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BROOKS' THEOREM FOR 2-FOLD COLORING

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Abstract. The two-fold chromatic number of a graph is the minimum number of colors needed to ensure that there is a way to color the graph so that each vertex gets two distinct colors, and adjacent vertices have no colors in common. The Ore degree is the maximum sum of degrees of an edge in a graph. We prove that, for 2-connected graphs, the two-fold chromatic number is at most the Ore degree, unless G is a complete graph or an odd cycle.

1. Introduction

A foundational result in graph coloring is given by Brooks' Theorem [2]:

Theorem 1.1. *For a connected graph G, $\chi(G) \leq \Delta(G)$, unless G is an odd cycle or a complete graph.*

There are many extensions of Brooks' Theorem, to list-coloring [5], online coloring, and online list coloring [8, 10]. There is a version of the theorem for Ore degree [6], which we mention below. In this paper, we prove a version of Brooks' theorem for 2-fold coloring. Recall that a b-fold k-coloring of G is an assignment of b *distinct* colors to every vertex, from a set of k colors, such that adjacent vertices do not have any colors in common. The b-fold chromatic number, $\chi_b(G)$, is the minimum number k such that a b-fold k-coloring exists. For more on b-fold chromatic numbers, see the book *Fractional Graph Theory* [9].

Kierstead and Kostochka [6] proved a version of Brooks' Theorem for the Ore degree. The Ore degree of an edge uv in G is $\theta(uv) = d(u) + d(v)$. The (maximum) Ore degree of a graph is defined to be

$$\theta(G) = \max\{\theta(uv) : uv \in E(G)\}.$$

Kierstead and Kostochka proved the following:

Theorem 1.2. *For every graph G, $\chi(G) \leq \lfloor \theta(G)/2 \rfloor + 1$. Moreover, if $7 \leq \chi(G) = \lfloor \theta(G)/2 \rfloor + 1$, then $\omega(G) = \chi(G)$.*

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They conjectured that the lower bound of 7 could be replaced with 6, which was shown by Rabern [7]. They give examples of graphs G where the bound is tight. These graphs appear in Figure 1, along with a 2-fold coloring for each of them, using $\theta(G)$ colors. We show that we can always use at most $\theta(G)$ colors, unless the graph is an odd cycle or a complete graph.

Theorem 1.3. *Let G be a 2-connected graph. Then $\chi_2(G) \leq \theta(G)$ unless G is an odd cycle, or a complete graph.*

We first prove an essentially equivalent result, which drops the assumption that G is 2-connected.

Theorem 1.4. *We have $\chi_2(G) \leq \max\{5, \theta(G), 2\omega(G)\}$.*

We prove our result by contradiction. A minimal counterexample turns out to either be regular, or satisfy $\delta(G) = \Delta(G) - 1$. For the regular graph, we can apply Brooks' Theorem twice to obtain the desired coloring. Thus, a minimal counterexample to the Theorem must satisfy $\delta(G) = \Delta(G) - 1$. We show that in this case, the maximum degree vertices form an independent set I. Then our main strategy is to color $G - I$ with $\theta(G) - 2$ colors and then color I with the remaining two colors. If this is not possible, then it is because G has a large clique. We show that in such a situation G has an independent set K which meets every maximum clique. We show that our strategy works when we decide to color $G - K$ with $\theta(G) - 2$ colors and then color K last. The only time this strategy fails to work is when $\theta(G) = 5$, which we prove using a separate argument.

Now we discuss the layout of the paper. In the next section, we discuss definitions and state the lemmas we use in our proof. One of our lemmas involves the existence of weak hitting sets, and the other involves 2-fold 5-colorability of subdivisions of cubic multigraphs. Then, we prove our main theorem. Then we prove our two lemmas. Finally, we end with some corollaries and open problems.

2. The Main Lemmas

Our proof relies on the existence of weak hitting sets and various lemmas which we prove in subsequent sections. In this section, we discuss the definitions and lemmas.

Recall that the strong product $G \boxtimes H$ of G and H has vertex set $V(G) \times V(H)$, and (a, b) is adjacent to (c, d) if and only if at least one of the following conditions are satisfied:

(1) $a = c$ and $bd \in E(H)$,
(2) $b = d$ and $ac \in E(G)$,
(3) $ac \in E(G)$ and $bd \in E(H)$.

A *hitting set* is an independent subset of G which intersects every maximal clique. However, we only need the weaker property of hitting every

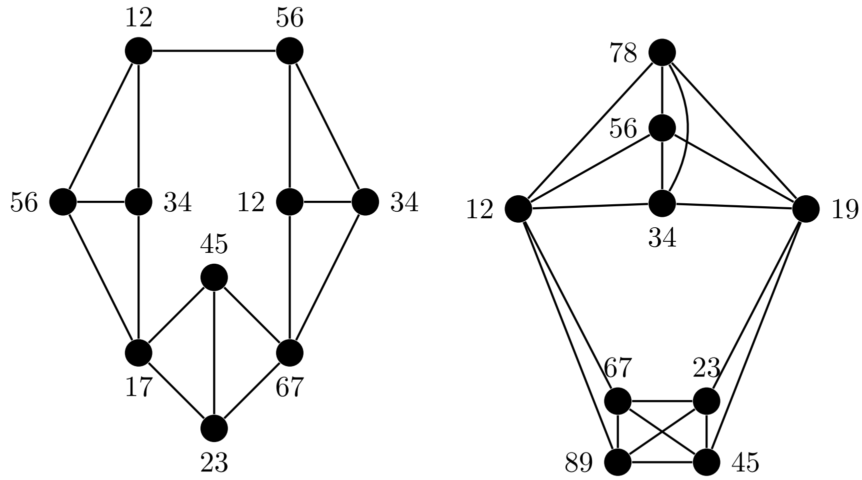


FIGURE 1. 2-fold coloring graphs with $\theta = 7$ and $\theta = 9$.

maximum clique. We define a *weak hitting set* as an independent subset of G which intersects every maximum clique. We use the following lemma about weak hitting sets proven by Christofides, Edwards, and King [3]:

Lemma 2.1. *Let G be a graph with*

$$\omega(G) \geq \frac{2}{3}(\Delta(G) + 1).$$

Then G has a weak hitting set, unless it is of the form $C_{2k+1} \boxtimes K_m$ for some $m \geq 1$, and $k \geq 2$.

We also need the following lemma about weak hitting sets which we prove in a later section.

Lemma 2.2. *Let G be a graph with $\Delta(G) \leq 4$, $\theta(G) \leq 7$, and $\omega(G) = 3$. Then G has a weak hitting set.*

Our proof involves induction, and the following lemma is used in the base case. It involves studying subdivisions of cubic multigraphs. Recall that, given a graph G, and an edge $f = xy \in E(G)$, subdividing f gives a new graph G' where we delete edge f, and add one new vertex, which is adjacent to x and y. We say a graph H is a subdivision of G if it is obtained from G by subdivision operations. The vertices from G are called *branch vertices*, and the new vertices obtained during the subdivision process are *subdividing vertices*. Similarly, *suppressing* a vertex v of degree 2 means that we contract one of the edges incident to v. Naturally, if we apply suppression operations to a graph G to obtain a graph H, then G is a subdivision of H.

Lemma 2.3. *Let H be a subdivision of a cubic multigraph G whose branch vertices form an independent set. Then H has a 2-fold 5-coloring.*

3. The Main Proof

In this section, we prove Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.4. Suppose the theorem is false, and consider a minimal counterexample G. First, suppose that $2\delta(G) \leq \theta(G) - 2$, and let v be a vertex of minimum degree. Then by minimality, we can 2-fold color $G - v$ using θ colors, and the number of distinct colors of $N(v)$ is at most $\theta(G) - 2$. Thus, we can complete the 2-fold coloring. So $2\delta(G) \geq \theta(G) - 1$.

Suppose that $2\delta(G) = \theta(G)$. Then $\delta(G) = \Delta(G)$, and G is regular. By Brooks' Theorem, $\chi(G) \leq \Delta(G)$. Thus

$$\chi_2(G) \leq 2\chi(G) \leq 2\Delta(G) = \theta(G).$$

So we assume that $2\delta(G) = \theta(G) - 1$. Let $k = \delta(G)$, so $\theta(G) = 2k + 1$.

Let A be the set of vertices of degree k, and B be the set of vertices of degree $k + 1$. Observe that $\theta(G) = 2k + 1, \Delta(G) = k + 1$, and B is an independent set.

Suppose that $k \geq 3$. Suppose that $\omega(G) < k$. Then if we extend B to a maximal independent set I, then by minimality there is 2-fold coloring of $G - I$ using at most

$$\max\{5, 2\omega(G - I), \theta(G - I)\}$$

colors. Since

$$\omega(G - I) \leq \omega(G) \leq k - 1, \text{ and } \theta(G - I) \leq \theta(G) - 2,$$

we see that we can 2-fold color $G - I$ using $2k - 1$ colors, and then extend the coloring to I using the remaining two colors.

So suppose that $\omega(G) = k$. We claim that G has a weak hitting set K. When $k = 3$, the existence of a weak hitting set K follows from Lemma 2.2. If $k \geq 4$, we have $k \geq 2(k + 2)/3$. Moreover, since

$$\delta(G) = k < k + 1 = \Delta(G),$$

G is not regular, and hence cannot be a strong product of two regular graphs. Thus, by Lemma 2.1, G has a weak hitting set K. In either case, if we extend the weak hitting set to a maximal independent set J. Then

$$\omega(G - J) \leq k - 1, \text{ and } \theta(G - J) \leq 2k - 1,$$

so by minimality, $G - J$ has a 2-fold coloring using at most $2k - 1$ colors. We can then extend the coloring to J using the remaining two colors.

Now suppose $k = 2$. The vertices of degree 3 form an independent set. Since $\delta(G) = 2$, there are vertices of degree 2. Moreover, if we suppress all vertices of degree 2, then we obtain a cubic multigraph. Hence G is a subdivision of a cubic multigraph and is 2-fold 5-colorable by Lemma 2.3.

The case $k = 1$ is trivial: the graph is then a disjoint union of paths, each of length at most 3. Such graphs are also 2-fold 4-colorable. $\square$

Proof of Theorem 1.3. Let G be a 2-connected graph, and suppose

$$\chi_2(G) > \theta(G).$$

Then

$$\chi_2(G) \leq \max\{5, 2\omega(G), \theta(G)\}.$$

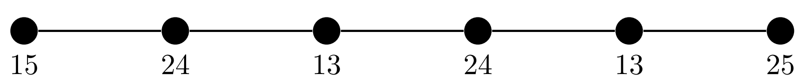
Thus either $5 > \theta(G)$ or $2\omega(G) > \theta(G)$. Since G is 2-connected, it must contain a cycle, and hence $\theta(G) \geq 4$. If $\theta(G) = 4$, then G is a cycle. Since an even cycle is 2-fold 4-colorable, we see that G must be an odd cycle. So now we are left with the case where $\theta(G) \geq 5$, and hence $2\omega(G) > \theta(G)$. Observe that $\theta(G) \geq 2\omega(G) - 2$. If equality holds, then every vertex on a maximum clique K does not have neighbors in $G \setminus K$. Hence $G = K$, and is a complete graph. So suppose $\theta(G) = 2\omega(G) - 1$. Then in a maximum clique K there is a unique vertex v with a neighbor in $G \setminus K$. Moreover v has only one such neighbor x. Then vx is a bridge of G, contradicting the fact that G is 2-connected. □

4. Coloring subdivisions of cubic multigraphs

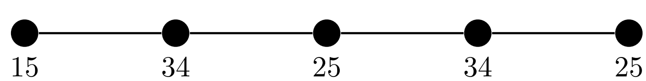
In this section, we prove Lemma 2.3.

Proof of Lemma 2.3. Let H be a subdivision of a cubic multigraph G such that the branch vertices B, form an independent set. Let $A = V(H) \setminus B$. Since G is cubic, then there exists a 4-coloring of G, provided we ignore loops. Let f denote this coloring. We define a 2-fold 5-coloring g of H. First, for $v \in B$, let $g(v) = \{f(v), 5\}$. The remaining vertices are subdividing vertices. Let $e = uv$ be an edge in G, and let P_e be the corresponding path of subdivided vertices. Note that P_e must have at one interval vertex, since B is an independent set. We define g on the internal vertices $V(P_e) \setminus \{u, v\}$. Without loss of generality, suppose $g(u) = \{1, 5\}$, and $g(v) = \{2, 5\}$.

If P_e has an even number of internal vertices (at least two), then, starting from u, color the subdividing vertices, by alternating between $\{2, 4\}$, and $\{1, 3\}$. For example, if P_e has 6 vertices, the result would be:



If P_e has an odd number of vertices, then, starting from u, color the subdividing vertices, by alternating between $\{3, 4\}$ and $\{2, 5\}$. For example, if P_e had five vertices:



We see then that if two subdivided vertices are adjacent, then they have no colors in common. Likewise, when a subdivided vertex is adjacent to a branch vertex, then they have no colors in common. Since branch vertices cannot be adjacent, we see that g is a 2-fold 5-coloring of H.

Now we address the issue of loops. A loop in G corresponds to a cycle C on at least 3 vertices in H. Because we assume $\omega(H) < 3$, the cycle must have at least four vertices. Since cycles on at least four vertices are 2-fold 5-colorable, we can 2-fold color the branch vertex of C, and extend the 2-fold coloring to the subdividing vertices on C. □

5. Existence of weak hitting sets when $\theta = 7$

We prove Lemma 2.2. We refer to a 3-clique as a triangle. A *triangle hitting set* is an independent set in G which meets every triangle of G. Recall some notation about neighborhoods which arise in the proof: for a vertex $v \in V(G)$, we let $N[v]$ denote the closed neighborhood. That is,

$$N[v] = \{v\} \cup \{u \in V : uv \in E(G)\}.$$

We let $G[N[v]]$ denote the induced subgraph on $N[v]$.

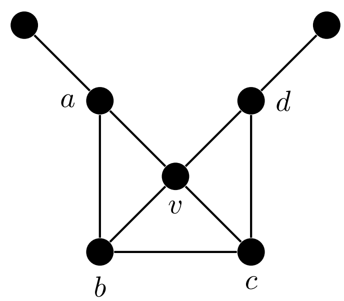


Figure 2. A graph with $t = 3$.

We prove the following Lemma, which implies Lemma 2.2.

Lemma 5.1. *Let G be a graph with $\Delta(G) \leq 4$, $\theta(G) \leq 7$, and $\omega(G) \leq 3$. Then G has a triangle hitting set.*

Proof of Lemma 2.2. Suppose that there exists such a graph G that does not have a triangle hitting set. Choose G to be a minimal counterexample. If $\omega(G) \leq 2$, then any independent set is a triangle hitting set. So we assume $\omega(G) = 3$.

Suppose that v is a vertex with $d(v) \leq 2$, then we can suppose that $G - v$ has a triangle hitting set K. Then K is also a triangle hitting set for G unless v is part of a triangle $\{v, a, b\}$, and $K \cap \{a, b\} = \emptyset$. In that case $K \cup \{v\}$ is a triangle hitting set for G.

Thus $\delta(G) = 3$. For a vertex v, let $t(v)$ be the number of triangles which contain v as a vertex. Let

$$t = \max\{t(v) : v \in V, d(v) = 4\}.$$

Since $\omega(G) = 3$, we have $t \leq 4$. Note that, since $\omega = 3$, no degree 3 vertex can lie on three triangles.

Suppose that $t = 4$, and let v be a vertex of degree 4 lying on four triangles. Then $G[N[v]] = W_4$, the wheel graph. By minimality, $G - G[N[v]]$ has a triangle hitting set K. Then $K \cup \{v\}$ is a triangle hitting set for G.

Suppose that $t = 3$, and let v be a vertex of degree 4 lying on three triangles. We routinely identify triangles by their vertex sets. Let $N(v) = \{a, b, c, d\}$, and suppose that the triangles are $\{a, b, v\}$, $\{b, c, v\}$, and $\{c, d, v\}$. That is, we have a subgraph like the one appearing in Figure 2. By minimality, $G - v$ has a triangle hitting set. Let K be a minimal triangle hitting set for $G - v$. We claim $K \cup \{v\}$ is a triangle hitting set for G. If not, then then v is adjacent to $u \in K$. Moreover, since K is a minimal triangle hitting set, there exists a triangle T such that $u \in T$ but $v \notin T$. However, u is on a triangle $\{u, v, w\}$. If $T \cap \{u, v, w\} = \{u\}$, the $d(u) = 4$, which is a contradiction. So $T \cap \{u, v, w\} = \{u, w\}$. However, $u \in \{b, c\}$, or $w \in \{b, c\}$. In either case, b or c lies on three triangles, and hence has degree 4, a contradiction.

Suppose that $t = 2$, and let v be a degree 4 vertex lying on two triangles T and T'. Suppose first that T and T' contain a common vertex $u \neq v$. Then $G - u$ has a triangle hitting set. Let K be a minimal triangle hitting set in $G - u$. We claim that $K \cup \{u\}$ is a triangle hitting set for G. Let $T \cup T' = \{u, v, w, x\}$. We are in the case of Figure 3. Since $t = 2$, v cannot lie on another triangle, and thus, by minimality of K, $v \notin K$. So suppose that, without loss of generality, $w \in K$, and, by minimality of K, lies on a triangle T''. If $T'' \cap \{u, v, w, x\} = \{w\}$, then w has degree 4. If $v \in T'' \cap \{u, v, w, x\}$, then $t = 3$. If $u \in T'' \cap \{u, v, w, x\}$, then u has degree 4. Finally, if $x \in T'' \cap \{u, v, w, x\}$, then G has a 4-clique. Hence $T'' \cap \{u, v, w, x\} = \{w\}$. This implies that w has degree 4, and that $\theta(G) \geq 8$, a contradiction. Thus $w \notin K$. By a similar argument, $x \notin K$. Hence we can add u to K and obtain a triangle hitting set for G.

Suppose that T and T' have no common vertices except v. Let

$$T = \{a, b, v\}, T' = \{c, d, v\}.$$

Then $N(c) = \{v, d, x\}$ for some x, and $N(d) = \{v, d, y\}$ for some y. Thus we have a subgraph like the one appearing on the left in Figure 4. Let

$$G' = G - \{v, c, d\} + ax + by.$$

If $\omega(G') \leq 3$, then G' has a triangle hitting set. Let K be a minimal triangle hitting set of G'. If $K \cap \{a, b\} = \emptyset$, then let $K' = K \cup \{v\}$. If $a \in K$, let $K' = K \cup \{c\}$. If $b \in K$, let $K' = K \cup \{d\}$. In all three cases K' is also a triangle hitting set.

If $\omega(G') = 4$, then we see that $x = y$, and there exists a vertex $z \neq v$ such that z is adjacent to a, b, and x. Suppose that $t(x) = 2$. If the two triangles containing x share an edge, then we are in the previous case. Hence they have no edge in common. This implies that $d(x) = 4$, and that x and z are vertices on a triangle. However, then $d(z) = 4$, and $\theta(x) = 8$, a contradiction. Thus $t(x) = 1$. Similarly, $t(q) = 1$. Now let

$$G'' = G - \{a, b, c, d, v, x, z\}.$$

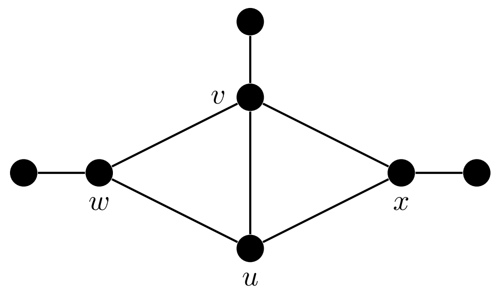


FIGURE 3. A graph with $t = 2$, with two triangles sharing an edge.

Then G'' has a triangle hitting set K, and $K \cup \{a, c\}$ is a triangle hitting set for G.

Suppose that $t = 1$. Suppose that there exists a vertex v of degree 4, and a triangle $T = \{a, b, v\}$ such that ab is the edge of another triangle $T' = \{a, b, x\}$. This is a subgraph like the one appearing in Figure 5. By minimality, $G - a$ has a triangle hitting set, and we let K be a minimal such triangle hitting set. We claim $K \cup \{a\}$ is a triangle hitting set for G. If not, then b, x, or $v \in K$. If $v \in K$, then by minimality of K v lies on a second triangle, and $t \geq 2$. If $b \in K$, then by minimality of K, b lies on another triangle distinct from T and T'. However, then b is incident to three triangles, and so has degree 4. Thus, $x \in K$. However, then x is on a triangle T''. If $T'' \cap T$ contains a or b, then the degree of a or b is 4. So $T'' \cap T = \{x\}$. Thus, x has degree 4, and $t \geq 2$. Therefore, $K \cap \{b, x, v\} = \emptyset$.

Now let v be a vertex of degree 4 lying on a triangle $T = \{c, d, v\}$. Let a and b be the remaining neighbors of v. Note that $N(c) = \{v, d, x\}$ for some x and $N(d) = \{v, c, y\}$ for some y. Thus we have a subgraph like the one on the left in Figure 6. Then we let

$$G' = G - \{v, c, d\} + ax + by.$$

Let K be a minimal triangle hitting set for G'. If $K \cap \{a, b\} = \emptyset$, then $K \cup \{v\}$ is a triangle hitting set for G. If $a \in K$, then $K \cup \{c\}$ is a triangle hitting set for G. Finally, if $a \notin K$, and $b \in K$, then $K \cup \{d\}$ is a triangle hitting set for G.

So $t = 0$. However, let G' be the subgraph induced by all vertices of degree 3. By Lemma 2.1, G' has a weak hitting set K. Then K is a triangle hitting set for G as well, as every triangle in G is contained in G'. $\square$

6. Open Questions and Further Results

First, one corollary of our theorem is a strengthening of the fact that $\alpha(G) \geq n/\Delta$. Recall that a b-fold clique is a subset $S \subset V$ such that, for every independent set I, $|I \cap S| \leq b$. The b-fold clique number, $\omega_b(G)$, is the maximum size of a b-fold clique.

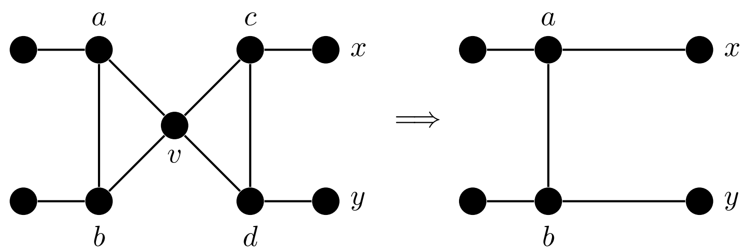


FIGURE 4. A graph G with two triangles meeting at a vertex, and the modified graph G'.

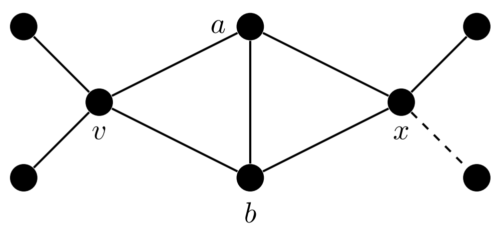


FIGURE 5. A graph where $t = 1$ that has a diamond.

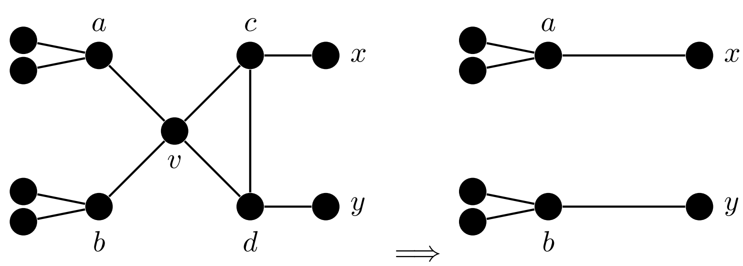


FIGURE 6. A graph with $t = 1$ and no diamonds, along with the modified graph G'.

Corollary 6.1. *Let G be a graph on n vertices, with $\omega_2(G) \leq \theta(G)$. Then $\alpha(G) \geq 2n/\theta$.*

There is also an analogue of the Borodin-Kostochka Conjecture [1]. Recall the Borodin-Kostochka Conjecture:

Conjecture. *Fix an integer $\Delta \geq 9$. If G is a graph with $\Delta(G) \leq \Delta$, and $\omega(G) \leq \Delta - 1$, then $\chi(G) \leq \Delta - 1$.*

A natural analogue for 2-fold coloring is the following:

Conjecture. *Fix an integer $\theta \geq 9$. If G is a graph with $\theta(G) \leq \theta$, and $\omega_2(G) \leq \theta - 1$, then $\chi_2(G) \leq \theta - 1$.*

This conjecture is true for odd $\theta \geq 9$. For $\theta \geq 11$, it follows from Theorem 1.2 and the generalization by Rabern [7]. That is, if $\chi_2(G) = \theta$, then

$\chi(G) \geq \theta/2$, so $\chi(G) \geq 6$, and thus must be a complete graph, whence $\omega_2(G) = \theta$. Note that the graph on the right in Figure 1 is an example where $\omega_2(G) = 9 = \theta(G)$. This conjecture could be used to show that, if $\omega(G) < \Delta(G)$ and $\Delta(G) \geq 6$, then the fractional chromatic number, $\chi_f(G)$, is at most $\Delta(G) - 1/2$, which is conjectured by Edwards and King [4].

Naturally, there is the question of determining simple upper bounds for b-fold chromatic numbers, for $b > 2$. One possibility is to let

$$\theta_b(v) = \max\left\{\sum_{u\in S} d(u) : S \subset N(v), |S| = b-1\right\},$$

and let

$$\theta_b = \max\{d(v) + \theta_b(v) : v \in V\}.$$

Conjecture. *Let G be a graph, and let $3 \leq b \leq \Delta(G)$. Then*

$$\chi_b(G) \leq \max\{2b+1, \omega_b(G), \theta_b(G)\}.$$

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