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LATIN SQUARES AND THEIR BRUHAT ORDER

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ABSTRACT. In this paper we investigate the Bruhat order on the class of Latin squares. We study its cover relation and minimal elements. We prove that the class of Latin squares of order n, with $n \notin \{1, 2, 4\}$, has at least two minimal elements, and we present a process to construct some minimal Latin squares for this relation.

1. INTRODUCTION

Let $R = (r_1, \dots, r_m)$ and $S = (s_1, \dots, s_n)$ be two nonincreasing positive integral vectors such that

$$r_1 + \ldots + r_m = s_1 + \ldots + s_n.$$

The *conjugate vector* of R is the nonincreasing positive integral vector R^* defined by

$$r_j^* = |\{i : \ m \geq i \geq 1, \ r_i \geq j\}|.$$

The vector S is *dominated* or *majorized* by the vector R^*, denoted by $S \preceq R^*$, when

$$s_1 + \ldots + s_i \leq r_1^* + \ldots + r_i^*,$$

for $1 \leq i \leq \min\{r_1, n\}$. As usual, $\mathcal{A}(R, S)$ denotes the class of all $(0,1)$-matrices whose row-sum sequence is R and column-sum sequence is S. If $R = (k, \dots, k)$, then the class $\mathcal{A}(R, R)$ is denoted by $\mathcal{A}(n, k)$. By the Gale–Ryser theorem (see [1, 2, 12, 13]) we know that the class $\mathcal{A}(R, S)$ is nonempty if and only if $S \preceq R^*$.

An important property of $\mathcal{A}(R, S)$ is due to Ryser [13], and states that if $A_1, A_2 \in \mathcal{A}(R, S)$, then A_1 can be transformed into A_2 by a finite sequence of *interchanges*

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftrightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where each interchange replaces a submatrix of A_1 equal to L_2 with I_2, or the other way around. Note that such a sequence of interchanges applied to a matrix of $\mathcal{A}(R, S)$ always results in a matrix of $\mathcal{A}(R, S)$.

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Let S_n be the symmetric group of degree n. If $\sigma \in S_n$, then we can represent σ as a word by $\sigma = \sigma_1 \ldots \sigma_n$, with $\sigma(i) = \sigma_i$, for $i = 1, \ldots, n$. A pair (i, j), $1 \le i, j \le n$, is called an *inversion* of σ if $i < j$, and $\sigma_i > \sigma_j$. The *Bruhat order*, $\preceq_B$, on S_n, states that $\sigma \preceq_B \tau$ if σ can be gotten from τ by a sequence of transformations of the form:

$$\tau_1 \ldots \tau_i \ldots \tau_j \ldots \tau_n \text{ replaced by } \tau_1 \ldots \tau_j \ldots \tau_i \ldots \tau_n,$$

where (i, j) is an inversion of τ.

The elements of S_n can be represented permutation matrices of order n. In fact, if $\sigma \in S_n$, then σ can be represented by the permutation matrix $P = [p_{i,j}]$, where $p_{i,j} = 1$ if and only if $j = \sigma(i)$. Let P and Q be two permutation matrices of order n corresponding to permutations π and τ. We write $P \preceq_B Q$ whenever $\pi \preceq_B \tau$. So, for permutation matrices, the Bruhat order is interpreted as a sequence of one sided interchanges

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

There is another way to define the Bruhat order on the class of permutation matrices. For an $m \times n$ matrix $A = [a_{i,j}]$, let Σ_A denote the $m \times n$ matrix whose (r, s)-entry is

$$\sigma_{r,s}(A) = \sum_{i=1}^{r} \sum_{j=1}^{s} a_{ij}, \quad 1 \le r \le m,\, 1 \le s \le n.$$

For permutation matrices P and Q of order n, we say $P \preceq_B Q$ if and only if $\Sigma_P \ge \Sigma_Q$ by the entrywise order.

In 2004, Brualdi and Hwang in [7] extended the Bruhat order from permutations matrices, matrices in the class $\mathcal{A}(n, 1)$, to more general nonempty classes, the classes $\mathcal{A}(R, S)$. They do it in two different ways. For $A_1, A_2 \in \mathcal{A}(R, S)$, they defined:

(1) $A_1 \preceq_B A_2$ (the Bruhat order), if by the entrywise order, $\Sigma_{A_1} \ge \Sigma_{A_2}$.
(2) $A_1 \preceq_{\widehat{B}} A_2$ (the Secondary Bruhat order), if A_1 can be obtained from A_2 by a sequence of one sided interchanges $L_2 \to I_2$.

They noted that the Bruhat order and the Secondary Bruhat order are different on $\mathcal{A}(R, S)$. Moreover, in [4], Brualdi and Deatt proved that the Bruhat order and the Secondary Bruhat order coincide in $\mathcal{A}(n, 2)$, as it happens in $\mathcal{A}(n, 1)$, but do not coincide in $\mathcal{A}(n, 3)$ (see [11]). However, it is straightforward to verify that if $A_1 \preceq_{\widehat{B}} A_2$, then $A_1 \preceq_B A_2$. So the Bruhat order is a refinement of the Secondary Bruhat order.

In the recent years, several authors extended the Bruhat order to other classes of matrices: Brualdi and Fritscher [6] studied the Bruhat order on the class of tournament matrices with a given score vector, Brualdi and Schroeder [8] on the class of alternating sign matrices, Brualdi and Dahl [3] on the class of doubly stochastic matrices, Fernandes, da Cruz, and Furtado [9] on the class of symmetric matrices, and Furtado [5] on the class of symmetric matrices with zero trace .

The aim of this paper is to investigate the Bruhat order on a new class of matrices: the class of Latin squares of order n.

A *Latin square* of order n is an $n \times n$ grid filled with n symbols so that each symbol appears once in each row and in each column. We use the integers $1, \dots, n$ for these symbols. Latin squares have a long history, and its applications can be found in many areas. For instance, the multiplication table of a finite group or the multiplication table of a quasigroup are Latin squares. Sudoku puzzles are 9×9 Latin squares with some additional constraints (see [10]).

The Bruhat order on the class of Latin squares of order n is defined similarly as in other classes of matrices. If A and C are Latin squares of order n, then we say that $A \preceq_B C$ whenever $\Sigma_A \geq \Sigma_C$ by the entrywise order.

Let J_n be the $n \times n$ matrix with all positions equal to 1. The notion of Latin square is closely related to permutations and, of course, with permutation matrices. If A is a Latin square of order n, then there are n permutation matrices $P_1, \dots, P_n$ such that $J_n = P_1 + P_2 + \cdots + P_n$, and

$$A = 1P_1 + 2P_2 + \cdots + nP_n.$$

Conversely, if $P(\rho_1), P(\rho_2), \dots, P(\rho_n)$, are n permutation matrices of order n such that $P(\rho_1) + P(\rho_2) + \cdots + P(\rho_n) = J_n$, then $1P(\rho_1) + 2P(\rho_2) + \cdots + nP(\rho_n)$ is a Latin square of order n. This implies that if $A = [a_{ij}]$ is a Latin square of order n, and $g \in \{1, \dots, n\}$, then there is a permutation of S_n, ρ_g, such that $a_{i,\rho_g(i)} = g$ for $i = 1, \dots, n$. For instance, if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \end{bmatrix},$$

then the integer 3 is associated with the permutation $\rho_3 = 3241 \in S_4$.

The paper is organized as follows: Section 2 studies the cover relation between two Latin squares for the Bruhat order. In Section 3, we present results that will be useful in the rest of the paper. In Section 4, we describe the classes of Latin squares that have a unique minimal element for the Bruhat order. A process to construct some minimal matrices for this relation is presented in Section 5.

2. The Cover Relation

Given a matrix A, we denote by $A[\{i_1, \dots, i_t\}|\{j_1, \dots, j_l\}]$ the submtrix of A that lies in the rows $i_1, \dots, i_t$, and in columns $j_1, \dots, j_l$.

Let $(X, \preceq)$ be a finite partially order set and $a, b \in X$. If $a \neq b$ and $a \preceq b$, then we write $a \prec b$. We say that b covers a if $a \prec b$, and there does not exist $c \in X$ such that $a \prec c \prec b$. Since the Latin squares of order n are related to the permutation matrices, we start by reviewing the cover relation for the

Bruhat order on the class of permutation matrices (see [4]). Let P and Q be permutation matrices of order n corresponding to permutations

$$\pi = \pi_1 \dots \pi_{i-1}\pi_i\pi_{i+1} \dots \pi_{j-1}\pi_j\pi_{j+1} \dots \pi_n$$

and

$$\tau = \tau_1 \dots \tau_{i-1}\tau_i\tau_{i+1} \dots \tau_{j-1}\tau_j\tau_{j+1} \dots \tau_n,$$

respectively, where $\tau_k = \pi_k$, whenever $k \in \{1, \dots, n\} \setminus \{i, j\}$, $\tau_i = \pi_j$, $\tau_j = \pi_i$ and (i, j) is an inversion of π. Thus, the permutation τ is obtained from π by interchanging π_i and π_j. Then P covers Q in the Bruhat order if and only if

$$P[\{i,j\},\{\pi_j,\pi_i\}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, P[\{i+1,\dots,j-1\},\{\pi_j+1,\dots,\pi_i-1\}]$$

is the null matrix, and the corresponding submatrices of Q are I_2 and the null matrix.

The cover relation for the Bruhat order on the class of Latin squares of order n is more complicated, and before we present some results on this matter, we consider an example:

Let

$$A = \begin{bmatrix} \colorbox{lightgray}{1} & \colorbox{lightgray}{2} & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \\ \colorbox{lightgray}{2} & \colorbox{lightgray}{1} & 4 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} \colorbox{lightgray}{2} & \colorbox{lightgray}{1} & 3 & 4 \\ 4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \\ \colorbox{lightgray}{1} & \colorbox{lightgray}{2} & 4 & 3 \end{bmatrix}$$

be Latin squares of order 4. So,

$$A = 1\begin{bmatrix} \colorbox{lightgray}{1} & \colorbox{lightgray}{0} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \colorbox{lightgray}{0} & \colorbox{lightgray}{1} & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} \colorbox{lightgray}{0} & \colorbox{lightgray}{1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \colorbox{lightgray}{1} & \colorbox{lightgray}{0} & 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$C = 1\begin{bmatrix} \colorbox{lightgray}{0} & \colorbox{lightgray}{1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \colorbox{lightgray}{1} & \colorbox{lightgray}{0} & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} \colorbox{lightgray}{1} & \colorbox{lightgray}{0} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \colorbox{lightgray}{0} & \colorbox{lightgray}{1} & 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We have

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ covers } Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$Q_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ covers } P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Note that P_1 and P_2 appear in the decomposition of C associated with the integers 1 and 2, respectively, and Q_1 and Q_2 appear in the decomposition of A associated with the integers 1 and 2, respectively. Although C is obtained from A by permuting the symbols 1 and 2 just in the four marked positions, if we consider the Latin square

$$D = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 4 & 3 \end{bmatrix},$$

then $C \prec_B D \prec_B A$. In fact,

$$\Sigma_C = \begin{bmatrix} 2 & 3 & 6 & 10 \\ 6 & 10 & 14 & 20 \\ 9 & 17 & 23 & 30 \\ 10 & 20 & 30 & 40 \end{bmatrix} \geq \Sigma_D = \begin{bmatrix} 2 & 3 & 6 & 10 \\ 5 & 10 & 14 & 20 \\ 9 & 17 & 23 & 30 \\ 10 & 20 & 30 & 40 \end{bmatrix}$$

and

$$\Sigma_D = \begin{bmatrix} 2 & 3 & 6 & 10 \\ 5 & 10 & 14 & 20 \\ 9 & 17 & 23 & 30 \\ 10 & 20 & 30 & 40 \end{bmatrix} \geq \Sigma_A = \begin{bmatrix} 1 & 3 & 6 & 10 \\ 5 & 10 & 14 & 20 \\ 9 & 17 & 23 & 30 \\ 10 & 20 & 30 & 40 \end{bmatrix}.$$

Proposition 1. *Let A and C be Latin squares of order n. Let p, v, l, t be integers with $1 \leq p < v \leq n$ and $1 \leq l < t \leq n$. If*

$$(C - A)[\{p, v\}, \{l, t\}] = \begin{bmatrix} a & -a \\ -a & a \end{bmatrix},$$

with $a \in \mathbb{N}$, and the other entries of $C - A$ are zero, then $C \preceq_B A$.

Proof. Using the hypothesis, we have

$$\sigma_{rs}(C - A) = \begin{cases} a, & \text{if } (r, s) \in \{p, \ldots, v - 1\} \times \{l, \ldots, t - 1\} \\ 0, & \text{otherwise.} \end{cases}$$

Since $\sigma_{rs}(C - A) = \sigma_{rs}(C) - \sigma_{rs}(A)$ then $C \preceq_B A$. $\square$

Remark 2. *If A and C are Latin squares of order n such that $\sigma_{rs}(A) = \sigma_{rs}(C)$, for all $(r, s) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$ then $A = C$.*

Theorem 3. *Let A and C be Latin squares of order n. Let p, v, l, t be integers with $1 \leq p < v \leq p+2 \leq n$ and $1 \leq l < t \leq l+2 \leq n$. If*

$$(C-A)[\{p,v\},\{l,t\}] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

and the other entries of $C-A$ are zero, then A covers C in the Bruhat order.

Proof. Using Lemma 1, $C \preceq_B A$.

If $D = [d_{ij}]$ is a Latin square of order n such that $C \preceq_B D \preceq_B A$, then $\sigma_{rs}(C-D)$ and $\sigma_{rs}(D-A)$ are nonnegative integers, for all $(r,s) \in \{1,\ldots,n\}\times\{1,\ldots,n\}$. Moreover, since $\sigma_{rs}(D-A)+\sigma_{rs}(C-D) = \sigma_{rs}(C-A)$ then

$$\sigma_{rs}(D-A)+\sigma_{rs}(C-D) = \begin{cases} 1, & \text{if } (r,s) \in \{p,\ldots,v-1\}\times\{l,\ldots,t-1\} \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $\sigma_{rs}(D-A) = \sigma_{rs}(C-D) = 0$ if $(r,s) \notin \{p,\ldots,v-1\}\times\{l,\ldots,t-1\}$.

If $v = p+1$ and $t = l+1$, then by Remark 2, $D = A$ or $C = D$. Therefore, A covers C.

If $v = p+1$ and $t = l+2$, then $\sigma_{rs}(C-A) \neq 0$ if and only if $(r,s) = (p,l)$ or $(r,s) = (p,l+1)$. We assume that $A = [a_{ij}]$ and $C = [c_{ij}]$.

Using Remark 2 we can suppose that $\sigma_{pl}(C-D) = 0$, and $\sigma_{p,l+1}(C-D) = 1$. The case $\sigma_{pl}(C-D) = 1$ and $\sigma_{p,l+1}(C-D) = 0$ is analogous. Then

$$c_{pl} = d_{pl},\ c_{p,l+1} = d_{p,l+1}+1,\ a_{pl} = d_{pl}-1,\ a_{p,l+1} = d_{p,l+1}+1.$$

Recall that $\sigma_{p,l+2}(C-D) = 0 = \sigma_{p,l+2}(D-A)$. So,

$$c_{p,l+2} = d_{p,l+2}-1 \text{ and } a_{p,l+2} = d_{p,l+2}.$$

Since A, C and D are Latin squares then

$$\{a_{pl}, a_{p,l+1}, a_{p,l+2}\} = \{c_{pl}, c_{p,l+1}, c_{p,l+2}\} = \{d_{pl}, d_{p,l+1}, d_{p,l+2}\}.$$

This implies that

$$\{a_{pl}, a_{p,l+1}, a_{p,l+2}\} = \{a_{pl}+1, a_{p,l+1}, a_{p,l+2}-1\} = \{a_{pl}+1, a_{p,l+1}-1, a_{p,l+2}\}.$$

Therefore, $a_{p,l+2} = a_{pl}+1 = a_{p,l+1}$. This is impossible. So, A covers C.

With a similar proof we get that A covers C if $v = p+2$ and $t = l+1$.

If $v = p+2$ and $t = l+2$, then $\sigma_{rs}(C-A) \neq 0$ if and only if $(r,s) \in \{p,p+1\}\times\{l,l+1\}$. Suppose that $\sigma_{pl}(C-D) = 1$. Using similar arguments as before, we have $\sigma_{p,l+1}(C-D) = 1$ and $\sigma_{p+1,l}(C-D) = 1$. Then,

$$c_{pl} = d_{pl}+1,\ c_{p,l+1} = d_{p,l+1},\ c_{p+1,l} = d_{p+1,l}.$$

By Remark 2, we can assume that $\sigma_{p+1,l+1}(C-D) = 0$. Then

$$c_{p+1,l+1} = d_{p+1,l+1}-1.$$

Since $\sigma_{p,l+2}(C-D) = \sigma_{p+1,l+2}(C-D) = \sigma_{p+2,l}(C-D) = 0$, and $\sigma_{p+2,l+1}(C-D) = \sigma_{p+2,l+2}(C-D) = 0$, then

$$c_{p,l+2} = d_{p,l+2}-1,\ c_{p+1,l+2} = d_{p+1,l+2}+1,$$

$$c_{p+2,l} = d_{p+2,l} - 1,\ c_{p+2,l+1} = d_{p+2,l+1} + 1, c_{p+2,l+2} = d_{p+2,l+2}.$$

Using similar arguments we get,

$$a_{p+2,l} = d_{p+2,l},\ a_{p+2,l+1} = d_{p+2,l+1} + 1,\ a_{p+2,l+2} = d_{p+2,l+2} - 1.$$

Since A, C, and D are Latin squares we have

$$\{a_{p+2,l}, a_{p+2,l+1}, a_{p+2,l+2}\} = \{c_{p+2,l}, c_{p+2,l+1}, c_{p+2,l+2}\},$$

and

$$\{a_{p+2,l}, a_{p+2,l+1}, a_{p+2,l+2}\} = \{d_{p+2,l}, d_{p+2,l+1}, d_{p+2,l+2}\}.$$

This implies that

$$\{a_{p+2,l}, a_{p+2,l+1}, a_{p+2,l+2}\} = \{a_{p+2,l} - 1, a_{p+2,l+1}, a_{p+2,l+2} + 1\},$$

and

$$\{a_{p+2,l}, a_{p+2,l+1}, a_{p+2,l+2}\} = \{a_{p+2,l}, a_{p+2,l+1} - 1, a_{p+2,l+2} + 1\}.$$

Therefore, $a_{p+2,l+2} = a_{p+2,l} - 1 = a_{p+2,l+1} - 1$. This is impossible. So, $\sigma_{p+1,l+1}(C - D) = 1$, and $D = A$. Consequently, A covers C. □

The converse of the last theorem does not hold.

Example 4. *Let*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

be Latin squares of order 3*. Although*

$$(C - A) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix},$$

by direct calculations we conclude that A covers C.

In the next results we use the decomposition of a Latin square in a sum of permutation matrices.

Proposition 5. *Let A and C be Latin squares of order n such that $(C-A) = (\alpha - \beta)(P_i - P_j)$ with P_i, P_j permutation matrices and α, $\beta \in \{1, \ldots, n\}$, $\alpha \neq \beta$. Then*

1. *If $P_i \prec_B P_j$ and $\alpha > \beta$, then $C \prec_B A$.*
2. *If $P_i \prec_B P_j$ and $\beta > \alpha$, then $A \prec_B C$.*
3. *If $P_i \not\prec_B P_j$, then $C \not\prec_B A$.*

Proof.

(1) Using the hypothesis, $\sigma_{rs}(P_i - P_j) \geq 0$, for all $(r, s) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$. If $\alpha > \beta$ then $\sigma_{rs}((\alpha - \beta)(P_i - P_j)) \geq 0$, for all $(r, s) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$. Thus, $C \prec_B A$.

(2) The proof is similar to (1).

(3) If $P_i \not\preceq_B P_j$, then there are (r,s), $(p,l) \in \{1,\ldots,n\}\times\{1,\ldots,n\}$ such that $\sigma_{rs}(P_i - P_j) > 0$, and $\sigma_{pl}(P_i - P_j) < 0$. So, $\sigma_{rs}((\alpha-\beta)(P_i-P_j))$ and $\sigma_{pl}((\alpha-\beta)(P_i-P_j))$ have different signal. Then $C \not\preceq_B A$.

□

Proposition 6. *Let A and C be Latin squares of order n such that $C = \sum_{i=1}^n iP_i$ and $A = \sum_{i=1}^n iT_i$ with permutation matrices P_i and T_i, $i = 1,\ldots,n$. If $P_i \preceq_B T_i$, for $i = 2,\ldots,n$, then $C \preceq_B A$.*

Proof. Since $\sum_{i=1}^n P_i = J_n = \sum_{i=1}^n T_i$ then

$$P_1 = J_n - \sum_{i=2}^n P_i \text{ and } T_1 = J_n - \sum_{i=2}^n T_i.$$

Consequently,

$$C - A = \sum_{i=2}^n (i-1)(P_i - T_i).$$

As $P_i \preceq_B T_i$, for $i = 2,\ldots,n$, then $\sigma_{rs}(C-A) \geq 0$, for all $(r,s) \in \{1,\ldots,n\}\times\{1,\ldots,n\}$. Thus, $C \preceq_B A$. □

3. Minimal Latin Squares

In this section, we present some results that we will use in the next sections. In addition we describe two processes to obtain minimal matrices for the Bruhat order, from a previous minimal matrix.

Remark 7. *Let A be a Latin square of order n. Then*

- $\sigma_{n,r}(A) = \sigma_{r,n}(A) = r\frac{(n+1)n}{2}$, *for $r = 1,\ldots,n$.*
- $\sigma_{nn}(A) = \frac{n^2(n+1)}{2}$.

As usual we denote by L_n the anti-identity matrix of order n. This matrix has in position (i,j) the element 1 if $j = n+1-i$ and 0 otherwise. Let A be an $n \times m$ matrix. The *reverse* of A, denoted by A^R, is the matrix $L_n A L_m$.

Remark 8. *If A is an $n \times m$ matrix, then*

- $(A^R)^R = A$.
- $(A^R)_{ij} = \sum_{k=1}^n \sum_{s=1}^m (L_n)_{ik} A_{ks} (L_m)_{sj} = A_{n+1-i,m+1-j}$.
- $(A^R)_{11} = A_{nm}$.

Proposition 9. *Let $A = [a_{kl}]$ be an $n \times m$ matrix, $i \in \{1,\ldots,n\}$, and $j \in \{1,\ldots,m\}$. Then*

$$\sigma_{ij}(A^R) = \sigma_{nm}(A) - \sigma_{n-i,m}(A) - \sigma_{n,m-j}(A) + \sigma_{n-i,m-j}(A).$$

Proof. By definition,

$$\begin{aligned}\sigma_{ij}(A^R) &= \sum_{l=1}^{i}\sum_{k=1}^{j} a_{n+1-l,m+1-k} = \sum_{t=n+1-i}^{n}\ \sum_{p=m+1-j}^{m} a_{tp}\\ &= \sum_{t=1}^{n}\sum_{p=1}^{m} a_{tp} - \sum_{t=1}^{n}\sum_{p=1}^{m-j} a_{tp} - \sum_{t=1}^{n-i}\sum_{p=1}^{m} a_{tp} + \sum_{t=1}^{n-i}\sum_{p=1}^{m-j} a_{tp}\\ &= \sigma_{nm}(A) - \sigma_{n-i,m}(A) - \sigma_{n,m-j}(A) + \sigma_{n-i,m-j}(A).\end{aligned}$$

□

Proposition 10. *Let A be a Latin square of order n. If A is a minimal matrix for the Bruhat order, $\preceq_B$, then A^R is a minimal matrix for $\preceq_B$.*

Proof. Suppose that A^R is not a minimal matrix for $\preceq_B$. Then there is a Latin square of order n, D, such that $D \neq A^R$ and $D \preceq_B A^R$. Therefore, there are $i, j \in \{1, \ldots, n\}$ with $\sigma_{ij}(D) > \sigma_{ij}(A^R)$. By Proposition 9,

$$\begin{aligned}&\sigma_{nn}(D^R) - \sigma_{n-i,n}(D^R) - \sigma_{n,n-j}(D^R) + \sigma_{n-i,n-j}(D^R)\\ &\qquad > \sigma_{nn}(A) - \sigma_{n-i,n}(A) - \sigma_{n,n-j}(A) + \sigma_{n-i,n-j}(A).\end{aligned}$$

By Remark 7 we have

$$\sigma_{nn}(D^R) = \sigma_{nn}(A), \sigma_{n-i,n}(D^R) = \sigma_{n-i,n}(A), \sigma_{n,n-j}(D^R) = \sigma_{n,n-j}(A).$$

So,

$$\sigma_{n-i,n-j}(D^R) > \sigma_{n-i,n-j}(A).$$

This is impossible because A is minimal. □

Let A^T be the transpose of the $n \times m$ matrix A. It is easy to check that

$$\sigma_{ij}(A^T) = \sigma_{ji}(A), \ \ i \in \{1, \ldots, m\}, \ j \in \{1, \ldots, n\}.$$

In general it is not possible to compare A and A^T by the Bruhat order unless $n = m$.

Proposition 11. *Let A be a minimal matrix for the Bruhat order, $\preceq_B$, on a subset S of the class of matrices of order n. Then A^T is a minimal matrix for $\preceq_B$ on S.*

Proof. Suppose that A^T is not a minimal matrix for $\preceq_B$ on S. Then there is $C \in S$ such that $C \neq A^T$ and $C \preceq_B A^T$. This implies that there are $i, j \in \{1, \ldots, n\}$ with $\sigma_{ij}(C) > \sigma_{ij}(A^T)$. Since $\sigma_{ij}(C) = \sigma_{ji}(C^T)$, $\sigma_{ij}(A^T) = \sigma_{ji}(A)$, then $\sigma_{ji}(C^T) > \sigma_{ji}(A)$. This is impossible because A is minimal. □

Remark 12. *From a minimal Latin square we can easily construct a maximal Latin square for the Bruhat order. In fact, if $A = [a_{i,j}]$ is a minimal Latin square of order n for the Bruhat order, then the Latin square $A' = [n - a_{i,j} + 1]$ is a maximal Latin square of this order.*

4. Classes With Minimum Element

In this section we investigate the possible integers n for which the class of Latin squares of order n has a unique minimal matrix for the Bruhat order.

If $n = 1$, then the class of Latin squares of order 1 has a unique matrix, the matrix $\begin{bmatrix} 1 \end{bmatrix}$. Consequently, there is a unique minimal matrix for the Bruhat order.

If $n = 2$, then the class of Latin squares of order 2 has two matrices, the matrices

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

By direct calculations we get its Hasse diagram, see Figure 1.

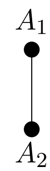

Figure 1. The Hasse diagram for Latin squares of order 2

So, A_2 is the unique minimal matrix of this class.

If $n = 3$, then the class of Latin squares of order 3 has twelve matrices, the matrices are:

$$C_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, C_3 = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix},$$

$$C_4 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}, C_5 = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}, C_6 = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix},$$

$$C_7 = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, C_8 = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, C_9 = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix},$$

$$C_{10} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}, C_{11} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}, C_{12} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}.$$

By direct calculations we get the Hasse diagram for $\preceq_B$, see Figure 2. Then, there are four minimal matrices for the Bruhat order.

If $n = 4$, then we can prove the next result.

Proposition 13. *The matrix*

$$A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

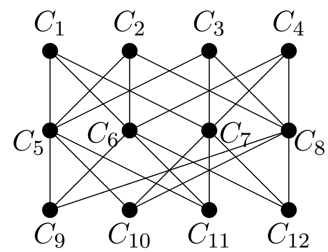

FIGURE 2. The Hasse diagram for Latin squares of order 3

is the unique minimal matrix for the Bruhat order on the class of Latin squares of order 4.

Proof. Suppose that there is a Latin square of order 4, C, such that $A \neq C$ and C is a minimal matrix for the Bruhat order. So, there are integers $i,\ j \in \{1, 2, 3, 4\}$ such that $\sigma_{ij}(C) > \sigma_{ij}(A)$. Since the vector $(4, 3, 2, 1)$ appears in the first row and in the first column of A, the vector $(3, 4, 1, 2)$ appears in the second row and in the second column of A. Using Remark 7, we conclude that the pair (i, j) is $(2, 3), (3, 2)$, or $(3, 3)$.

Suppose that $(i, j) = (2, 3)$. Because $a_{1,4} = 1$, $a_{2,4} = 2$ and C is a Latin square of order 4, then by Remark 7, $\sigma_{2,3}(C) \leq 18 = \sigma_{2,3}(A)$. Thus, $(i, j) \neq (2, 3)$.

Using similar arguments we prove that $(i, j) \neq (3, 2)$ and $(i, j) \neq (3, 3)$. □

As we will see there is no $n > 4$ such that the class of Latin squares of order n has a unique minimal element for the Bruhat order.

Proposition 14. *If n is an odd positive integer, and $n \neq 1$, then the class of Latin squares of order n has at least two minimal elements for the Bruhat order.*

Proof. Suppose that there is an odd positive integer n, $n \neq 1$, such that the class of Latin squares of order n has a unique minimal element for the Bruhat order. By Proposition 11, if A is a minimal element for $\preceq_B$, then A^T is also a minimal element for $\preceq_B$. Therefore, if the class of Latin squares has a unique minimal element A, then $A = A^T$, and then A is a symmetric matrix. So, if g is an integer, $1 \leq g \leq n$, and the entry (i, j) of A has the element g, then the entry (j, i) of A also has the element g. Consequently, if ρ_g is the permutation associated with g in A, then ρ_g is a product of disjoint transpositions. Since n is odd, then there is an integer k such that $\rho_g(k) = k$. This implies that all integers between 1 and n appear in the main diagonal of A. Moreover, each integer between 1 and n appears only once in the main diagonal of A because A is a matrix of order n.

On the other hand, by Proposition 11, L_nAL_n is minimal for the Bruhat order. So, $L_nAL_n = A$. But the entry $(1, 1)$ of L_nAL_n is the entry (n, n) of

A. Then A has, at least, two equal elements in the main diagonal. This is impossible and the result follows. $\square$

Theorem 15. *Let n be an even positive integer, and A, C, E, F be minimal Latin squares of order $n/2$ for the Bruhat order. Then the matrix*

$$D = \left[\begin{array}{c|c} A + \frac{n}{2}J_{\frac{n}{2}} & C \\ \hline E & F + \frac{n}{2}J_{\frac{n}{2}} \end{array}\right],$$

is a minimal Latin square of order n for the Bruhat order.

Proof. Let P be a Latin square of order n such that $P \preceq_B D$, and

$$P = \left[\begin{array}{c|c} P_1 & P_2 \\ \hline P_3 & P_4 \end{array}\right],$$

where P_1, P_2, P_3, P_4 are matrices of order $n/2$.

Since A is a Latin square of order $n/2$, then $A+(n/2)J_{n/2}$ has all integers k with $(n/2)+1 \leq k \leq n$. Moreover, each integer appears once in each column and in each row. So,

$$\sigma_{\frac{n}{2},\frac{n}{2}}(D) \geq \sigma_{\frac{n}{2},\frac{n}{2}}(P).$$

But $P \preceq_B D$, consequently, $\sigma_{n/2,n/2}(D) = \sigma_{n/2,n/2}(P)$. This implies that in each row and in each column of P_1, the integer l appears with $(n/2)+1 \leq l \leq n$, and $P_1 - (n/2)J_{n/2}$ is a Latin square of order $n/2$. Since $P \preceq_B D$ and A is minimal for the Bruhat order, then

$$\sigma_{ij}(A) \geq \sigma_{ij}(P_1 - \frac{n}{2}J_{\frac{n}{2}}), \quad \text{for all } i,j \in \left\{1, \ldots, \frac{n}{2}\right\}.$$

Therefore,

$$\sigma_{i,j}(D) = \sigma_{i,j}(A + \frac{n}{2}J_{\frac{n}{2}}) \geq \sigma_{i,j}(P_1) = \sigma_{i,j}(P), \quad \text{for all } i,j \in \left\{1, \ldots, \frac{n}{2}\right\}.$$

Since $P \preceq_B D$, we conclude that $A + (n/2)J_{n/2} = P_1$. This implies that P_2 is a Latin square of order $n/2$. Since C is minimal for the Bruhat order,

$$\sigma_{i,j}(C) \geq \sigma_{i,j}(P_2), \quad \text{for all } i,j \in \left\{1, \ldots, \frac{n}{2}\right\}.$$

If $l \in \{1, \ldots, n/2\}$ and $k \in \{(n/2)+1, \ldots, p\}$, then

$$\sigma_{l,k}(D) = \sigma_{l\frac{n}{2}}(A + \frac{n}{2}J_{\frac{n}{2}}) + \sigma_{l,k-\frac{n}{2}}(C) \geq \sigma_{l\frac{n}{2}}(P_1) + \sigma_{l,k-\frac{n}{2}}(P_2) = \sigma_{lk}(P).$$

Since $P \preceq_B D$, we conclude that $C = P_2$. Using a similar argument we have $E = P_3$. By the previous equalities we get $P_4 - (n/2)J_{n/2}$ is a Latin square of order $n/2$. Since F is minimal for the Bruhat order then

$$\sigma_{i,j}(F) \geq \sigma_{i,j}(P_4 - \frac{n}{2}J_{\frac{n}{2}}), \qquad \text{for all } i,j \in \left\{1, \ldots, \frac{n}{2}\right\}.$$

Consequently,

$$\begin{aligned}\sigma_{h,v}(D) &= \sigma_{\frac{n}{2},\frac{n}{2}}(A+\frac{n}{2}J_{\frac{n}{2}}) + \sigma_{\frac{n}{2},v-\frac{n}{2}}(C) + \sigma_{h-\frac{n}{2},\frac{n}{2}}(E)\\ &\quad + \sigma_{h-\frac{n}{2},v-\frac{n}{2}}(F+\frac{n}{2}J_{\frac{n}{2}})\\ &\geq \sigma_{\frac{n}{2},\frac{n}{2}}(P_1) + \sigma_{\frac{n}{2},v-\frac{n}{2}}(P_2) + \sigma_{h-\frac{n}{2},\frac{n}{2}}(P_3) + \sigma_{h-\frac{n}{2},v-\frac{n}{2}}(P_4) = \sigma_{hv}(P),\end{aligned}$$

for all $h, v \in \{n/2, \ldots, n\}$. Since $P \preceq_B D$ and by the previous equalities, we conclude that $F + (n/2)J_{n/2} = P_4$. Thus, $D = P$ and D is minimal. □

The next result is used in Section 5, where we present a process to construct minimal Latin squares of order n, when n is odd, and $n = 2^k - 1$ with $k \geq 2$.

Corollary 16. *Let n be a positive integer with $n = 2^k$, $k \geq 1$. There exists a Latin square of order n, which is minimal for the Bruhat order, whose last row is $[1, 2, \ldots, n]$, the last column is $[1, 2, \ldots, n]^T$, and it has n in all entries of the main diagonal.*

Proof. The result is true if $n = 2$ or $n = 4$. Let $k \geq 2$, and assume that the corollary holds for $n = 2^k$. Now we prove that the result holds for $n = 2^{k+1}$. Let A_0 be a minimal Latin square of order 2^k in the conditions of the corollary, and let

$$A_1 = A_0 + 2^k\, J_{2^k}.$$

Then, A_1 is a Latin square in the integers $2^k + 1, \ldots, 2^{k+1}$, with 2^{k+1} in the main diagonal. The last row of A_1 is $[2^k + 1, \ldots, 2^{k+1}]$, and the last column is $[2^k + 1, \ldots, 2^{k+1}]^T$. Let

$$A = \begin{bmatrix} A_1 & A_0 \\ A_0 & A_1 \end{bmatrix}.$$

Then, A is a Latin square in the integers $1, \ldots, 2^k, 2^k + 1, \ldots, 2^{k+1}$. The last row of A is $[1, \ldots 2^k, 2^k + 1, \ldots, 2^{k+1}]$, the last column is $[1, \ldots 2^k, 2^k + 1, \ldots, 2^{k+1}]^T$, and all the entries in the main diagonal A are equal to 2^{k+1}. By Theorem 15, A is minimal for the Bruhat order and the proof is complete. □

Proposition 17. *The class of Latin squares of order 8 has at least two minimal elements for the Bruhat order.*

Proof. By Proposition 13, the matrix

$$A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

is a minimal Latin square of order 4 for the Bruhat order. Then, using the previous theorem, we get that the matrix

$$P = \left[\begin{array}{c|c} A + 4J_4 & A \\ \hline A & A + 4J_4 \end{array}\right]$$

is minimal for the Bruhat order.

Let

$$H = \begin{bmatrix} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 8 & 7 & 3 & 5 & 4 & 1 & 2 \\ 7 & 6 & 8 & 2 & 1 & 5 & 4 & 3 \\ 1 & 4 & 2 & 8 & 7 & 6 & 3 & 5 \\ 3 & 2 & 1 & 6 & 8 & 7 & 5 & 4 \\ 2 & 1 & 5 & 4 & 3 & 8 & 6 & 7 \\ 5 & 3 & 4 & 7 & 2 & 1 & 8 & 6 \\ 4 & 5 & 3 & 1 & 6 & 2 & 7 & 8 \end{bmatrix}$$

be a Latin square of order 8. Since $\sigma_{2,3}(H) = 42 > 41 = \sigma_{2,3}(P)$, we conclude that there is a Latin square of order 8, K, which is minimal for the Bruhat order, $K \preceq_B H$, and K is not related to P by the Bruhat order. Therefore, the result follows. $\square$

Proposition 18. *Let n be an even positive integer, $n \neq 2$ and $n \neq 4$. Then the class of Latin squares of order n has at least two minimal elements for the Bruhat order.*

Proof. Let n be an even positive integer, $n \neq 2$, $n \neq 4$. Then there are positive integers k and u such that $n = 2^k \cdot u$, where u is an odd integer and $k \geq 1$ if $u \geq 3$ or $k \geq 3$ if $u = 1$. So, we have two cases:

Case 1: $u \geq 3$.

By Proposition 14, let V and X be two distinct minimal Latin squares of order u for the Bruhat order. Using Theorem 15, the matrices

$$Q_1 = \left[\begin{array}{c|c} V + uJ_u & V \\ \hline V & V + uJ_u \end{array}\right] \text{ and } R_1 = \left[\begin{array}{c|c} X + uJ_u & X \\ \hline X & X + uJ_u \end{array}\right]$$

are two distinct minimal Latin squares of order $2u$ for the Bruhat order. If $k = 1$, then the result follows. If $k > 1$, repeating this process with Q_1 and R_1 we obtain two distinct minimal Latin squares of order $2^2 \cdot u$, Q_2 and R_2 for the Bruhat order (note that the block on the top right of Q_2 is V, and the same block in R_2 is X). Again, if $k = 2$ the result follows. If $k > 2$, then we repeat this process until we have two Latin squares of order n.

Case 2: $u = 1$.

In this case, $k \geq 3$ and by Proposition 17, there are at least two minimal Latin squares of order 8, for the Bruhat order. If $n = 2^3$, then the result holds. Otherwise, we use similar arguments as in Case 1 until we have the desired matrices.

$\square$

We conclude this section with a result that summarizes our study.

Theorem 19. *Let n be a positive integer. Then the class of Latin squares of order n has a unique minimal element for the Bruhat order if and only if $n = 1,\ 2,\ 4$.*

Remark 20. *By Theorem 19, and by Remark 12 we conclude that*

$$A' = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

is the unique maximal Latin square of order 4.

5. A Construction of Minimal Latin Squares

Since the proof of Proposition 18 gives us a process that constructs minimal Latin squares of order n for the Bruhat order when n is an even positive integer, we only need to obtain a process when n is odd. In this section we describe how to construct a minimal Latin square of order n for the Bruhat order when n is an odd positive integer and $n = 2^k - 1$ with $k \geq 2$. First we write the process described in Theorem 15.

5.1. **Minimal Latin squares when n is even.** Let $n = 2^k \cdot u$, where u is an odd integer and $k \geq 1$. Let V be a minimal Latin square of order u for the Bruhat order. To construct a minimal Latin square of order n we proceed as follows:

(1) Let $p = u$ and $A = V$.
(2) Let $G = \left[\begin{array}{c|c} A + pJ_p & A \\ \hline A & A + pJ_p \end{array}\right]$.
(3) If $n = 2p$, stop. Otherwise, go to step 1 with $2p$ instead of u and G instead of V.

This process gives us a minimal Latin square of order n for the Bruhat order because each time we repeat it we obtain a minimal matrix (see Theorem 15).

Example 21. *Let* $n = 12 = 2^2 \times 3$. *The matrix*

$$C_9 = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

is a minimal Latin square of order 3 *for the Bruhat order (see Section 4). Using this process, we consider* $p = 3$ *and* $A = C_9$ *and obtain the matrix*

$$G = \left[\begin{array}{c|c} C_9 + 3J_3 & C_9 \\ \hline C_9 & C_9 + 3J_3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 6 & 4 & 5 & 3 & 1 & 2 \\ 5 & 6 & 4 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ \hline 3 & 1 & 2 & 6 & 4 & 5 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}\right].$$

Since $12 = n \neq 2p = 6$, we repeat the process with $p = 6$ and $A = G$. We obtain the matrix

$$H = \left[\begin{array}{c|c} G + 6J_6 & G \\ \hline G & G + 6J_6 \end{array}\right]$$

$$= \left[\begin{array}{cccccc|cccccc}
12 & 10 & 11 & 9 & 7 & 8 & 6 & 4 & 5 & 3 & 1 & 2 \\
11 & 12 & 10 & 8 & 9 & 7 & 5 & 6 & 4 & 2 & 3 & 1 \\
10 & 11 & 12 & 7 & 8 & 9 & 4 & 5 & 6 & 1 & 2 & 3 \\
9 & 7 & 8 & 12 & 10 & 11 & 3 & 1 & 2 & 6 & 4 & 5 \\
8 & 9 & 7 & 11 & 12 & 10 & 2 & 3 & 1 & 5 & 6 & 4 \\
7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
6 & 4 & 5 & 3 & 1 & 2 & 12 & 10 & 11 & 9 & 7 & 8 \\
5 & 6 & 4 & 2 & 3 & 1 & 11 & 12 & 10 & 8 & 9 & 7 \\
4 & 5 & 6 & 1 & 2 & 3 & 10 & 11 & 12 & 7 & 8 & 9 \\
3 & 1 & 2 & 6 & 4 & 5 & 9 & 7 & 8 & 12 & 10 & 11 \\
2 & 3 & 1 & 5 & 6 & 4 & 8 & 9 & 7 & 11 & 12 & 10 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12
\end{array}\right].$$

Since $n = 12 = 2p$, the process ends and H is a minimal matrix.

5.2. **Minimal Latin squares of order n when $n = 2^k - 1$ is odd, with $k \geq 2$.** We now give a procedure to construct minimal Latin squares of order $n = 2^k - 1$, $k \geq 2$:

Since $n = 2^k - 1 = 2^{k-1} + (2^{k-1} - 1)$, let $s = 2^{k-1}$, $v = 2^{k-1} - 1$, X be a minimal Latin square of order v for the Bruhat order, and Y be a minimal Latin square of order s for the Bruhat order, with the above conditions of Corollary 16 (the last column of Y is equal to the last row of Y and are equal to the vector $(1, 2, \ldots, s)$, and the main diagonal of Y is filled with the integer s). Let $T = X + sJ_v$. Let Z be the matrix obtained by Y removing the last row and W be the matrix obtained by Y removing the last column. Let G be the matrix obtained by Y changing

- the integer u by $u + s$, for $u = 1, \ldots, s - 2$,
- the integer $s - 1$ by n,
- the integer s in row i by i, for $i = 1, \ldots, s$.

Proposition 22. *With the above conditions, the matrix*

$$A = \left[\begin{array}{c|c} T & Z \\ \hline W & G \end{array}\right]$$

is a minimal Latin square of order n for the Bruhat order.

Proof. Let $s = 2^{k-1}$, $v = 2^{k-1} - 1$. First we prove that this process produces a Latin square. Since Z is obtained from Y by removing the last row, W is obtained from Y by removing the last column, and $T = X + sJ_v$, there are no repeated integers in the first $v = 2^{k-1} - 1$ rows or in the first $v = 2^{k-1} - 1$ columns of A. Keeping in mind that Y is a minimal Latin square of order s for the Bruhat order in the conditions of Corollary 16 (the last row and the last column of Y are $[1, \ldots 2^{k-1}]$ and $[1, \ldots 2^{k-1}]^T$, respectively, and the integer 2^{k-1} is on the main diagonal), then the matrix G has in row and column i, the integers $i, s+1, \ldots, n$, for $i = 1, \ldots, s$, and the matrix A is a Latin square on the integers $1, \ldots, n = 2^k - 1$.

Let D be a Latin square of order n and assume that $D \preceq_B A$. Write

$$D = [d_{i,j}] = \begin{bmatrix} D_0 & D_1 \\ D_2 & D_3 \end{bmatrix}$$

where D_0 is the $v \times v$ submatrix of D. Since $D \preceq_B A$,

$$\sigma_{i,j}(D) \geq \sigma_{i,j}(A),$$

for all $i, j \in \{1, \ldots, n\}$.

We have $d_{i,j} \geq 2^{k-1} + 1 = s + 1$, for all $i, j \in \{1, \ldots, 2^{k-1} - 1\}$. Otherwise we have

$$\sigma_{v,v}(D) < \sigma_{v,v}(A),$$

which is impossible because $D \preceq_B A$.

Since D is a Latin square then $D_0 - sJ_v$ is a Latin square of order v. Using the fact that

$$\sigma_{i,j}(D_0) = \sigma_{i,j}(D) \geq \sigma_{i,j}(A) = \sigma_{i,j}(T),$$

$i, j \in \{1, \ldots, v\}$, and $X \preceq_B (D_0 - sJ_v)$ we conclude that $D_0 = T$.

Therefore D_1 is an $v \times s$ matrix whose entries are in the set $\{1, \ldots, s\}$. Since D is a Latin square, there are no two equal integers in the same row or in the same column of D_1. Assume $D_1 \neq Z$, and let $i \in \{1, \ldots, v\}$ and $j \in \{s, \ldots, n\}$. Then

$$\sigma_{i,j}(D) = \sigma_{i,v}(D_0) + \sigma_{i,j'}(D_1),$$

where $j' = j - v$.

Similarly

$$\sigma_{i,j}(A) = \sigma_{i,v}(T) + \sigma_{i,j'}(Z),$$

and since $T = D_0$ and $D \preceq_B A$ we have

$$\sigma_{i,j'}(D_1) \geq \sigma_{i,j'}(Z),$$

for all $i \in \{1, \ldots, v\}$ and $j' \in \{1, \ldots, s\}$. Since $D_1 \neq Z$, let (i, j') be the smaller pair by the lexicographic order satisfying $\sigma_{i,j'}(D_1) > \sigma_{i,j'}(Z)$.

Let D_1' be the matrix obtained from D_1 by adding the $(s+1)$th row whose ith position is an integer in the set $\{1, \ldots, s\}$ and is not in the ith column of D_1. Then D_1' is a Latin square on the integers $\{1, \ldots, s\}$ and using Remark 7, we conclude that $D_1' \prec_B Y$. This is impossible because Y is a minimal Latin square. Then $D_1 = Z$. Similarly we have $W = D_2$. To complete the proof we only have to show that $G = D_3$. Since $D_1 = Z$, $D_2 = W$, noting how Z and W are constructed from Y, and since D is Latin square, we conclude that the entry (i, i) of D_3 has the integer i, for $i = 1, \ldots, s$, and the other entries of D_3 are filled with the integers $s+1, \ldots, 2s-1$. So, D_3 and G are Latin squares on the same set of integers. Moreover, $D_3 \preceq_B G$.

Because D is a Latin square there are no two integers in the same row or in the same column of D_3. Therefore there is a unique Latin square V on the integers $2^{k-1}, \ldots, n$ such that if we replace the entry (i,i) of V by i, for $i = 1, \ldots, 2^{k-1}$ we obtain D_3. Since Y is minimal then $Y \preceq_B (V - vJ_s)$. Consequently, $G = D_3$ and $D = A$. Then A is minimal. $\square$

Example 23. *Let $n = 7 = 2^3 - 1 = 2^2 + (2^2 - 1) = 3 + 2^2$. The matrix*

$$X = C_9 = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

is a minimal element for the Bruhat order (see Section 4). By Proposition 13,

$$Y = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

is the unique minimal Latin square of order 4, for the Bruhat order.

Using previous process, we consider

$$T = X + 4J_4 = \begin{bmatrix} 7 & 5 & 6 \\ 6 & 7 & 5 \\ 5 & 6 & 7 \end{bmatrix}$$

and we obtain the minimal Latin square of order 7

$$A = \left[\begin{array}{ccc|cccc} 7 & 5 & 6 & 4 & 3 & 2 & 1 \\ 6 & 7 & 5 & 3 & 4 & 1 & 2 \\ 5 & 6 & 7 & 2 & 1 & 4 & 3 \\ \hline 4 & 3 & 2 & 1 & 7 & 6 & 5 \\ 3 & 4 & 1 & 7 & 2 & 5 & 6 \\ 2 & 1 & 4 & 6 & 5 & 3 & 7 \\ 1 & 2 & 3 & 5 & 6 & 7 & 4 \end{array}\right].$$

Remark 24. *In the procedures above, the construction of minimal Latin squares of order n requires minimal Latin squares of orders less than n. Since for $n = 4$ we can only use minimal Latin squares of order 2 and since there is a unique minimal Latin square of order 2 for the Bruhat order, then we get a unique minimal Latin square of order 4.*

References

1. R. A. Brualdi, *Matrices of zeros and ones with fixed row and column-sum vectors*, Linear Algebra Appl. **33** (1980), 159–231.
2. ———, *Combinatorial Matrix Classes*, Encyclopedia of Mathematics and its Applications, vol. 108, Cambridge University Press, Cambridge, 2006.
3. R. A. Brualdi and G. Dahl, *Doubly stochastic matrices and the Bruhat order*, Czechoslovak Math. J. **66** (2016), 681–700.

4. R. A. Brualdi and L. Deaett, *More on the Bruhat order for* $(0,1)$*-matrices*, Linear Algebra Appl. **421** (2007), 219–232.
5. R. A. Brualdi, R. Fernandes, and S. Furtado, *On the Bruhat order of labeled graphs*, Discrete Appl. Math. **258** (2019), 49–64.
6. R. A. Brualdi and E. Fritscher, *Bruhat order of tournaments*, Linear Algebra Appl. **458** (2014), 261–279.
7. R. A. Brualdi and S.-G. Hwang, *A Bruhat order for the class of* $(0,1)$*-matrices with row sum vector* R *and column sum vector* S, Electron. J. Linear Algebra **12** (2004), 6–16.
8. R. A. Brualdi and M. W. Schroeder, *Alternating sign matrices and their Bruhat order*, Discrete Math. **340** (2017), 1996–2019.
9. H. F. Cruz, R. Fernandes, and S. Furtado, *Minimal matrices in the Bruhat order for symmetric* $(0,1)$*-matrices*, Linear Algebra Appl. **530** (2017), 160–184.
10. G. Dahl, *Permutation matrices related to Sudoku*, Linear Algebra Appl. **430** (2009), 2457–2463.
11. R. Fernandes, H. F. Cruz, and D. Salomão, *Classes of* $(0,1)$*-matrices where the Bruhat order and the Secondary Bruhat order coincide*, Order (2019), https://doi.org/10.1007/s11083-019-09500-8.
12. D. Gale, *A theorem on flows in networks*, Pacific J. Math. **7** (1957), 1073–1082.
13. H. J. Ryser, *Combinatorial properties of matrices of zeros and ones*, Can. J. Math. **9** (1957), 371–377.

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