COLORING PERMUTATION-GAIN GRAPHS

DANIEL SLILATY

ABSTRACT. Correspondence colorings of graphs were introduced in 2018 by Dvořák and Postle as a generalization of list colorings of graphs which generalizes ordinary graph coloring. Kim and Ozeki observed that correspondence colorings generalize various notions of signed-graph colorings which again generalizes ordinary graph colorings. In this note we state how correspondence colorings generalize Zaslavsky’s notion of gain-graph colorings and then formulate a new coloring theory of permutation-gain graphs that sits between gain-graph coloring and correspondence colorings. Like Zaslavsky’s gain-graph coloring, our new notion of coloring permutation-gain graphs has well defined chromatic polynomials and lifts to colorings of the regular covering graph of a permutation-gain graph.

1. Introduction

Dvořák and Postle very recently introduced [2] the concept of correspondence colorings (also called DP-colorings) of graphs as a generalization of list colorings of graphs which is a generalization of ordinary colorings of graphs. A signed graph is a pair \((G, \sigma)\) in which \(\sigma : E(G) \to \{+, -\}\). Zaslavsky [7] defined two notions of coloring signed graphs: one has colors corresponding to the elements of \(\{0, \pm 1, \ldots, \pm k\}\) and the other with colors corresponding to the elements of \(\{\pm 1, \ldots, \pm k\}\) (also see Mácajová, Raspaud, and Škoviera [6]). Call this notion of coloring signed graphs integer coloring of signed graphs. Kang and Steffen explored [3, 4] the analogous notion of coloring signed graphs with elements of the group \(\mathbb{Z}_k\). We will call this modular coloring of signed graphs. The notions of modular coloring and integer coloring of signed graphs correspond when the number of colors is odd but not when the number of colors is even. Kim and Ozeki noted [5] that correspondence colorings generalize both modular coloring and integer coloring of signed graphs. Zaslavsky also developed [7, 9] a notion of gain-graph colorings which generalizes integer colorings of signed graphs. The purpose of this note is to define a concept of coloring permutation-gain graphs which generalizes gain-graph colorings and both notions of signed graph colorings and which is in turn generalized by correspondence colorings (see Figure 1).
The main focus of [9] is the development of a rich theory of chromatic polynomials of gain graphs which are closely related to the very important Tutte polynomials of different matroids associated with gain graphs (see Theorems 5.1 and 5.2 in [9]). This connection with Tutte polynomials and its associated matroid invariants is one of the main attractions of Zaslavsky’s concept of gain-graph colorings. In this note, we give two consequences of our definition of coloring permutation-gain graphs. In Section 3, we show that there is a well-defined chromatic polynomial for a permutation-gain graph along with a limited deletion-contraction relation. This chromatic polynomial, however, is in general not an invariant of matroids normally associated with the permutation-gain graph. In Section 4 we also show how proper colorings of permutation-gain graphs lift to proper colorings of their derived graphs (i.e., regular covering graphs). It is not clear whether or not correspondence colorings have chromatic polynomials or associations with colorings of covering graphs.

2. Formal Definitions

Graphs. In a graph $G$, an edge $e$ with its ends on distinct endpoints is called a link and an edge with its ends on the same vertex is called a loop. An oriented edge is an edge $e$ along with a choice of direction along $e$. We usually use $e$ to refer to an oriented edge whose underlying edge is $e$; it usually causes no confusion. The reverse orientation of an oriented edge $e$ is denoted by $e^{-1}$. The collection of oriented edges of $G$ is denoted by $\vec{E}(G)$. The tail and head endpoints of an oriented edge $e$ are sometimes denoted, respectively, by $t(e)$ and $h(e)$. A walk $w$ is a product of oriented edges $e_1 \cdots e_n$ for which $h(e_i) = t(e_{i+1})$ for each $i \in \{1, \ldots, n-1\}$. The walk $w = e_1 \cdots e_n$ is closed when $h(e_n) = t(e_1)$. The reverse walk of $w$ is $w^{-1} = e_n^{-1} \cdots e_1^{-1}$.
Correspondence Colorings. Given a graph $G$ and a positive integer $k$, associate to each vertex $v \in V(G)$ the set $\{1, \ldots, k\} \times \{v\}$. A $k$-correspondence assignment is a function $M$ assigning to each edge $e \in E(G)$ a partial matching $M_e$ between the sets $\{1, \ldots, k\} \times \{u\}$ and $\{1, \ldots, k\} \times \{v\}$ where $u$ and $v$ are the endpoints of $e$. In the case that $e$ is a loop, we can use two separate copies of $\{1, \ldots, k\} \times \{v\}$ for the matching $M_e$. An $M$-coloring of $G$ is a $k$-coloring $\kappa$ of $G$ such that for each edge $e$ with endpoints $u$ and $v$, $(\kappa(u), u)$ and $(\kappa(v), v)$ are not connected in the matching $M_e$. Note that when the matchings of $M$ are all complete matchings of $k$ edges, then $M$ corresponds to a gain function on $G$ of some subgroup of the symmetric group $S_k$. This observation lead the author to consider coloring permutation-gain graphs as a special case of correspondence colorings.

Permutation-Gain Graphs. Let $\Gamma$ be a subgroup of the symmetric group $S_k$. We will be looking at functions $\pi: \bar{E}(G) \to \Gamma$ and $\eta: V(G) \to \Gamma$ from $G$ to $\Gamma$ which will always be denoted with lowercase Greek letters; furthermore, the values of these functions at arguments $e$ and $v$ will be denoted by $\pi_e$ and $\eta_v$ rather than the far more common notation $\pi(e)$ and $\eta(v)$. The reason for this is because we will also be considering the action of $\Gamma \subseteq S_k$ on the set $\{1, \ldots, k\}$ and denoting the action of $\pi_e \in \Gamma$ on $i \in \{1, \ldots, k\}$ by $\pi_e(i)$. We will also consider the action of $\Gamma$ on the set $\{1, \ldots, l\}$ defined by right multiplication on the $\Gamma$-coordinate of each pair. Recall that all fixed-point free actions of $\Gamma$ on a finite set are essentially of this type. So for each positive integer $t$, there is an isomorphic copy of $\Gamma$, again call it $\Gamma$, in the symmetric group $S_k$ where $k = l + t|\Gamma|$ for which: the symbols $1, \ldots, l$ are as before, the action of $\Gamma$ on $\{1, \ldots, l\}$ is as before, and the action of $\Gamma$ on the remaining $t|\Gamma|$ elements is free. Thus fix($\pi$) is the same subset of $\{1, \ldots, l\}$ no matter which $k = l + t|\Gamma|$ we consider.

A $k$-coloring of $(G, \pi)$ (again where $k = l + t|\Gamma|$) is a function $\kappa: V(G) \to \{1, \ldots, k\}$. The $k$-coloring $\kappa$ is proper when for each oriented edge $e$ (including loops) with tail endpoint $u$ and head endpoint $v$ we have that
does as well because \( \kappa(u) \neq \pi_e \kappa(v) \). Note that \( e \) satisfies the propriety condition if and only if \( e^{-1} \) does as well because \( \kappa(u) \neq \pi_e \kappa(v) \) if and only if \( \pi_{e^{-1}} \kappa(u) \neq \kappa(v) \). We now see how \( k \)-colorings of permutation-graph graphs generalizes proper-coloring concepts of gain graphs and signed graphs in \([3, 4, 7, 9]\). In gain-graph colorings and integer signed-graph colorings, the action of \( \Gamma \) either: acts freely on a color set of size \( t|\Gamma| \) or acts freely on a color set of size \( t|\Gamma| + 1 \) save at one color which is fixed by all of the elements of \( \Gamma \). In modular signed-graph colorings with even color set \( \mathbb{Z}_{2k} \), the action of \( \Gamma = \{+, -\} \) fixes 0 and \( k \) and acts freely on \( \mathbb{Z}_{2k} - \{0, k\} \).

If \( \kappa \) is a \( k \)-coloring of \((G, \pi)\) and \( \eta \) switching function, then define the \( k \)-coloring \( \eta \kappa \) by \( \eta \kappa(v) = \eta(v) \kappa(v) \). Note that \( \eta \kappa(u) \neq \pi_e^0 \eta \kappa(v) \) if and only if \( \eta_u \kappa(u) \neq \eta_u \pi_e \eta_e^{-1} \eta_u \kappa(v) \) if and only if \( \kappa(u) \neq \pi_e \kappa(v) \). This yields Proposition 2.1.

**Proposition 2.1.** If \( \eta : V(G) \to \Gamma \) is a switching function and \( \kappa \) is a \( k \)-coloring of \((G, \pi)\), then \( \kappa \) is a proper \( k \)-coloring of \((G, \pi)\) if and only if \( \eta \kappa \) is a proper \( k \)-coloring of \((G, \pi^0)\). Furthermore, \( \kappa \mapsto \eta \kappa \) defines a bijection between the proper \( k \)-colorings of \((G, \pi)\) and \((G, \pi^0)\).

3. A chromatic polynomial

In this section we show that there is a well-defined chromatic polynomial \( p(G, \pi, k) \) for \((G, \pi)\) where the value of \( p(G, \pi, k) \) at any \( k = l + t|\Gamma| \) is the number of proper \( k \)-colorings of \((G, \pi)\). For example, if \((G, \pi)\) is a graph in which every edge is a loop, then for each \( v \in V(G) \) let \( \text{fix}(v) \subseteq \{1, \ldots, l\} \) denote the union of all \( \text{fix}(e) \) in which \( e \) is a loop incident to \( v \). We now get that

\[
p(G, \pi, k) = \prod_{v_j \in V(G)} (k - |\text{fix}(v_j)|)
\]

This example is interesting in that it shows that the chromatic polynomial of a permutation-gain graph is dependent on the structure of fixed points in \( \{1, \ldots, l\} \subseteq \{1, \ldots, k\} \) under the action of \( \Gamma \). As such, this chromatic polynomial is not an invariant of any of the matroids normally associated with \((G, \pi)\): the frame matroid, lift matroid, and complete lift matroid (see, for example, [8]).

**Theorem 3.1.** Let \( p(G, \pi, k) \) be the number of proper \( k \)-colorings of \((G, \pi)\) for any \( k = l + t|\Gamma| \).

1. If \( e \) is any link in \( G \) and \( \eta \) a switching function such that \( \pi^0(e) = 1 \) (here 1 is the identity permutation) then \( p(G, \pi, k) = p(G, \pi^0, k) = p(G \setminus e, \pi^0, k) - p(G/e, \pi^0, k) \).
2. If \((G, \pi)\) has no balanced loops (i.e., a loop \( e \) for which \( \pi_e = 1 \)), then \( p(G, \pi, k) \) is a well-defined, monic polynomial in \( k \) of degree \( |V(G)| \).

**Proof.** (1) By Proposition 2.1, the proper \( k \)-colorings of \((G, \pi)\) correspond to the proper \( k \)-colorings of \((G, \pi^0)\) by \( \kappa \mapsto \eta \kappa \). Say that \( e \) has endpoints \( u \) and \( v \). If \( \kappa \) is any proper \( k \)-coloring of \((G \setminus e, \pi^0)\), then \( \kappa \) is a proper \( k \)-coloring of
(G, π") if and only if \(κ(u) \neq κ(v)\). Those proper colorings of \((G\setminus e, \pi")\) with \(κ(u) = κ(v)\) correspond precisely to the proper colorings of \((G/e, \pi")\).

(2) Repeatedly applying the deletion-contraction identity in (1) reduces \(p(G, π, k)\) to

\[
p(G, π, k) = \sum_i (-1)^{|V(G)| - |V(G_i)|} p(G_i, π_i, k)
\]

where each \((G_i, π_i)\) is a linkless graph (that is, any edge of \(G_i\) is a loop) with \(|V(G_i)| \leq |V(G)|\) and exactly one such term having \(|V(G_i)| = |V(G)|\), that is, \(G\) with all of its links deleted, call it \((G_0, π_0)\). Since \((G, π)\) has no balanced loops, our calculation using Part (1) yields a polynomial in \(k\) of degree \(|V(G)|\).

This polynomial is well-defined despite different ways of using part (1) to calculate it. The reason for this is as follows. The resulting polynomial of degree \(|V(G)|\) counts the number of proper \(k\)-colorings for any \(k = l + t|Γ|\) despite the way in which it is calculated. Now using polynomial interpolation over \(|V(G)| + 1\) distinct values for \(k = l + t|Γ|\) implies the uniqueness of the polynomial. □

4. LIFTING COLORINGS TO COVERING GRAPHS

Given a \(Γ\)-gain graph \((G, π)\), its derived graph \(G^n\) is an ordinary graph defined as follows: \(V(G^n) = V(G) \times Γ\) and \(E(G^n) = E(G) \times Γ\) in which \(t(\ell, γ) = (t(\ell), γ)\) and \(h(\ell, γ) = (h(\ell), γπ_e)\). The derived graph \(G^n\) is known as a regular covering graph of the ordinary graph \(G\). Now if \(κ\) is a proper \(k\)-coloring of \((G, π)\), then \(κ\) induces a \(k\)-coloring, also call it \(κ\), on \(G^n\) by \(κ(γ) = γκ(γ)\).

**Theorem 4.1.** If \(κ\) is a proper \(k\)-coloring of \((G, π)\), then the \(k\)-coloring on the derived graph \(G^n\) induced by \(κ\) is a proper \(k\)-coloring of \(G^n\).

**Proof.** Given an oriented edge \(e\) (a link or a loop) in \((G, π)\) we have \(κ(t(\ell)) ≠ π_eκ(h(\ell))\). Now for any \(γ \in Γ\) we also have \(γκ(t(\ell)) ≠ γπ_eκ(h(\ell))\). Also, \(κ(t(\ell, γ)) = κ(t(\ell), γ) = γκ(t(\ell))\) and \(κ(h(\ell, γ)) = κ(h(\ell), γπ_e) = γπ_eκ(h(\ell))\) which implies our result. □

**References**


**Department of Mathematics and Statistics, Wright State University, Dayton OH, USA**

*E-mail address*: daniel.slilaty@wright.edu