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PROOF OF A CONJECTURE OF Z. W. SUN

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Abstract. Recently, Sun defined a new sequence

$$a(n) = \sum_{k=0}^{n} \binom{n}{2k} \binom{2k}{k} \frac{1}{2k-1},$$

which can be viewed as an analogue of the Motzkin numbers. Sun conjectured that the sequence $\{a(n+1)/a(n)\}_{n\geq 5}$ is strictly increasing with limit 3, and the sequence $\left\{ \sqrt[n+1]{a(n+1)}/\sqrt[n]{a(n)} \right\}_{n\geq 9}$ is strictly decreasing with limit 1. In this paper, we prove Sun's conjecture.

1. Introduction

Recently, Sun defined a new sequence

$$a(n) = \sum_{k=0}^{n} \binom{n}{2k} \binom{2k}{k} \frac{1}{2k-1}, \qquad n \geq 0,$$

which can be viewed as an analogue of Motzkin numbers; see [3, A295112]. Below are the values of $a(0)$, $a(1)$, $a(2)$, …, $a(10)$, respectively:

$$-1, -1, 1, 5, 13, 29, 63, 139, 317, 749, 1827.$$

Since $\binom{2k}{k} = 2(2k-1)C(k-1)$ for $k \geq 1$, where $C(k)$ are the well-known Catalan numbers, we see that $a(n)$ is always an odd integer. Applying the Zeilberger algorithm [2, pp. 101-119], we get the following recurrence relations for $a(n)$:

$$(n+1)a(n+1) = (3n+1)a(n) + (n-7)a(n-1) - 3(n-1)a(n-2). \tag{1.1}$$

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Recall that a sequence $\{a_i\}_{0\le i\le m}$ of positive numbers is said to be log-concave if

$$\frac{a_0}{a_1} \le \frac{a_1}{a_2} \le \cdots \le \frac{a_{m-1}}{a_m}.$$

and a sequence $\{b_i\}_{0\le i\le m}$ of positive numbers is said to be log-convex if

$$\frac{b_0}{b_1} \ge \frac{b_1}{b_2} \ge \cdots \ge \frac{b_{m-1}}{b_m}.$$

Motivated by some conjectures of Sun [4], Sun presented the following conjecture (see [3, A295112]):

Conjecture 1.1. *The sequence* $\{a(n+1)/a(n)\}_{n\ge 5}$ *is strictly increasing with limit* 3*, and the sequence* $\left\{ \sqrt[n+1]{a(n+1)}/\sqrt[n]{a(n)} \right\}_{n\ge 9}$ *is strictly decreasing with limit* 1*.*

The aim of this paper is to prove Conjecture 1.1.

2. A lower bound and an upper bound for $a(n)/a(n-1)$

In order to prove Conjecture 1.1, we first give a lower bound and an upper bound for $a(n)/a(n-1)$. These bounds are discovered by using the method given by Chen and Xia in [1].

Lemma 2.1. *For* $n \ge 13$*,*

$$f(n) < \frac{a(n)}{a(n-1)} < f(n+1), \tag{2.1}$$

where

$$f(n) = 3 - \frac{9}{2(n-1)} - \frac{9}{8(n-1)^2} - \frac{1089}{32(n-1)^3}. \tag{2.2}$$

Proof. We prove this lemma by induction. It is easy to check that Lemma 2.1 is true when $n = 13, 14, 15$. Suppose that Lemma 2.1 holds for all $13 \le n \le m$, that is,

$$\begin{aligned} f(13) < \frac{a(13)}{a(12)} < f(14) < \cdots < \frac{a(m-1)}{a(m-2)} \\ < f(m) < \frac{a(m)}{a(m-1)} < f(m+1). \end{aligned} \tag{2.3}$$

In order to prove Lemma 2.1, it suffices to prove (2.1) holds when $n = m+1$.

From (2.3),

$$\begin{aligned} \frac{1}{f(13)} > \frac{a(12)}{a(13)} > \frac{1}{f(14)} > \cdots > \frac{1}{f(m-1)} > \frac{a(m-2)}{a(m-1)} \\ > \frac{1}{f(m)} > \frac{a(m-1)}{a(m)} > \frac{1}{f(m+1)}. \end{aligned} \tag{2.4}$$

In view of (1.1) and (2.4),

$$\begin{aligned}\frac{a(m+1)}{a(m)}-f(m+1)&=\frac{3m+1}{m+1}+\frac{m-7}{m+1}\frac{a(m-1)}{a(m)}\\&\quad-\frac{3(m-1)}{m+1}\frac{a(m-2)}{a(m-1)}\frac{a(m-1)}{a(m)}-f(m+1)\\&>\frac{3m+1}{m+1}+\frac{m-7}{m+1}\frac{1}{f(m+1)}\\&\quad-\frac{3(m-1)}{m+1}\frac{1}{f(m-1)}\frac{1}{f(m)}-f(m+1)\\&=\frac{p_1(m)}{32m^3(m+1)p_2(m)}>0,\end{aligned}$$

where

$$\begin{aligned}p_1(m)=&\,15859712m^{10}-312705024m^9+1891893248m^8-5844299776m^7\\&+13594221568m^6-24253871232m^5+32556598368m^4\\&-36216659680m^3-8146605888m^2\\&+9139795137m-134086948479\end{aligned}$$

and

$$\begin{aligned}p_2(m)=&(32m^3-48m^2-12m-363)(32m^3-240m^2+564m\\&-787)(32m^3-144m^2+180m-431).\end{aligned}$$

Therefore, for $m\geq 13$,

$$\frac{a(m+1)}{a(m)}>f(m+1). \tag{2.5}$$

Moreover,

$$\begin{aligned}\frac{a(m+1)}{a(m)}-f(m+2)=&\frac{3m+1}{m+1}+\frac{m-7}{m+1}\frac{a(m-1)}{a(m)}\\&-\frac{3(m-1)}{m+1}\frac{a(m-2)}{a(m-1)}\frac{a(m-1)}{a(m)}-f(m+2)\\<&\frac{3m+1}{m+1}+\frac{m-7}{m+1}\frac{1}{f(m)}\\&-\frac{3(m-1)}{m+1}\frac{1}{f(m)}\frac{1}{f(m+1)}-f(m+2)\\=&\frac{p_3(m)}{32(m+1)^3p_4(m)}<0,\end{aligned}$$

where

$$p_3(m) = -98304m^7 + 970752m^6 - 7003136m^5 + 13598720m^4 - 31614848m^3 + 87566144m^2 - 40875092m + 187658537$$

and

$$p_4(m) = (32m^3 - 144m^2 + 180m - 431)(32m^3 - 48m^2 - 12m - 363).$$

Thus, for $m \geq 13$,

$$\frac{a(m+1)}{a(m)} < f(m+2). \tag{2.6}$$

It follows from (2.5) and (2.6) that (2.1) holds when $n = m+1$ and Lemma 2.1 is proved by induction. □

3. Proof of Conjecture 1.1

It is easy to check that

$$\frac{a(n)}{a(n-1)} < \frac{a(n+1)}{a(n)} \qquad 4 \leq n \leq 13 \tag{3.1}$$

and

$$\frac{\sqrt[n+1]{a(n+1)}}{\sqrt[n]{a(n)}} > \frac{\sqrt[n+2]{a(n+2)}}{\sqrt[n+1]{a(n+1)}} \qquad 9 \leq n \leq 13. \tag{3.2}$$

It follows from (2.1), (2.2), and (3.1) that the sequence $\{a(n+1)/a(n)\}_{n\geq 5}$ is strictly increasing with limit 3.

From (2.1), we find that for $n \geq 13$,

$$\left(\frac{a(n+1)}{a(n)}\right)^{(n+1)(n+2)} > [f(n+1)]^{(n+1)(n+2)}, \tag{3.3}$$

$$\frac{a(n+1)}{a(n)} < f(n+2) < \frac{a(n+2)}{a(n+1)} < f(n+3) \tag{3.4}$$

and

$$\frac{a(14)}{a(13)} < f(15) < \frac{a(15)}{a(14)} < f(16) < \cdots < \frac{a(n)}{a(n-1)} < f(n+1) < \frac{a(n+1)}{a(n)}. \tag{3.5}$$

In view of (3.3)–(3.5),

$$\begin{aligned}(3.6)\qquad &\left(\frac{a(n+2)}{a(n+1)}\right)^{n(n+1)} a(n+1)^2 < [f(n+3)]^{n(n+1)}a(n+1)^2\\ &= [f(n+3)]^{n(n+1)}a(13)^2\left(\frac{a(14)}{a(13)}\frac{a(15)}{a(14)}\cdots\frac{a(n)}{a(n-1)}\right)^2\frac{a(n+1)^2}{a(n)^2}\\ &< [f(n+3)]^{n(n+1)}a(13)^2[f(n+1)]^{2n-26}\frac{a(n+1)^2}{a(n)^2}\\ &< [f(n+3)]^{n^2+n+2}a(13)^2[f(n+1)]^{2n-26}.\end{aligned}$$

It is easy to check that for $n \geq 13$,

$$\frac{f(n+1)}{f(n+3)} > 1 - \frac{4}{n^2+n+2}. \tag{3.7}$$

Combining (3.3), (3.6), and (3.7), we see that for $n \geq 13$,

$$\begin{aligned}(3.8)\qquad &\left(\frac{a(n+1)}{a(n)}\right)^{(n+1)(n+2)} - \left(\frac{a(n+2)}{a(n+1)}\right)^{n(n+1)} a(n+1)^2\\ &> [f(n+1)]^{(n+1)(n+2)} - [f(n+3)]^{n^2+n+2}a(13)^2[f(n+1)]^{2n-26}\\ (3.9)\qquad &= [f(n+1)]^{2n-26}[f(n+3)]^{n^2+n+2}\\ &\quad\times\left(\left(\frac{f(n+1)}{f(n+3)}\right)^{n^2+n+2}[f(n+1)]^{26} - a(13)^2\right)\\ &> [f(n+1)]^{2n-26}[f(n+3)]^{n^2+n+2}\\ &\quad\times\left(\left(1-\frac{4}{n^2+n+2}\right)^{n^2+n+2}[f(n+1)]^{26} - a(13)^2\right).\end{aligned}$$

It is easy to check that for $d > c > 0$,

$$\begin{aligned}(3.10)\qquad d^{n+1} - c^{n+1} &= (d-c)(d^n + d^{n-1}c + \cdots + dc^{n-1} + c^n)\\ &> (n+1)(d-c)c^n.\end{aligned}$$

For any fixed positive number $n > 4$, setting $d = 1 - \frac{4}{n+1}$ and $c = 1 - \frac{4}{n}$ in (3.10), we obtain

(3.11)

$$\begin{aligned}\left(1-\frac{4}{n+1}\right)^{n+1} - \left(1-\frac{4}{n}\right)^{n+1} &> (n+1)\left(\frac{4}{n}-\frac{4}{n+1}\right)\left(1-\frac{4}{n}\right)^n\\ &= \frac{4}{n}\left(1-\frac{4}{n}\right)^n,\end{aligned}$$

which yields

$$\left(1-\frac{4}{n+1}\right)^{n+1} > \left(1-\frac{4}{n}\right)^{n}. \tag{3.12}$$

So, the sequence $\{(1-\frac{4}{n})^n\}_{n\geq 4}$ is increasing and the subsequence

$$\left\{\left(1-\frac{4}{n^2+n+2}\right)^{n^2+n+2}\right\}_{n\geq 13}$$

is increasing. This implies that for $n \geq 13$,

$$\left(1-\frac{4}{n^2+n+2}\right)^{n^2+n+2} \geq \left(1-\frac{4}{13^2+13+2}\right)^{184} = \left(\frac{45}{46}\right)^{184}. \tag{3.13}$$

Based on (2.1), (3.8), and (3.13),

(3.14)

$$\begin{aligned}
&\left(\frac{a(n+1)}{a(n)}\right)^{(n+1)(n+2)} - \left(\frac{a(n+2)}{a(n+1)}\right)^{n(n+1)} a(n+1)^2 \\
&> [f(n+1)]^{2n-26}[f(n+3)]^{n^2+n+2}\left(\left(\frac{45}{46}\right)^{184} f(n+1)^{26} - a(13)^2\right) \\
&> [f(n+1)]^{2n-26}[f(n+3)]^{n^2+n+2}\left(\left(\frac{45}{46}\right)^{184} f(13)^{26} - a(13)^2\right).
\end{aligned}$$

It is easy to verify that

$$\left(\frac{45}{46}\right)^{184} f(13)^{26} - a(13)^2 > 0. \tag{3.15}$$

In view of (3.14) and (3.15),

$$\left(\frac{a(n+1)}{a(n)}\right)^{(n+1)(n+2)} > \left(\frac{a(n+2)}{a(n+1)}\right)^{n(n+1)} a(n+1)^2. \tag{3.16}$$

We can rewrite (3.16) as follows

$$\frac{\sqrt[n+1]{a(n+1)}}{\sqrt[n]{a(n)}} > \frac{\sqrt[n+2]{a(n+2)}}{\sqrt[n+1]{a(n+1)}} \qquad n \geq 13. \tag{3.17}$$

From (3.2) and (3.17), the sequence $\left\{\sqrt[n+1]{a(n+1)}/\sqrt[n]{a(n)}\right\}_{n\geq 9}$ is strictly decreasing.

By (2.1),

$$\begin{aligned}
a(n+1) = a(12)\frac{a(13)}{a(12)}\frac{a(14)}{a(13)}\cdots\frac{a(n+1)}{a(n)} &< a(12)f(14)f(15)\cdots f(n+2) \\
&< a(12)[f(n+2)]^{n-11}
\end{aligned}$$

and

$$0 < \ln a(n+1)^{\frac{1}{n(n+1)}} = \frac{\ln a(n+1)}{n(n+1)} < \frac{(n-11)\ln a(12) f(n+2)}{n(n+1)},$$

which yields

$$\lim_{n\to+\infty} \ln a(n+1)^{\frac{1}{n(n+1)}} = 0$$

and

$$\lim_{n\to+\infty} a(n+1)^{-\frac{1}{n(n+1)}} = 1. \tag{3.18}$$

In view of (2.1),

$$f(n)^{\frac{1}{n}} < \left(\frac{a(n+1)}{a(n)}\right)^{\frac{1}{n}} < f(n+1)^{\frac{1}{n}}. \tag{3.19}$$

Moreover,

$$\lim_{n\to+\infty} f(n)^{\frac{1}{n}} = \lim_{n\to+\infty} f(n+1)^{\frac{1}{n}} = 1. \tag{3.20}$$

Thanks to (3.19) and (3.20),

$$\lim_{n\to+\infty} \left(\frac{a(n+1)}{a(n)}\right)^{\frac{1}{n}} = 1. \tag{3.21}$$

It follows from (3.18) and (3.21) that

$$\lim_{n\to+\infty} \frac{a(n+1)^{\frac{1}{n+1}}}{a(n)^{\frac{1}{n}}} = \lim_{n\to+\infty} \left(\frac{a(n+1)}{a(n)}\right)^{\frac{1}{n}} a(n+1)^{-\frac{1}{n(n+1)}}$$

$$= \lim_{n\to+\infty} \left(\frac{a(n+1)}{a(n)}\right)^{\frac{1}{n}} \lim_{n\to+\infty} a(n+1)^{-\frac{1}{n(n+1)}} = 1.$$

This completes the proof.

References

1. W. Y. C. Chen and E. X. W. Xia, *The 2-log-convexity of the Apéry numbers*, Proc. Amer. Math. Soc. **139** (2011), 391-400.
2. M. Petkovšek, H. S. Wilf and D. Zeilberger, $A = B$, A. K. Peters, Wellesley, 1996.
3. N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, `http://oeis.org` (A295112).
4. Z. W. Sun, *Conjectures involving arithmetical sequences*, Numbers Theory: Arithmetic in Shangri-La (eds., S. Kanemitsu, H. Li and J. Liu), Proc. 6th China–Japan Seminar (Shanghai, August 15–17, 2011), World Sci., Singapore, 2013, pp. 244–258.

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