}[F_n, F_{n+1}, F_{n+2}, F_{n+3}, F_{n+4}, F_{n+5}] &= [[F_n, F_{n+1}, F_{n+2}, F_{n+3}, F_{n+4}], F_{n+5}] \\ &= \left[\frac{F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}}{F_{(n,4)}}, F_{n+5}\right] \\ &= \frac{F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}F_{n+5}}{F_{(n,4)}\left(\frac{F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}}{F_{(n,4)}}, F_{n+5}\right)} \\ &= \frac{F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}F_{n+5}}{(F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}, F_{(n,4)}F_{n+5})} \\ &= \frac{P_5}{(P_4, F_{(n,4)}F_{n+5})}.\end{aligned}$$

Since $\left(F_{(n,4)}, F_{n+5}\right) = F_{((n,4),n+5)} = F_{(n,(4,n+5))} = F_{(n,4,n+1)} = 1$ and $n \equiv 1 \pmod 3$, we obtain by (2.4) that

$$(P_4, F_{(n,4)}F_{n+5}) = 2(P_4, F_{(n,4)})F_{(n,5)}F_{(n+1,4)}. \qquad (2.5)$$

It is easy to check that if $n \equiv 1, 2 \pmod 4$, then the right hand side of (2.5) is equal to $2F_{(n,5)}$, and if $n \equiv 0, 3 \pmod 4$, then it is equal to $6F_{(n,5)}$.

Case 2: $n \equiv 0, 2 \pmod 3$.

Similar to Case 1, we have

$$[F_n, F_{n+1}, F_{n+2}, F_{n+3}, F_{n+4}, F_{n+5}] = \frac{P_5}{(P_4, 2F_{(n,4)}F_{n+5})}.$$

It is easy to check using (2.2) that $2 = F_3$ is relatively prime to $F_{(n,4)}$ and F_{n+5}, and that $(F_{(n,4)}, F_{n+5}) = F_{((n,4),n+5)} = 1$. This and (2.4) implies that

$$(P_4, 2F_{(n,4)}F_{n+5}) = 2(P_4, F_{(n,4)})F_{(n,5)}F_{(n+1,4)},$$

which is the same as (2.5). So if $n \equiv 1, 2 \pmod 4$, then it is equal to $2F_{(n,5)}$, and if $n \equiv 0, 3 \pmod 4$, then it is equal to $6F_{(n,5)}$. This proves (v).

Next we give a proof of (vi).

Case 1: $n \equiv 1, 2 \pmod 4$.

Similar to the proof of (v), we have

$$[F_n, F_{n+1}, F_{n+2}, F_{n+3}, F_{n+4}, F_{n+5}, F_{n+6}] = \frac{P_6}{(P_5, 2F_{(n,5)}F_{n+6})}.$$

It is easy to see that $F_{(n,5)}$ is relatively prime to 2. This implies that $(F_{(n,5)}, 2F_{n+6}) = (F_{(n,5)}, F_{n+6}) = F_{((n,5),n+6)} = 1$. So

$$\begin{aligned}(P_5, 2F_{(n,5)}F_{n+6}) &= (P_5, F_{(n,5)})(P_5, 2F_{n+6}) \\ &= (F_nF_{n+5}, F_{(n,5)})(P_5, 2F_{n+6}).\end{aligned}$$

We see that if $5 \mid n$, then $(F_nF_{n+5}, F_{(n,5)}) = 5$, and if $5 \nmid n$, then $(F_nF_{n+5}, F_{(n,5)}) = 1$. This implies that $(F_nF_{n+5}, F_{(n,5)}) = F_{(n,5)}$. Thus the above equation becomes

$$(P_5, 2F_{(n,5)}F_{n+6}) = F_{(n,5)}(P_5, 2F_{n+6}). \tag{2.6}$$

Consider $(2, F_{n+6}) = (F_3, F_{n+6}) = F_{(3,n+6)} = F_{(3,n)}$.

Subcase 1.1: $3 \nmid n$.

Then $(2, F_{n+6}) = 1$, and F_{n+6} is relatively prime to F_{n+5}, F_{n+4}, and F_{n+3}. So (2.6) becomes

$$\begin{aligned}(P_5, 2F_{(n,5)}F_{n+6}) &= 2F_{(n,5)}(P_5, F_{n+6}) \\ &= 2F_{(n,5)}(F_nF_{n+1}F_{n+2}, F_{n+6}) \\ &= 2F_{(n,5)}(F_n, F_{n+6})(F_{n+1}, F_{n+6})(F_{n+2}, F_{n+6}) \\ &= 2F_{(n,5)}F_{(n,6)}F_{(n+1,5)}F_{(n+2,4)} \\ &= 2F_{(n(n+1),5)}F_{(n,6)}F_{(n+2,4)}. \qquad (2.7)\end{aligned}$$

Subcase 1.2: $3 \mid n$.

Then 2 and F_{n+6} are relatively prime to F_{n+4} and F_{n+5}. In addition, $(F_nF_{n+1}F_{n+2}, F_{n+3}) = (F_n, F_{n+3}) = F_{(n,3)} = 2$. So

$$\left(\frac{F_nF_{n+1}F_{n+2}}{2}, \frac{F_{n+3}}{2}\right) = 1.$$

Therefore

$$\begin{aligned}(P_5, 2F_{n+6}) &= (F_nF_{n+1}F_{n+2}F_{n+3}, 2F_{n+6})\\ &= 4\left(\frac{F_nF_{n+1}F_{n+2}F_{n+3}}{4}, \frac{F_{n+6}}{2}\right)\\ &= 4\left(\frac{F_nF_{n+1}F_{n+2}}{2}\frac{F_{n+3}}{2}, \frac{F_{n+6}}{2}\right)\\ &= 4\left(\frac{F_nF_{n+1}F_{n+2}}{2}, \frac{F_{n+6}}{2}\right)\left(\frac{F_{n+3}}{2}, \frac{F_{n+6}}{2}\right)\\ &= (F_nF_{n+1}F_{n+2}, F_{n+6})(F_{n+3}, F_{n+6})\\ &= (F_n, F_{n+6})(F_{n+1}, F_{n+6})(F_{n+2}, F_{n+6})(F_{n+3}, F_{n+6})\\ &= F_{(n,6)}F_{(n+1,5)}F_{(n+2,4)}F_{(n+3,3)} = 2F_{(n,6)}F_{(n+1,5)}F_{(n+2,4)}.\end{aligned}$$

Thus (2.6) becomes

$$\begin{aligned}(P_5, 2F_{(n,5)}F_{n+6}) &= 2F_{(n,5)}F_{(n,6)}F_{(n+1,5)}F_{(n+2,4)}\\ &= 2F_{(n(n+1),5)}F_{(n+2,4)}F_{(n,6)},\end{aligned}$$

which is the same as (2.7).

We conclude that Subcases 1.1 and 1.2 lead to the same formula for $[F_n, F_{n+1}, F_{n+2}, F_{n+3}, F_{n+4}, F_{n+5}, F_{n+6}]$. Observe that if $n \equiv 1 \pmod 4$, then $F_{(n+2,4)} = 1$, and if $n \equiv 2 \pmod 4$, then $F_{(n+2,4)} = 3$. This leads to the desired formula in (vi).

Case 2: $n \equiv 0, 3 \pmod 4$.

Similar to the proof of (v), we have

$$[F_n, F_{n+1}, F_{n+2}, F_{n+3}, F_{n+4}, F_{n+5}, F_{n+6}] = \frac{P_6}{(P_5, 6F_{(n,5)}F_{n+6})}.$$

It is easy to see that $F_{(n,5)}$ is relatively prime to 2 and 3. So $(F_{(n,5)}, 6F_{n+6}) = (F_{(n,5)}, F_{n+6}) = F_{((n,5),n+6)} = 1$. Thus

$$(P_5, 6F_{(n,5)}F_{n+6}) = (P_5, F_{(n,5)})(P_5, 6F_{n+6}) = F_{(n,5)}(P_5, 6F_{n+6}). \tag{2.8}$$

Subcase 2.1: $3 \nmid n$.

Then $(6, F_{n+6}) = 1$ and $(F_{n+3}F_{n+4}F_{n+5}, F_{n+6}) = 1$. So

$$\begin{aligned}(P_5, 6F_{n+6}) &= 6(P_5, F_{n+6}) = 6(F_nF_{n+1}F_{n+2}, F_{n+6})\\ &= 6(F_n, F_{n+6})(F_{n+1}, F_{n+6})(F_{n+2}, F_{n+6})\\ &= 6F_{(n,6)}F_{(n+1,5)}.\end{aligned}$$

So we obtain by (2.8) that

$$(P_5, 6F_{(n,5)}F_{n+6}) = 6F_{(n,5)}F_{(n,6)}F_{(n+1,5)} = 6F_{(n,6)}F_{(n(n+1),5)}. \tag{2.9}$$

Subcase 2.2: $3 \mid n$.

Then $(F_{n+5}, 6F_{n+6}) = (F_{n+5}, 6) = (F_4, F_{n+5})(F_3, F_{n+5}) = F_{(4,n+1)}$.

We obtain similarly that $(F_{n+4}, 6F_{n+6}) = F_{(4,n)}$ and $(F_{n+3}, 6F_{n+6})$

$= (F_{n+3}, 3)(F_{n+3}, 2F_{n+6}) = (F_{n+3}, 2F_{n+6}) = (F_{n+3}, 4)$, where the last equality is obtained from the fact that $(F_{n+3}, F_{n+6}) = 2$. So

$$(F_{n+3}F_{n+4}F_{n+5}, 6F_{n+6}) = F_{(4,n+1)}F_{(4,n)}(F_{n+3}, 4). \tag{2.10}$$

From this we obtain by Lemma 2.1 that

$$(F_{n+3}F_{n+4}F_{n+5}, 6F_{n+6}) = \begin{cases} 6, & \text{if } n \equiv 0 \pmod{12}; \\ 12, & \text{if } n \equiv 3 \pmod{12}. \end{cases}$$

Subsubcase 2.2.1: $n \equiv 0 \pmod{12}$.
Then $\left(\frac{F_{n+3}F_{n+4}F_{n+5}}{6}, F_{n+6}\right) = 1$. So

$$\begin{aligned}(P_5, 6F_{n+6}) &= 6\left(F_nF_{n+1}F_{n+2}\frac{F_{n+3}F_{n+4}F_{n+5}}{6}, F_{n+6}\right) \\ &= 6(F_nF_{n+1}F_{n+2}, F_{n+6}) \\ &= 6(F_n, F_{n+6})(F_{n+1}, F_{n+6})(F_{n+2}, F_{n+6}) \\ &= 6F_{(n,6)}F_{(n+1,5)}.\end{aligned}$$

Thus we obtain by (2.8) that

$$(P_5, 6F_{(n,5)}F_{n+6}) = 6F_{(n,6)}F_{(n+1,5)}F_{(n,5)} = 6F_{(n,6)}F_{(n(n+1),5)}, \tag{2.11}$$

which is the same as (2.9).
Subsubcase 2.2.2: $n \equiv 3 \pmod{12}$.
Then $\left(\frac{F_{n+3}F_{n+4}F_{n+5}}{12}, \frac{F_{n+6}}{2}\right) = 1$. So

$$\begin{aligned}(P_5, 6F_{n+6}) &= 12\left(F_nF_{n+1}F_{n+2}\frac{F_{n+3}F_{n+4}F_{n+5}}{12}, \frac{F_{n+6}}{2}\right) \\ &= 12\left(F_nF_{n+1}F_{n+2}, \frac{F_{n+6}}{2}\right) \\ &= 12\left(F_n, \frac{F_{n+6}}{2}\right)\left(F_{n+1}, \frac{F_{n+6}}{2}\right)\left(F_{n+2}, \frac{F_{n+6}}{2}\right).\end{aligned}$$

Consider $(F_{n+2}, F_{n+6}) = F_{(n+2,4)} = 1$, $(F_{n+1}, F_{n+6}) = F_{(n+1,5)}$, $(F_n, F_{n+6}) = F_{(n,6)} = F_{(3,6)} = 2$, and $v_2(F_n) = v_2(F_{n+6}) = 1$. Therefore $(P_5, 6F_{n+6}) = 12F_{(n+1,5)}$, and thus $(P_5, 6F_{(n,5)}F_{n+6}) = 12F_{(n,5)}F_{(n+1,5)} = 6F_{(n,6)}F_{(n(n+1),5)}$, which is the same as (2.11) and (2.9).

So Subcases 2.1 and 2.2 lead to the same formula for

$$[F_n, F_{n+1}, F_{n+2}, F_{n+3}F_{n+4}, F_{n+5}F_{n+6}].$$

This completes the proof of (vi).

□

3. Main Results

As mentioned in the introduction, our method of proof gives a general idea on how to obtain $z(F_nF_{n+1}\cdots F_{n+k})$ for every $k \geq 1$. In fact, the next theorem describes a general strategy for obtaining a formula for $z(F_nF_{n+1}\cdots F_{n+k})$.

Theorem 3.1. *Let $n \geq 3$, $k \geq 1$, $a = [n, n+1, \dots, n+k]$, $b = F_nF_{n+1}\cdots F_{n+k}$ and*

$$f_k(n) = \frac{F_nF_{n+1}F_{n+2}\cdots F_{n+k}}{[F_n, F_{n+1}, F_{n+2}, \dots, F_{n+k}]}$$

Then the following holds.

(i) *$b \mid f_k(n)F_{aj}$ for every $j \geq 1$.*
(ii) *$z(b) = aj$ where j is the smallest positive integer such that $b \mid F_{aj}$. In fact, j is the smallest positive integer such that $v_p(b) \leq v_p(F_{aj})$ for every prime p dividing $f_k(n)$.*

Proof. Since $n+i \mid a$ for all $0 \leq i \leq k$, we obtain by (2.1) that $F_{n+i} \mid F_a$ for all $0 \leq i \leq k$. So $[F_n, F_{n+1}, \dots, F_{n+k}] \mid F_a$. By the definition of $f_k(n)$, we see that $b \mid f_k(n)F_a$. Since $F_a \mid F_{aj}$,

$$b \mid f_k(n)F_{aj} \quad \text{for every } j \geq 1.$$

This proves (i). Next let $z(b) = \ell$. Then $b \mid F_\ell$. Therefore $F_{n+i} \mid F_\ell$ for all $0 \leq i \leq k$. Since $n \geq 3$, we obtain by (2.1) that $n+i \mid \ell$ for all $0 \leq i \leq k$, which implies that $a \mid \ell$. Thus $\ell = aj$ for some $j \in \mathbb{N}$. By the definition of $z(b)$, we see that j is the smallest positive integer such that

$$b \mid F_{aj}. \tag{3.1}$$

Note that (3.1) is equivalent to $v_p(b) \leq v_p(F_{aj})$ for every prime p. But by (i), if p is a prime and $p \nmid f_k(n)$, then

$$v_p(b) \leq v_p(f_k(n)F_{aj}) = v_p(F_{aj}).$$

Therefore (3.1) is equivalent to

$$v_p(b) \leq v_p(F_{aj}) \text{ for every prime } p \text{ dividing } f_k(n). \tag{3.2}$$

Hence $z(b) = \ell = aj$ and j is the smallest positive integer satisfying (3.2). This proves (ii). □

Theorem 3.2. *Let $n \geq 1$, $a = [n, n+1, n+2, n+3, n+4]$, and $b = F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}$. Then*

$$z(b) = \begin{cases} a, & \text{if } n \equiv 1,2,3,4,5,6,7,10 \pmod{12}, \text{ or } n \equiv 8,60 \pmod{72}; \\ 2a, & \text{if } n \equiv 9,11 \pmod{12}, \text{ or } n \equiv 24,44 \pmod{72}; \\ 3a, & \text{if } n \equiv 12,32,36,56 \pmod{72}; \\ 6a, & \text{if } n \equiv 0,20,48,68 \pmod{72}. \end{cases}$$

Proof. It is easy to check that the result holds for $n = 1, 2$. So assume that $n \geq 3$.

Case 1: $n \equiv 1 \pmod 3$.

Then by Lemma 2.4 and Theorem 3.1, we have $b \mid F_{(n,4)} F_{aj}$ for every $j \geq 1$ and we would like to find the smallest j such that $b \mid F_{aj}$. If $n \equiv 1, 2, 3 \pmod 4$, then $F_{(n,4)} = 1$, so we can choose $j = 1$ and obtain $z(b) = a$. So assume that $n \equiv 0 \pmod 4$. Then $F_{(n,4)} = 3$ and by Theorem 3.1 we only need to consider $v_3(b)$ and $v_3(F_{aj})$. Since $n \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 4$, we obtain by Lemma 2.1 that $v_3(b) = v_3(F_n) + v_3(F_{n+4}) = v_3(n) + v_3(n+4) + 2 = 2$. Since $4 \mid n$ and $n \mid aj$, $4 \mid aj$. So we obtain by Lemmas 2.1 and 2.3 that for every $j \geq 1$,

$$\begin{aligned} v_3(F_{aj}) &= v_3(a) + v_3(j) + 1 \\ &= v_3\left(\frac{n(n+1)(n+2)(n+3)(n+4)}{8}\right) + v_3(j) + 1 \\ &= v_3(n+2) + v_3(j) + 1 \geq 2 + v_3(j) \geq 2 = v_3(b). \end{aligned}$$

Thus we can choose $j = 1$ and obtain $z(b) = a$. This shows $z(b) = a$ whenever $n \equiv 1 \pmod 3$. We remark that the idea that will be used in the following case is still the same as that in the previous case. So our argument will be shorter.

Case 2: $n \equiv 2 \pmod 3$.

Then by Lemma 2.4 and Theorem 3.1, we have b divides $2F_{(n,4)} F_{aj}$ for every $j \geq 1$ and our problem is reduced to finding the smallest positive integer j such that $v_p(b) \leq v_p(F_{aj})$ for every prime p dividing $2F_{(n,4)}$. Let $j \geq 1$. Since $3 \mid n+1$ and $n+1 \mid a$, we see that $3 \mid aj$. Similarly $2 \mid aj$. Therefore $6 \mid aj$. By Lemma 2.1, $v_2(F_{aj}) = v_2(aj) + 2$. In addition, $v_2(b) = v_2(F_{n+1}) + v_2(F_{n+4})$.

Subcase 2.1: $n \equiv 1 \pmod 4$.

Then by Lemmas 2.1 and 2.3, we obtain

$$\begin{aligned} v_2(F_{aj}) &= v_2(a) + v_2(j) + 2 \\ &= v_2(n+1) + v_2(n+3) - v_2(2) + v_2(j) + 2 \\ &= v_2(n+3) + v_2(j) + 2 \geq 4 = v_2(n+1) + 3 \\ &= v_2(F_{n+1}) + v_2(F_{n+4}) = v_2(b). \end{aligned}$$

So in this case, we can choose $j = 1$ and obtain $z(b) = a$.

Subcase 2.2: $n \equiv 2 \pmod 4$.

Similar to Subcase 2.1, we see that

$$\begin{aligned} v_2(F_{aj}) &= v_2(n) + v_2(n+2) + v_2(n+4) - v_2(4) + v_2(j) + 2 \\ &= v_2(n+2) + v_2(j) + 2 \geq 4 = v_2(b), \text{ and } z(b) = a. \end{aligned}$$

Subcase 2.3: $n \equiv 3 \pmod 4$.

Then $v_2(b) = v_2(n+1)+3$, and $v_2(F_{aj}) = v_2(n+1)+v_2(j)+2$. So $v_2(F_{aj}) \geq v_2(b)$ if and only if $v_2(j) \geq 1$. So we choose $j = 2$ and obtain $z(b) = 2a$.

Subcase 2.4: $n \equiv 0 \pmod 4$.

Then $2F_{(n,4)} = 6$ and we need to consider 2-adic and 3-adic orders of b and F_{aj}. By Lemmas 2.1 and 2.3, we obtain similarly to the other cases that

$$\begin{aligned} v_2(b) &= v_2(n+4)+3 \\ v_2(F_{aj}) &= v_2(n)+v_2(n+4)+v_2(j), \\ v_3(b) &= v_3(F_n)+v_3(F_{n+4}) \\ &= v_3(n)+v_3(n+4)+2 = v_3(n+4)+2, \text{ and} \\ v_3(F_{aj}) &= v_3(aj)+1 = v_3(n+1)+v_3(n+4)+v_3(j). \end{aligned}$$

So we need to find the smallest $j \geq 1$ such that

$$v_2(n)+v_2(j) \geq 3 \text{ and } v_3(n+1)+v_3(j) \geq 2.$$

Note that $n \equiv 0, 4 \pmod 8$ and $n+1 \equiv 0, 3, 6 \pmod 9$.

(i) If $n \equiv 0 \pmod 8$ and $n+1 \equiv 0 \pmod 9$, then $v_2(j) = v_3(j) = 0$, so $j = 1$ and
$$z(b) = a = \frac{72a}{(8,n)(9,n+1)}.$$

(ii) If $n \equiv 0 \pmod 8$ and $n+1 \equiv 3, 6 \pmod 9$, then $v_2(j) = 0$ and $v_3(j) = 1$, so $j = 3$ and
$$z(b) = 3a = \frac{72a}{(8,n)(9,n+1)}.$$

(iii) If $n \equiv 4 \pmod 8$ and $n+1 \equiv 0 \pmod 9$, then $v_2(j) = 1$ and $v_3(j) = 0$, so $j = 2$ and
$$z(b) = 2a = \frac{72a}{(8,n)(9,n+1)}.$$

(iv) If $n \equiv 4 \pmod 8$ and $n+1 \equiv 3, 6 \pmod 9$, then $v_2(j) = v_3(j) = 1$, so $j = 6$ and
$$z(b) = 6a = \frac{72a}{(8,n)(9,n+1)}.$$

Case 3: $n \equiv 0 \pmod 3$.

Similar to Case 2, $b \mid 2F_{(n,4)}F_{aj}$ for every $j \geq 1$ and we need to find the smallest j such that $v_p(b) \leq v_p(F_{aj})$ for every prime p dividing $2F_{(n,4)}$.

Subcase 3.1: $n \equiv 1 \pmod 4$.

Then $2F_{(n,4)} = 2$, $v_2(b) = v_2(n+3)+3$, and $v_2(F_{aj}) = v_2(n+3)+v_2(j)+2$. So we need $j = 2$ and therefore $z(b) = 2a$.

Subcase 3.2: $n \equiv 2 \pmod 4$.

Then $2F_{(n,4)} = 2$, $v_2(b) = 4$, and $v_2(F_{aj}) = v_2(n+2) + v_2(j) + 2 \geq 4 = v_2(b)$. So $j = 1$ and $z(b) = a$.

Subcase 3.3: $n \equiv 3 \pmod 4$.

Then $2F_{(n,4)} = 2$, $v_2(b) = 4$, and $v_2(F_{aj}) = v_2(n+1) + v_2(j) + 2 \geq 4 = v_2(b)$. So $j = 1$ and $z(b) = a$.

Subcase 3.4: $n \equiv 0 \pmod 4$.

Then $2F_{(n,4)} = 6$. So we need to consider 2-adic and 3-adic orders of b and F_{aj}. By Lemmas 2.1 and 2.3, we obtain that

$$\begin{aligned} v_2(b) &= v_2(n) + 3, \\ v_2(F_{aj}) &= v_2(n) + v_2(n+4) + v_2(j), \\ v_3(b) &= v_3(F_n) + v_3(F_{n+4}) \\ &= v_3(n) + v_3(n+4) + 2 = v_3(n) + 2, \quad \text{and} \\ v_3(F_{aj}) &= v_3(aj) + 1 = v_3(n) + v_3(n+3) + v_3(j). \end{aligned}$$

So we need to find the smallest $j \geq 1$ such that

$$v_2(n+4) + v_2(j) \geq 3 \text{ and } v_3(n+3) + v_3(j) \geq 2.$$

Note that $n + 4 \equiv 0, 4 \pmod 8$ and $n + 3 \equiv 0, 3, 6 \pmod 9$.

(i) If $n + 4 \equiv 0 \pmod 8$ and $n + 3 \equiv 0 \pmod 9$, then $v_2(j) = v_3(j) = 0$, so $j = 1$ and

$$z(b) = a = \frac{72a}{(8, n+4)(9, n+3)}.$$

(ii) If $n+4 \equiv 0 \pmod 8$ and $n+3 \equiv 3, 6 \pmod 9$, then $v_2(j) = 0$ and $v_3(j) = 1$, so $j = 3$ and

$$z(b) = 3a = \frac{72a}{(8, n+4)(9, n+3)}.$$

(iii) If $n + 4 \equiv 4 \pmod 8$ and $n + 3 \equiv 0 \pmod 9$, then $v_2(j) = 1$ and $v_3(j) = 0$, so $j = 2$ and

$$z(b) = 2a = \frac{72a}{(8, n+4)(9, n+3)}.$$

(iv) If $n + 4 \equiv 4 \pmod 8$ and $n + 3 \equiv 3, 6 \pmod 9$, then $v_2(j) = v_3(j) = 1$, so $j = 6$ and

$$z(b) = 6a = \frac{72a}{(8, n+4)(9, n+3)}.$$

This completes the proof.

□

We can state Theorem 3.2 in another form as follows.

Corollary 3.3. *Let* $n \geq 1$*,* $a = [n, n+1, n+2, n+3, n+4]$*, and* $b = F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}$*. Then*

$$z(b) = \begin{cases} a, & \text{if } n \equiv 1 \pmod 3 \text{ or } n \equiv 2,3,5,6 \pmod{12}; \\ 2a, & \text{if } n \equiv 9,11 \pmod{12}; \\ \frac{72a}{(8,n)(9,n+1)}, & \text{if } n \equiv 8 \pmod{12}; \\ \frac{72a}{(8,n+4)(9,n+3)}, & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Proof. This can be obtained from the proof of Theorem 3.2, or by comparing the result with Theorem 3.2. □

Corollary 3.4. *Let* $n \geq 1$ *and* $b = F_nF_{n+1}F_{n+2}F_{n+3}F_{n+4}$*. Then*

$$z(b) = \begin{cases} \frac{n(n+1)(n+2)(n+3)(n+4)}{2}, & \text{if } n \equiv 1,7 \pmod{12}; \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{3}, & \text{if } n \equiv 9,11 \pmod{12}; \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{4}, & \text{if } n \equiv 10 \pmod{12} \\ & \text{or } n \equiv 0,20,48,68 \pmod{72}; \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{6}, & \text{if } n \equiv 3,5 \pmod{12}; \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{8}, & \text{if } n \equiv 4 \pmod{12} \\ & \text{or } n \equiv 12,32,36,56 \pmod{72}; \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{12}, & \text{if } n \equiv 2,6 \pmod{12} \\ & \text{or } n \equiv 24,44 \pmod{72}; \\ \frac{n(n+1)(n+2)(n+3)(n+4)}{24}, & \text{if } n \equiv 8,60 \pmod{72}. \end{cases}$$

Proof. This follows from Theorem 3.2 and Lemma 2.3. □

Theorem 3.5. *Let* $n \geq 1$*,* $a = [n, n+1, \ldots, n+5]$*,* $b = F_nF_{n+1}\cdots F_{n+5}$*, and* $c = (5, n)$*. Then*

$$z(b) = \begin{cases} ac, & \text{if } n \equiv 1,2,3,4,5,6 \pmod{12}, \text{ or} \\ & n \equiv 7,8,59,60 \pmod{72}; \\ 2ac, & \text{if } n \equiv 9,10 \pmod{12}, \text{ or } n \equiv 23,24,43,44 \pmod{72}; \\ 3ac, & \text{if } n \equiv 11,12,31,32,35,36,55,56 \pmod{72}; \\ 6ac, & \text{if } n \equiv 0,19,20,47,48,67,68,71 \pmod{72}. \end{cases}$$

Proof. The proof of this theorem is similar to that of Theorem 3.2. So we will be brief here. It is easy to check that the result holds for $n = 1, 2$. So assume that $n \geq 3$. By Lemma 2.4 and Theorem 3.1, we obtain that $b \mid \ell F_{(n,5)} F_{aj}$ for every $j \geq 1$ where $\ell = 2, 6$. So we need to consider only v_2, v_3, and v_5 of b and F_{aj}. It is easy to check using Lemmas 2.1 and 2.3 that when $5 \mid n$, $v_5(b) \leq v_5(F_{aj})$ if and only if $v_5(j) \geq 1$, and when $5 \nmid n$, $v_5(b) \leq v_5(F_{aj})$ for every $j \geq 1$.

In addition, v_2 and v_3 of b and F_{aj} are

$$v_2(b) = \begin{cases} 4, & \text{if } n \equiv 1,2,3,4,5,6 \pmod{12}; \\ v_2(n+12-r)+3, & \text{if } n \equiv r \pmod{12} \text{ and } 7 \leq r \leq 12, \end{cases}$$

$$v_2(F_{aj}) = \begin{cases} v_2(n+4-r)+v_2(j)+2, & \text{if } n \equiv r \pmod 4 \\ & \text{and } 1 \le r \le 2; \\ v_2(n+4-r)+v_2(n+8-r)+v_2(j), & \text{if } n \equiv r \pmod 4 \\ & \text{and } 3 \le r \le 4, \end{cases}$$

$$v_3(b) = \begin{cases} 1, & \text{if } n \equiv 1,2,5,6 \pmod{12}; \\ 2, & \text{if } n \equiv 3,4 \pmod{12}; \\ v_3(n+12-r)+1, & \text{if } n \equiv r \pmod{12} \text{ and } r \in \{9,10\}; \\ v_3(n+12-r)+2, & \text{if } n \equiv r \pmod{12} \text{ and } r \in \{7,8,11,12\}, \end{cases}$$

$v_3(F_{aj}) = v_3(n+3-r)+v_3(n+6-r)+v_3(j)$, if $n \equiv r \pmod 3$ and $1 \le r \le 3$.

Case 1: $n \equiv 1 \pmod 4$.

Then $b \mid 2F_{(n,5)}F_{aj}$ for every $j \ge 1$ and we only need to consider $v_p(b)$ and $v_p(F_{aj})$ for $p = 2, 5$. If $n \equiv 1 \pmod 3$, then $v_2(F_{aj}) \ge v_2(b)$. So if $5 \nmid n$, we can choose $j = 1$ and obtain $z(b) = a$, and if $5 \mid n$, we can choose $j = 5$ and obtain $z(b) = 5a$. Therefore $z(b) = (5, n)a$. If $n \equiv 2 \pmod 3$, then $v_2(F_{aj}) \ge v_2(b)$ and we similarly obtain that $z(b) = (5, n)a$. If $n \equiv 0 \pmod 3$, then $v_2(F_{aj}) \ge v_2(b)$ if and only if $v_2(j) \ge 1$. Thus if $5 \nmid n$, we can choose $j = 2$ and obtain $z(b) = 2a$, and if $5 \mid n$, we can choose $j = 10$ and obtain $z(b) = 10a$. Therefore $z(b) = 2(5, n)a$.

Case 2: $n \equiv 2 \pmod 4$.

This case is similar to Case 1 and we obtain

$$z(b) = \begin{cases} (5,n)a, & \text{if } n \equiv 0,2 \pmod 3; \\ 2(5,n)a, & \text{if } n \equiv 1 \pmod 3. \end{cases}$$

Case 3: $n \equiv 3 \pmod 4$.

Then $b \mid 6F_{(n,5)}F_{aj}$ for every $j \ge 1$, and we need to consider $v_p(b)$ and $v_p(F_{aj})$ for $p = 2, 3, 5$.

Subcase 3.1: $n \equiv 1 \pmod 3$.

Then

$$v_2(b) \le v_2(F_{aj}) \Leftrightarrow v_2(n+1)+v_2(j) \ge 3, \text{ and}$$
$$v_3(b) \le v_3(F_{aj}) \Leftrightarrow v_3(n+2)+v_3(j) \ge 2.$$

Note that $n+1 \equiv 0,4 \pmod 8$ and $n+2 \equiv 0,3,6 \pmod 9$.

(i) If $n+1 \equiv 0 \pmod 8$ and $n+2 \equiv 0 \pmod 9$, then $v_2(j) = v_3(j) = 0$, and so

$$z(b) = (5,n)a = \frac{72(5,n)a}{(8,n+1)(9,n+2)}.$$

(ii) If $n+1 \equiv 0 \pmod 8$ and $n+2 \equiv 3,6 \pmod 9$, then $v_2(j) = 0$ and $v_3(j) = 1$, and so

$$z(b) = 3(5,n)a = \frac{72(5,n)a}{(8,n+1)(9,n+2)}.$$

(iii) If $n+1 \equiv 4 \pmod 8$ and $n+2 \equiv 0 \pmod 9$, then $v_2(j) = 1$ and $v_3(j) = 0$, and so

$$z(b) = 2(5,n)a = \frac{72(5,n)a}{(8,n+1)(9,n+2)}.$$

(iv) If $n+1 \equiv 4 \pmod 8$ and $n+2 \equiv 3, 6 \pmod 9$, then $v_2(j) = v_3(j) = 1$, and so

$$z(b) = 6(5,n)a = \frac{72(5,n)a}{(8,n+1)(9,n+2)}.$$

Subcase 3.2: $n \equiv 2 \pmod 3$.

This case is similar to Subcase 3.1 and we obtain

$$\begin{aligned} &v_2(b) \le v_2(F_{aj}) \Leftrightarrow v_2(n+5) + v_2(j) \ge 3, \\ &v_3(b) \le v_3(F_{aj}) \Leftrightarrow v_3(n+4) + v_3(j) \ge 2, \text{ and} \\ &z(b) = \frac{72(5,n)a}{(8,n+5)(9,n+4)}. \end{aligned}$$

Subcase 3.3: $n \equiv 0 \pmod 3$.

This case leads to $z(b) = (5,n)a$.

Case 4: $n \equiv 0 \pmod 4$.

Similar to Case 3, we obtain

$$z(b) = \begin{cases} (5,n)a, & \text{if } n \equiv 1 \pmod 3; \\ \frac{72(5,n)a}{(8,n)(9,n+1)}, & \text{if } n \equiv 2 \pmod 3; \\ \frac{72(5,n)a}{(8,n+4)(9,n+3)}, & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

This completes the proof. □

We can obtain the following result from the proof of Theorem 3.5.

Corollary 3.6. *Let $n \ge 1$, $a = [n, n+1, \dots, n+5]$, $b = F_nF_{n+1}\cdots F_{n+5}$, and $c = (5,n)$. Then*

$$z(b) = \begin{cases} ac, & \text{if } n \equiv 1,2,3,4,5,6 \pmod{12} \\ 2ac, & \text{if } n \equiv 9,10 \pmod{12}; \\ \frac{72(5,n)a}{(8,n+|r-8|)(9,n+|r-9|)}, & \text{if } n \equiv r \pmod{12} \text{ and } r \in \{7,8,12\}; \\ \frac{72(5,n)a}{(8,n+5)(9,n+4)}, & \text{if } n \equiv 11 \pmod{12}. \end{cases}$$

Next we give the formula of $z(F_nF_{n+1}\cdots F_{n+6})$. It is shorter to state it in the form similar to Corollary 3.6 than Theorem 3.5.

Theorem 3.7. *Let $n \geq 1$, $a = [n, n+1, \ldots, n+6]$, $b = F_nF_{n+1}\cdots F_{n+6}$, and $c = (5, n(n+1))$. Then $z(b) =$*

$$\begin{cases} ac, & \text{if } n \equiv 1,2,3,4,5 \pmod{12};\\ \frac{(64)(27)ac}{(64,n+2)(27,n(n+3))}, & \text{if } n \equiv 6 \pmod{24};\\ \frac{(8)(27)ac}{(27,n(n+3))}, & \text{if } n \equiv 18 \pmod{24};\\ \frac{72ac}{(8,n-r)(9,n-r)}, & \text{if } n \equiv r \pmod{12} \text{ and } r \in \{7,8\};\\ 4ac, & \text{if } n \equiv 9 \pmod{12};\\ \frac{72ac}{(8,n+6)(9,n+5)}, & \text{if } n \equiv 10 \pmod{12};\\ \frac{72ac}{(8,n+5)(9,n+4)}, & \text{if } n \equiv 11 \pmod{12};\\ \frac{(64)(27)ac}{(64,n+4)(27,(n+3)(n+6))}, & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Proof. The proof of this theorem follows the same ideas used previously. So we will only give the evaluation of v_2, v_3, and v_5 of b and F_{aj}. Similar to the proof of Theorem 3.5, we have when $5 \mid n(n+1)$, $v_5(b) \leq v_5(F_{aj})$ if and only if $v_5(j) \geq 1$, when $5 \nmid n(n+1)$, $v_5(b) \leq v_5(F_{aj})$ for every $j \geq 1$,

$$v_2(F_{aj}) = \begin{cases} v_2(n+3) + v_2(j) + 2, & \text{if } n \equiv 1 \pmod{4};\\ v_2(n+6) + v_2(j) + 3, & \text{if } n \equiv 2 \pmod{8};\\ v_2(n+2) + v_2(j) + 2, & \text{if } n \equiv 6 \pmod{8};\\ v_2(n+1) + v_2(n+5) + v_2(j), & \text{if } n \equiv 3 \pmod{4};\\ v_2(n) + v_2(n+4) + v_2(j), & \text{if } n \equiv 0 \pmod{4}, \end{cases}$$

$$v_3(F_{aj}) = \begin{cases} v_3(n+2) + v_3(n+5) + v_3(j), & \text{if } n \equiv 1 \pmod{3};\\ v_3(n+1) + v_3(n+4) + v_3(j), & \text{if } n \equiv 2 \pmod{3};\\ v_3(n) + v_3(n+3) + v_3(n+6) + v_3(j) - 1, & \text{if } n \equiv 0 \pmod{3}, \end{cases}$$

$$v_2(b) = \begin{cases} 4, & \text{if } n \equiv 1,2,4,5 \pmod{12};\\ 5, & \text{if } n \equiv 3 \pmod{12};\\ v_2(n+12-r) + 3, & \text{if } n \equiv r \pmod{12} \text{ and } r \in \{7,8,10,11\};\\ v_2(n+3) + 4, & \text{if } n \equiv 9 \pmod{12};\\ v_2(n+12-r) + 6, & \text{if } n \equiv r \pmod{12} \text{ and } r \in \{6,12\}, \end{cases}$$

$$v_3(b) = \begin{cases} 1, & \text{if } n \equiv 1,5 \pmod{12};\\ 2, & \text{if } n \equiv 2,3,4 \pmod{12};\\ v_3(n+12-r) + 2, & \text{if } n \equiv r \pmod{12} \text{ and}\\ & \quad r \in \{6,7,8,10,11,12\};\\ v_3(n+3) + 1, & \text{if } n \equiv 9 \pmod{12}. \end{cases}$$

□

4. Conclusion

In this article, we give a systematic method in calculating the order of appearance of products of consecutive Fibonacci numbers. We also obtain the corresponding results for the Lucas numbers in [7]. The converse of the results in [18] is given in [16] and the order of appearance of factorials is obtained in [23]. For other closedly related results, see for example in [17, 21, 22, 20].

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