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ALTERNATING SERIES OF APÉRY-TYPE FOR THE RIEMANN ZETA FUNCTION

WENCHANG CHU

Abstract. By making use of a transformation formula for the non-terminating well-poised ${}_5F_4$-series, we investigate a class of alternating series of Apéry-type and establish several identities for the Riemann zeta function, including three identities conjectured by Sun (2015).

1. Introduction and Motivation

The Riemann zeta function is defined by the series

$$\zeta(x)=\sum_{n=1}^{\infty}\frac{1}{n^x}\quad\text{for}\quad\Re(x)>1.$$

Euler discovered the well-known formula that this function at the even positive integers can be expressed in terms of π:

$$\zeta(2n)=(-1)^{n-1}\frac{(2\pi)^{2n}}{(2n)!}B_{2n},$$

where the even index Bernoulli numbers B_{2n} are defined by the generating function:

$$y\cot y=\sum_{n=0}^{\infty}(-1)^n\frac{(2y)^{2n}}{(2n)!}B_{2n}.$$

In his irrationality proof for $\zeta(2)$ and $\zeta(3)$, Apéry [3] (see also [19]) found the following surprising infinite series identities:

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{3}{n^2\binom{2n}{n}}\qquad\text{and}\qquad\zeta(3)=\frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

More formulae of Apéry-type involving central binomial coefficients can be found in numerous articles, for example, [1, 2, 5, 8, 11, 12, 14, 16, 18, 20, 22], as well as the monographs by Berndt [6, §9], Borwein and Borwein [7, §11] and Comtet [15, p. 89].

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Email address: chu.wenchang@unisalento.it.

Define the generalized harmonic numbers of higher order by

$$H_n^{\langle m\rangle} := \sum_{k=1}^{n} \frac{1}{k^m} \quad \text{with} \quad H_n := H_n^{\langle 1\rangle}.$$

The purpose of this paper is to investigate alternating series of Apéry-type involving both harmonic numbers of higher order and central binomial coefficients. According to Chu [10, 11] and Chu–Zhang [13], numerous π-related series identities (such as Riemann zeta series, Apéry-type series, and Ramanujan-like series) can be derived by examining classical hypergeometric functions. By employing a transformation formula for the nonterminating well-poised ${}_5F_4$-series appearing in [13, Theorem 10], we shall establish several further infinite series identities. For instance, we find two Apéry-like series for $\zeta(4)$ (see Examples 3.2 and 3.3)

$$\frac{\pi^4}{30} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{10nH_n - 3}{n^4\binom{2n}{n}}, \tag{1.1}$$

$$\frac{2\pi^4}{75} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n} + 4H_n}{n^3\binom{2n}{n}}; \tag{1.2}$$

plus another series for $\zeta^2(3)$ (see Example 3.11)

$$2\zeta^2(3) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1 + 5n^3H_n^{\langle 3\rangle}}{n^6\binom{2n}{n}}. \tag{1.3}$$

They are among the list of the conjectured series detected experimentally by Sun [21].

2. Hypergeometric Series transformation

Let $\mathbb{N}$ be the set of natural numbers. The shifted factorials are defined by

$$(x)_0 \equiv 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}.$$

In their derivation of Ramanujan's π-formulae, Chu and Zhang [13, Theorem 10] proved the following transformation.

Lemma 2.1. *For the five complex parameters* $\{a, b, c, d, e\}$ *subject to the convergence condition* $\Re(1+2a-b-c-d-e) > 0$, *the following transformation formula holds*

$$\begin{aligned}
&\sum_{k=0}^{\infty}(a+2k)\frac{(b)_k(c)_k(d)_k(e)_k}{(1+a-b)_k(1+a-c)_k(1-a-d)_k(1+a-e)_k}\\
&=\sum_{n=0}^{\infty}(-1)^n\frac{(1+a-b-c)_n(1+a-b-d)_n(1+a-b-e)_n}{(1+2a-b-c-d-e)_{2+2n}}\\
&\times\frac{(1+a-c-d)_n(1+a-c-e)_n(1+a-d-e)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-e)_n}\\
&\times\left\{\begin{matrix}(1+2a-b-c-d+2n)(2+2a-b-c-d-e+2n)(a-e+n)\\ +(1+a-b-c+n)(1+a-b-d+n)(1+a-c-d+n)\end{matrix}\right\}.
\end{aligned}$$

Letting $e=a$ and then applying the summation theorem for the well-poised ${}_5F_4$-series due to Dougall (cf. Bailey [4, §4.4])

$$\begin{aligned}
&\sum_{k=0}^{\infty}(a+2k)\frac{(b)_k(c)_k(d)_k(e)_k}{(1+a-b)_k(1+a-c)_k(1-a-d)_k(1+a-e)_k}\\
&\quad=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}
\end{aligned}$$

we can express the resulting equation, under the parameter replacements

$$a\to 1-ax,\qquad b\to 1-bx,\qquad c\to 1-cx,\qquad d\to 1-dx$$

as the following theorem involving the variable x and four parameters $\{a, b, c, d\}$.

Theorem 2.2. *For the five complex parameters* $\{a, b, c, d, x\}$ *subject to the condition that none of the linear combinations*

$$\{ax+bx,\ ax+cx,\ ax+dx,\ ax+bx+cx+dx\}$$

is a negative integer (so that the series is well-defined), *the following summation formula holds*

$$\begin{aligned}
&\frac{\Gamma(1+ax+bx)\Gamma(1+ax+cx)\Gamma(1+ax+dx)\Gamma(1+ax+bx+cx+dx)}{\Gamma(1+ax)\Gamma(1+ax+bx+cx)\Gamma(1+ax+bx+dx)\Gamma(1+ax+cx+dx)}\\
&=1+\sum_{n=1}^{\infty}(-1)^n\frac{bcdx^3}{n^6\binom{2n}{n}}\\
&\times\left\{\begin{matrix}(n+ax+bx+cx)(n+ax+bx+dx)(n+ax+cx+dx)\\ +\,n(2n+2ax+bx+cx+dx)(2n+ax+bx+cx+dx)\end{matrix}\right\}\\
&\times\frac{\binom{n-1+bx}{n-1}\binom{n-1+cx}{n-1}\binom{n-1+dx}{n-1}\binom{n-1+ax+bx+cx}{n-1}\binom{n-1+ax+bx+dx}{n-1}\binom{n-1+ax+cx+dx}{n-1}}{\binom{n+ax+bx}{n}\binom{n+ax+cx}{n}\binom{n+ax+dx}{n}\binom{2n+ax+bx+cx+dx}{2n}}.
\end{aligned}$$

By comparing the coefficients of x^3 across the last identity, we recover immediately the following important identity.

Example 2.3 (Hjortnaes (1954: see also [8, 17, 20])**.**

$$\frac{2}{5}\zeta(3)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

Throughout the paper, we shall utilize the modified harmonic numbers of higher order, slightly different from the usual one, defined by

$$\mathbf{H}_m(n)=\sum_{k=1}^{n}\Big(\frac{n}{k}\Big)^m=n^m H_n^{\langle m\rangle}\quad\text{for}\quad m,\,n\in\mathbb{N}.$$

The reason to introduce the above notation $\mathbf{H}_m(n)$, instead of $H_n^{\langle m\rangle}$, is not only for brevity to reduce lengthy and ugly expressions containing

$$\mathbf{H}_m^{\lambda}(n)=n^{m\lambda}\big(H_n^{\langle m\rangle}\big)^{\lambda},$$

but also justified by another advantage that all the infinite series expressions for Riemann zeta function in this paper have the form

$$\zeta(m)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^m\binom{2n}{n}}\times\mathcal{P}\quad\text{for}\quad m\geq 3,$$

where $\mathcal{P}$ is a multivariate polynomial of $\{\mathbf{H}_k(n)\}_{k=1}^{m-3}$ with each $\mathbf{H}_k(n)$ behaving like a constant, as shown in Examples 3.5–3.7 for $\zeta(5)$ and Example 3.16 for $\zeta(7)$.

In order to derive further infinite series identities by extracting the coefficients of higher powers of the variable x in Theorem 2.2, it is necessary to record some basic facts about the Bell polynomials and generalized harmonic numbers as well as power series expansions of the Γ-function.

For the indeterminates $y=\{y_k\}_{k\in\mathbb{N}}$, we define the Bell polynomials $\mathcal{B}_m(y)$ (cf. Comtet [15, §3.3]) by the generating function

$$\sum_{m\geq 0}\mathcal{B}_m(y)x^m=\exp\Big\{\sum_{k\geq 1}\frac{x^k}{k}y_k\Big\}. \tag{2.1}$$

There is the following explicit expression

$$\mathcal{B}_m(y)=\mathcal{B}_m(y_1,y_2,\ldots,y_m)\;=\;\sum_{\Omega(m)}\prod_{k=1}^{m}\frac{y_k^{\ell_k}}{\ell_k!\,k^{\ell_k}} \tag{2.2}$$

where the multiple sum runs over $\Omega(m)$, the set of m-partitions represented by m-tuples of $(\ell_1,\ell_2,\ldots,\ell_m)\in\mathbb{N}_0^m$ subject to the condition $\sum_{k=1}^m k\ell_k=m$.

Let $[x^m]f(x)$ stand for the coefficient of x^m in a formal power series $f(x)$. By means of the generating function method, it is not hard to show that (cf. Chen–Chu [9]) there hold the relations:

$$[x^m]\binom{n-\lambda x}{n} = \mathcal{B}_m(u), \qquad u_k := -\Big(\frac{\lambda}{n}\Big)^k \mathbf{H}_k(n); \tag{2.3}$$

$$[x^m]\binom{n-\lambda x}{n}^{-1} = \mathcal{B}_m(v), \qquad v_k := \Big(\frac{\lambda}{n}\Big)^k \mathbf{H}_k(n); \tag{2.4}$$

$$[x^m]\binom{n-1-\lambda x}{n-1} = \mathcal{B}_m(w), \qquad w_k := \Big(\frac{\lambda}{n}\Big)^k \big\{1-\mathbf{H}_k(n)\big\}. \tag{2.5}$$

Then the Bell polynomials corresponding to (2.3), (2.4), and (2.5) can be expressed as

$$\mathcal{B}_m(u) = \frac{\lambda^m}{n^m} \sum_{\Omega(m)} \prod_{k=1}^{m} \frac{\big\{-\mathbf{H}_k(n)\big\}^{\ell_k}}{\ell_k!\, k^{\ell_k}}, \tag{2.6}$$

$$\mathcal{B}_m(v) = \frac{\lambda^m}{n^m} \sum_{\Omega(m)} \prod_{k=1}^{m} \frac{\mathbf{H}_k^{\ell_k}(n)}{\ell_k!\, k^{\ell_k}}, \tag{2.7}$$

$$\mathcal{B}_m(w) = \frac{\lambda^m}{n^m} \sum_{\Omega(m)} \prod_{k=1}^{m} \frac{\big\{1-\mathbf{H}_k(n)\big\}^{\ell_k}}{\ell_k!\, k^{\ell_k}}. \tag{2.8}$$

They will be employed to compute coefficients of x^m for $m \in \mathbb{N}$ from the right hand side of the equation displayed in Theorem 2.2. Instead, for the left hand side of the same equation, we shall utilize the power series expansions of the Γ-function [10]

$$\Gamma(1-x) = \exp\Big\{\sum_{k\geq 1} \frac{\sigma_k}{k} x^k\Big\}, \tag{2.9}$$

$$\Gamma(\tfrac{1}{2}-x) = \sqrt{\pi}\exp\Big\{\sum_{k\geq 1} \frac{\tau_k}{k} x^k\Big\}; \tag{2.10}$$

where σ_k and τ_k are defined respectively by

$$\begin{aligned} &\sigma_1 = \gamma && \text{and} \quad \sigma_m = \zeta(m) && \text{for} \quad m \geq 2;\\ &\tau_1 = \gamma + 2\ln 2 && \text{and} \quad \tau_m = (2^m-1)\zeta(m) && \text{for} \quad m \geq 2; \end{aligned}$$

with γ being the Euler–Mascheroni constant given by $\gamma = \lim_{n\to\infty}\big(H_n - \ln n\big)$.

3. Alternating series of Apéry-type

In this section, we shall establish several alternating series identities of Apéry-type for the Riemann zeta function by extracting coefficients of x^m for $m \in \mathbb{N}$ across the identity given in Theorem 2.2. Because the computations are entirely routine, we highlight only the parameter settings in the example headers, where for simplicity, we shall make use of the following three power sums

$$\rho_1 = b+c+d, \quad \rho_2 = b^2+c^2+d^2, \quad \text{and} \quad \rho_3 = b^3+c^3+d^3.$$

3.1. **Infinite Series for $\zeta(4)$ from $[x^4]$ in Theorem 2.2.**

Example 3.1 $(\rho_1 = 0)$**.**

$$\frac{2\pi^4}{15} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{5\mathbf{H}_1^1(2n)+12}{n^4\binom{2n}{n}}.$$

Example 3.2 $(a = -\rho_1)$**.**

$$\frac{\pi^4}{30} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{10\mathbf{H}_1^1(n)-3}{n^4\binom{2n}{n}}.$$

Example 3.3 $(a = -\frac{3}{2}\rho_1)$**.**

$$\frac{4\pi^4}{75} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\mathbf{H}_1^1(2n)+8\mathbf{H}_1^1(n)}{n^4\binom{2n}{n}}.$$

The above two identities confirm the conjectured values detected experimentally by Sun [21, Equations 3.13 and 3.14]. The next one follows from the linear combination "$2 \times \mathbf{E}(3.1) - 5 \times \mathbf{E}(3.3)$", where $\mathbf{E}(3.1)$ and $\mathbf{E}(3.3)$ stand for the equations in Example 3.1 and Example 3.3, respectively.

Example 3.4 $(a = -\frac{\rho_1}{2})$**.**

$$0 = \sum_{n=1}^{\infty}(-1)^{n}\frac{5\mathbf{H}_1^1(2n)-40\mathbf{H}_1^1(n)+24}{n^4\binom{2n}{n}}.$$

3.2. **Infinite series for $\zeta(5)$ from $[x^5]$ in Theorem 2.2.** Koecher [17] (see also [2, 5, 8, 19]) found the following infinite series identity

$$\zeta(5) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5\binom{2n}{n}} + \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^3\binom{2n}{n}}\sum_{k=1}^{n-1}\frac{1}{k^2}$$

which can be restated equivalently in the example below.

Example 3.5 $(a = \rho_1 = 0)$**.**

$$2\zeta(5) = \sum_{n=1}^{\infty}(-1)^{n}\frac{5\mathbf{H}_2^1(n)-9}{n^5\binom{2n}{n}}.$$

Example 3.6 $(a = -\rho_1$ and $\rho_2 = \rho_1^2)$**.**

$$4\zeta(5) = \sum_{n=1}^{\infty}(-1)^{n}\frac{6\mathbf{H}_1^1(n)-10\mathbf{H}_1^2(n)-5}{n^5\binom{2n}{n}}.$$

Example 3.7 $(\rho_1 = \rho_2 = 0)$**.**

$$96\zeta(5) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{24\mathbf{H}_1^1(2n)+5\mathbf{H}_1^2(2n)+5\mathbf{H}_2^1(2n)+64}{n^5\binom{2n}{n}}.$$

Example 3.8 ($\rho_1 = -\frac{2a}{3}$ and $\rho_2 = -\frac{16a^2}{3}$).

$$0 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^5\binom{2n}{n}} \left\{ \begin{array}{r} 320\mathbf{H}_1^2(n) + 5\mathbf{H}_1^2(2n) + 5\mathbf{H}_2^1(2n) - 1664 \\ +1120\mathbf{H}_2^1(n) + 80\mathbf{H}_1^1(n)\mathbf{H}_1^1(2n) \end{array} \right\}.$$

As done for Example 3.4, the next two "redundant series" can also be derived by combining the two equations displayed in Examples 3.5 and 3.6.

Example 3.9 ($a = -\rho_1$ and $\rho_2 = -3\rho_1^2$).

$$0 = \sum_{n=1}^{\infty} (-1)^n \frac{6\mathbf{H}_1^1(n) - 10\mathbf{H}_1^2(n) - 10\mathbf{H}_2^1(n) + 13}{n^5\binom{2n}{n}}.$$

Example 3.10 ($a = -\rho_1$ and $\rho_2 = -\frac{2\rho_1^2}{9}$).

$$26\zeta(5) = \sum_{n=1}^{\infty} (-1)^n \frac{54\mathbf{H}_1^1(n) - 90\mathbf{H}_1^2(n) - 25\mathbf{H}_2^1(n)}{n^5\binom{2n}{n}}.$$

3.3. **Infinite series for $\zeta(6)$ from $[x^6]$ in Theorem 2.2.** Firstly, we confirm the following identity conjectured experimentally by Sun [21, Equation 4.5].

Example 3.11 ($a = \rho_1 = 0$).

$$2\zeta^2(3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{5\mathbf{H}_3^1(n) + 1}{n^6\binom{2n}{n}}.$$

Example 3.12 ($\rho_1 = -a,\ \rho_2 = a^2$ and $\rho_3 = -a^3$).

$$\frac{\pi^6}{63} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{30\mathbf{H}_1^1(n) - 18\mathbf{H}_1^2(n) + 20\mathbf{H}_1^3(n) + 10\mathbf{H}_3^1(n) - 9}{n^6\binom{2n}{n}}.$$

Example 3.13 ($\rho_1 = -a,\ \rho_2 = a^2$ and $\rho_3 = -3a^3$).

$$\frac{\pi^6}{63} - 4\zeta^2(3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{30\mathbf{H}_1^1(n) - 18\mathbf{H}_1^2(n) + 20\mathbf{H}_1^3(n) - 11}{n^6\binom{2n}{n}}.$$

Example 3.14 ($\rho_1 = 0$ and then $[a]$).

$$\frac{4\pi^6}{189} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{40 + 9\mathbf{H}_1^1(2n) - 12\mathbf{H}_2^1(n) - 5\mathbf{H}_2^1(n)\mathbf{H}_1^1(2n)}{n^6\binom{2n}{n}}.$$

Example 3.15 ($\rho_1 = 0$ and then $[a^3]$).

$$\frac{64\pi^6}{63} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^6\binom{2n}{n}} \left\{ \begin{array}{r} 480 + 192\mathbf{H}_1^1(2n) + 36\mathbf{H}_1^2(2n) + 36\mathbf{H}_2^1(2n) \\ +5\mathbf{H}_1^3(2n) + 15\mathbf{H}_1^1(2n)\mathbf{H}_2^1(2n) + 10\mathbf{H}_3^1(2n) \end{array} \right\}.$$

3.4. **Infinite series for $\zeta(7)$ from $[x^7]$ in Theorem 2.2.**
Koecher [17] (cf. [8, Equation 9-3]) discovered also the following interesting series for $\zeta(7)$.

Example 3.16 ($a = \rho_1 = 0$)**.**

$$4\zeta(7) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^7\binom{2n}{n}} \Big\{18\mathbf{H}_2^1(n) - 5\mathbf{H}_2^2(n) + 5\mathbf{H}_4^1(n) - 26\Big\}.$$

Example 3.17 ($\rho_1 = \rho_2 = 0$ and then $[a]$)**.**

$$\frac{4\pi^4}{3}\zeta(3) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^7\binom{2n}{n}} \Big\{132 - 300\mathbf{H}_4(n) - \big\{1 + 5\mathbf{H}_3(n)\big\}\big\{12 + 5\mathbf{H}_1(2n)\big\}\Big\}.$$

3.5. **Infinite series for $\zeta(8)$ from $[x^8]$ in Theorem 2.2.**

Example 3.18 ($a = \rho_1 = 0$)**.**

$$4\zeta(3)\zeta(5) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^8\binom{2n}{n}} \Big\{2 - \mathbf{H}_2^1(n) + 9\mathbf{H}_3^1(n) + 5\mathbf{H}_5^1(n) - 5\mathbf{H}_2^1(n)\mathbf{H}_3^1(n)\Big\}.$$

Example 3.19 ($\rho_1 = \rho_2 = 0$ and then $[a]$)**.**

$$\frac{4\pi^8}{675} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^8\binom{2n}{n}} \times \left\{ \begin{array}{c} 5\mathbf{H}_2^2(n)\mathbf{H}_1^1(2n) - 18\mathbf{H}_2^1(n)\mathbf{H}_1^1(2n) - 5\mathbf{H}_4^1(n)\mathbf{H}_1^1(2n) \\ +168 - 80\mathbf{H}_2^1(n) + 12\mathbf{H}_2^2(n) - 12\mathbf{H}_4^1(n) + 26\mathbf{H}_1^1(2n) \end{array} \right\}.$$

3.6. **Infinite series for $\zeta(9)$ from $[x^9]$ in Theorem 2.2.**

Example 3.20 ($a = \rho_1 = \rho_2 = 0$)**.**

$$\frac{4\zeta(9)}{3} + \frac{8\zeta^3(3)}{3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^9\binom{2n}{n}} \Big\{5\mathbf{H}_6^1(n) - 5\mathbf{H}_3^2(n) - 2\mathbf{H}_3^1(n) - 10\Big\}.$$

Example 3.21 ($a = c = 0$ and $b = -d = 1$)**.**

$$12\zeta(9) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^9\binom{2n}{n}} \left\{ \begin{array}{c} 27\mathbf{H}_4^1(n) + 10\mathbf{H}_6^1(n) - 15\mathbf{H}_2^1(n)\mathbf{H}_4^1(n) \\ -102 + 78\mathbf{H}_2^1(n) - 27\mathbf{H}_2^2(n) + 5\mathbf{H}_2^3(n) \end{array} \right\}.$$

By carrying out the same procedure, it is possible to derive alternating series expressions for $\zeta(m)$ with $m > 9$. However, we shall not produce them further because the resulting series are too complicated.

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School of Mathematics and Statistics
Zhoukou Normal University
Zhoukou 466001, P. R. China

Corresponding address
Department of Mathematics and Physics
University of Salento (P. O. Box 193)
73100 Lecce, Italy

E-mail address: chu.wenchang@unisalento.it