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MINIMAL GRAPHS FOR COMPLETELY INDEPENDENT SPANNING TREES AND COMPLETELY INDEPENDENT SPANNING TREES IN COMPLETE T-PARTITE GRAPH

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ABSTRACT. Let $T_1, T_2, \dots, T_k$ be spanning trees of a graph G. For any two vertices u, v of G, if the paths from u to v in these k trees are pairwise openly disjoint, then we say that $T_1, T_2, \dots, T_k$ are completely independent spanning trees. In this paper, we give the definition of *minimal graph* for k completely independent spanning trees and we characterized all *minimal graphs* for k completely independent spanning trees. Finally, we obtain the number of completely independent spanning trees in complete $t(t \geq 2)$-partite graph $K_{n_1,n_2,\dots,n_t}$, which is generalizes the known result.

1. INTRODUCTION

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). The *vertex set* and the *edge set* of G are denoted by $V(G)$ and $E(G)$, respectively. Denote $e(G) = |E(G)|$. For a vertex $v \in V(G)$, the *neighbor set* $N_G(v)$ is the set of vertices adjacent to v, $d_G(v) = |N_G(v)|$ is the *degree* of v. For a subgraph H of G, $N_H(v)$ is the set of neighbors of v which are in H, and $d_H(v) = |N_H(v)|$ is the degree of v in H. The set of neighbors (resp. close neighbors) of an edge e in G is denoted by $N(e)$(resp. $N[e]$). When no confusion occurs, we write $N(v)$ instead of $N_G(v)$. $\delta(G)$ =min$\{d(v) : v \in V(G)\}$ is the *minimum degree* of G. For a subset $U \subseteq V(G)$, the subgraph induced by U is denoted by

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$G[U]$, which is the graph on U whose edges are precisely the edges of G with both ends in U. Let $K_{n_1,n_2,\ldots,n_t}$ be a complete $t(t \geq 2)$-partite graph with $|V_1| = n_1, |V_2| = n_2, \ldots, |V_t| = n_t$. Denote $[n] = \{1, 2, \ldots, n\}$.

A tree T of G is a *spanning tree* of G if $V(T) = V(G)$. A leaf is a vertex of degree 1. An internal vertex is a vertex of degree at least 2. Let x, y be two vertices of G. An (x, y)*-path* is a path with the two ends x and y. Two (x, y)-paths P_1, P_2 are *openly disjoint* if they have no common edge and no common vertex except for the two ends x and y. Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph G. For any two vertices u, v of G, if the paths from u to v in these k trees are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are *completely independent spanning trees* (CISTs) in G. The concept of CISTs was proposed by Hasunuma [5].

In [5], Hasunuma gave a characterization for CISTs and proved that the underlying graph of a k-connected line digraph always contains k CISTs. It is well known [12][16] that every $2k$-edge-connected graph has k edge disjoint spanning trees. Motivated by this, Hasunuma [6] conjectured that every $2k$-connected graph has k CISTs. However, Péterfalvi [15] disproved the conjecture by constructing a k-connected graph, for each $k \geq 2$, which does not have two CISTs. Recently, sufficient conditions have been determined in order to guarantee the existence of two CISTs. These conditions are inspired by the sufficient conditions for Hamiltonicity: Fleischner's condition [1], Dirac's condition [1], Ore's condition [4] and Neighborhood Union and Intersection Conditions [10]. Moreover, Dirac's condition has been generalized to $k(\geq 2)$ CISTs [3][7][17] and has been independently improved for two CISTs [7][17]. In [9], Hong proved that the k-th power of a k-connected graph G on n vertices with $n \geq 2k$ has k CISTs. Constructing CISTs has many applications on interconnection networks such as fault-tolerant broadcasting and secure message distribution [2][14][11][8].

In this paper, we give the definition of *minimal graph* for k CISTs and we characterized all *minimal graphs* for k CISTs. Finally, we obtain the number of CISTs in complete $t(t \geq 2)$-partite graphs $K_{n_1,n_2,\ldots,n_t}$, which is generalizes the known result [13].

2. Preliminaries

Definition 2.1 ([5])**.** *Let $T_1, T_2, \ldots, T_k$ be spanning trees in a graph G. For any two vertices u, v of G, if the paths from u to v in $T_1, T_2, \ldots, T_k$ are pairwise openly disjoint, then we say that $T_1, T_2, \ldots, T_k$ are completely independent spanning trees(CISTs) in G.*

The following result obtained by Hasunuma [5] plays a key role in our proof.

Lemma 2.1 ([5])**.** *Let $k \geq 2$ be an integer. $T_1, T_2, \ldots, T_k$ are CISTs in a graph G if and only if they are edge disjoint spanning trees of G and for any $v \in V(G)$, there is at most one T_i such that $d_{T_i}(v) > 1$.*

Hasunuma [6] showed that whether there exist two CISTs in an arbitrary graph G is NP-complete, and proved the following result.

Lemma 2.2 ([6])**.** *There are two CISTs in any 4-connected maximal plane graph.*

Kung-Jui Pai [13] showed that the following results.

Lemma 2.3 ([13])**.** *There are $\lfloor \frac{n}{2} \rfloor$ CISTs in complete graph K_n for all $n \geq 4$.*

Lemma 2.4 ([13])**.** *There are $\lfloor \frac{n}{2} \rfloor$ CISTs in complete bipartite graph $K_{m,n}$ for all $m \geq n \geq 4$.*

Lemma 2.5 ([13])**.** *There are $\lfloor \frac{n_2+n_1}{2} \rfloor$ CISTs in complete tripartite graph K_{n_3,n_2,n_1} for all $n_3 \geq n_2 \geq n_1$ and $n_2 + n_1 \geq 4$.*

In [1], Araki provided a new characterization of the existence of k CISTs. Let $(V_1, V_2, \ldots, V_k)$ be a partition of the vertex set $V(G)$ and, for $i \neq j$, $B(V_i, V_j, G)$ be a bipartite graph with the edge set $\{uv | uv \in E(G), u \in V_i \text{ and } v \in V_j\}$. If the graph G is clear from the context, we may use $B(V_1, V_2)$ instead of $B(V_1, V_2, G)$. A partition $(V_1, V_2, \ldots, V_k)$ is called a CIST-partition of G if it satisfies the following two conditions:

(1) for $i \in [k]$, the induced subgraph $G[V_i]$ is connected and
(2) for any $i \neq j$, the bipartite graph $B(V_i, V_j)$ has no tree components, that is, every connected component H of $B(V_i, V_j)$ satisfies $|E(H)| \geq |V(H)|$.

The following result obtained by Araki [1] plays a key role in our proof.

Lemma 2.6 ([1])**.** *A connected graph G has k CISTs if and only if there is a CIST-partition $(V_1, \ldots, V_k)$ of $V(G)$.*

Now, based on the concept of CISTs, we give the definition of *minimal graphs* for k CISTs.

Definition 2.2. *Let G be a graph for which there exist k CISTs. Then G is called a minimal graphs for k CISTs if there exists a set of k CISTs $T_1, T_2, \ldots, T_k$ in G such that $E(G) = E(T_1) \cup E(T_2) \cup \cdots \cup E(T_k)$.*

If $k = 1$, then a minimal graph is a tree. So we consider a minimal graph for k CISTs for $k \geq 2$. According to Definition 2.2, we obtain the following two propositions.

Proposition 2.1. *Let G be a connected graph with $n(n \geq 1)$ vertices. We suppose that G has k edge disjoint spanning trees $T_1, T_2, \ldots, T_k$ and $E(G) = E(T_1) \cup E(T_2) \cup \cdots \cup E(T_k)$. If it satisfies one of the following two conditions, then G is a minimal graph for k CISTs.*

(a) *$n = 2k$ and there exists only one spanning tree $T_i (1 \leq i \leq k)$ such that $d_{T_i}(x) = k$ for each vertex $x \in V(G)$.*
(b) *The subgraph induced by all internal vertex of $T_i (1 \leq i \leq k)$ is a path $P = x_1 x_2 \cdots x_r$ and $d_{T_i}(x_1) = d_{T_i}(x_r) = k$, $d_{T_i}(x_i) = k + 1$ for $i \neq 1, r$, where r is the number of all internal vertices of T_i.*

Proof. (a) Let $T_1, T_2, \ldots, T_k$ be k edge disjoint spanning trees of G. Since $E(G) = E(T_1) \cup E(T_2) \cup \cdots \cup E(T_k)$, we have $e(G) = k(n-1)$. If there exists only one spanning tree T_i such that $d_{T_i}(x) = k$ for any vertex $x \in V(G)$, then $\sum_{j \neq i} d_{T_j}(x) \leq n - 1 - k = k - 1$. Because G has $k-1$ edge disjoint spanning trees except for T_i and $T_j (j \neq i)$ is a spanning tree. Thus, $d_{T_j}(x) = 1 (j \neq i)$. By Lemma 2.1, G has k CISTs. Hence, G is a *minimal graphs* for k CISTs.

Figure 1 illustrates K_6 is a *minimal graphs* for 3 CISTs if $n = 6$ and there exists only one spanning tree $T_i (1 \leq i \leq 3)$ such that $d_{T_i}(x) = 3$ for each vertex $x \in V(G)$.

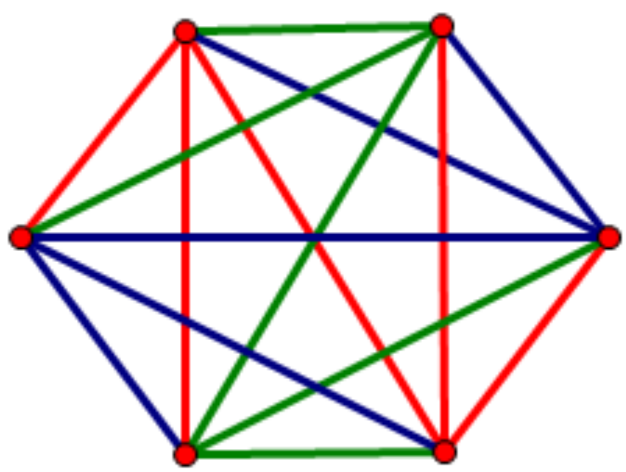

FIGURE 1. Minimal graph for 3 CISTs in K_6 (red line: edge of T_1, blue line: edge of T_2, green line: edge of T_3).

(b) Let $T_1, T_2, \ldots, T_k$ be k edge disjoint spanning trees of G. Since $E(G) = E(T_1) \cup E(T_2) \cup \cdots \cup E(T_k)$, we have $e(G) = k(n-1)$. It is sufficient to prove that k CISTs.

Suppose that $U_j = \{u_1^j, u_2^j, \ldots, u_{r_j}^j\}$ are internal vertices of T_j and $d_{T_j}(u_i^j) \in \{k, k+1\}$ for $u_i^j \in U_j, 1 \le i \le r_j, 1 \le j \le k$. Assume that T_i has a l_i leaves for $1 \le i \le k$, then

$$r_i = n - l_i. \tag{1}$$

According to the assumptions of the proposition, it follows that

$$2e(T_i) = 2k + (r_i - 2)(k+1) + l_i, \qquad 1 \le i \le k. \tag{2}$$

Summing over i in (2), we get

$$2k^2 + (k+1)(\sum_{i=1}^{k} r_i - 2k) + \sum_{i=1}^{k} l_i = 2k(n-1). \tag{3}$$

Substituting (1) into (3), we obtain

$$2k^2 + (k+1)(\sum_{i=1}^{k} r_i - 2k) + (nk - \sum_{i=1}^{k} r_i) = 2k(n-1). \tag{4}$$

Simplifying (4), and implies that

$$\sum_{i=1}^{k} r_i = n.$$

So, $U_i \cap U_j = \emptyset$ for $1 \le i \ne j \le k$ and $U_1 \cup U_2 \cup \cdots \cup U_k = V(G)$. It follows that if $d_{T_i}(u) \in \{k, k+1\}$ for any vertex $u \in U_i$, then $d_{T_j}(u) = 1 (j \ne i)$. By Lemma 2.1, G has k CISTs. Hence, G is a *minimal graphs* for k CISTs.

Figure 2 and Figure 3 illustrates the graphs is a *minimal graphs* for k CISTs if the subgraph induced by all internal vertex of $T_i (1 \le i \le k)$ is a path $P = x_1 x_2 \cdots x_r$ and $d_{T_i}(x_1) = d_{T_i}(x_r) = k$, $d_{T_i}(x_i) = k+1$ for $i \ne 1, r$, where $r = 3$ (or $r = 4$) and $k = 2$ (or $k = 3$).

□

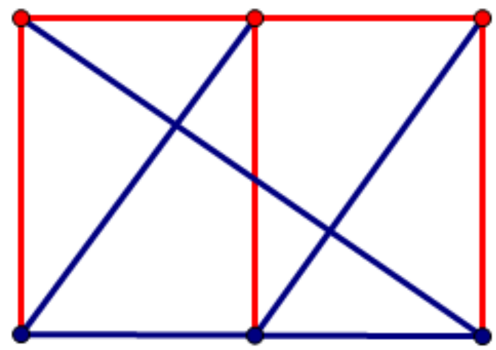

FIGURE 2. Minimal graph for 2 CISTs (red line: edge of T_1, blue line: edge of T_2).

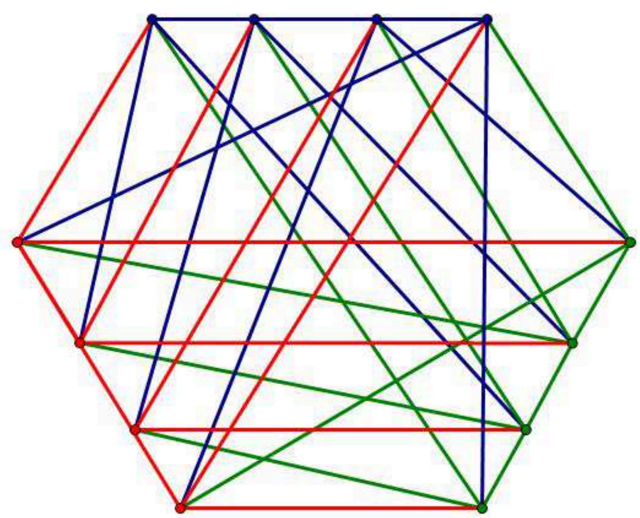

FIGURE 3. Minimal graph for 3 CISTs (red line: edge of T_1, blue line: edge of T_2, green line: edge of T_3).

Proposition 2.2. *Let G be a connected graph with $n(n \geq 1)$ vertices and G has k edge disjoint spanning trees. Suppose that $V(G)$ can be partitioned into A and B, where $A = \{u|d(u) = 2k\}$,$B = \{u|d(u) = 2k-1\}$. Then G is minimal graph for k CISTs if and only if $|A| = n - 2k, |B| = 2k$.*

Proof. Let G be a *minimal graph* for k CISTs, then $e(G) = k(n-1)$. We suppose that $|A| = s, |B| = t$, then

$$2k \cdot s + (2k-1) \cdot t = 2k(n-1),$$
$$s + t = n.$$

Consequently,

$$s = n - 2k, \qquad t = 2k.$$

Suppose that $|A| = n - 2k, |B| = 2k$, then we have

$$2k(n-2k) + (2k-1)2k = 2e(G).$$

It follows that

$$e(G) = k(n-1). \tag{5}$$

We only need to show that G has k CISTs.

For $1 \leq j \leq k$, we suppose that $U_j = \{u_1^j, u_2^j, \ldots, u_{r_j}^j\}$ are internal vertices of T_j and $A_j = U_j \cap A, B_j = U_j \cap B, |A_j| = s_j, |B_j| = t_j$, then

$$s_j + t_j = r_j.$$

and

$$\sum_{j=1}^{k} s_j \leq n - 2k, \qquad \sum_{j=1}^{k} t_j \leq 2k. \tag{6}$$

As $d_{T_l}(u) \geq 1 (l \neq j)$ for $u \in U_j$ and there are $k-1$ edge disjoint spanning trees except for T_j. So, we have

$$\left| N(u) \cap \bigcup_{l \neq j} U_l \right| \geq k - 1. \tag{7}$$

Therefore,

$$d_{T_j}(u) \leq k+1, \qquad u \in A_j,$$
$$d_{T_j}(u) \leq k, \qquad u \in B_j.$$

Claim. $d_{T_j}(u_i^j) = k+1$ for $u_i^j \in A_j$, $d_{T_j}(u_i^j) = k$ for $u_i^j \in B_j$. Suppose, to the contrary, that there exists a vertex $u \in B_j$ such that $d_{T_j}(u) < k$ or there exists a vertex $u \in A_j$ such that $d_{T_j}(u) < k+1$. Without loss of generality, we may assume there exists a vertex $u_i^1 \in B_1$ such that $d_{T_1}(u_i^1) < k$, then

$$
\begin{aligned}
e(T_1) &\le (k-1) + (t_1 - 1)k + s_1(k+1) + n - r_1 \\
&= t_1 k + s_1 k + n - t_1 - 1, \\
e(T_2) &\le t_2 k + s_2(k+1) + n - r_2, \\
&\cdots \\
e(T_k) &\le t_k k + s_k(k+1) + n - r_k. \qquad (8)
\end{aligned}
$$

Summing over k in (8) and by (6), we obtain

$$\sum_{j=1}^{k} e(T_j) \le 2k(n-1) - 1.$$

This is a contradiction to (5). The claim is proved.

By Claim, the equation in (7) is true and we have

$$d_{T_l}(u) = 1 (l \neq j), \qquad u \in U_j. \qquad (9)$$

Thus, $U_i \cap U_j = \emptyset$ for $1 \le i \neq j \le k$, $U_1 \cup U_2 \cup \cdots \cup U_k = V(G)$ and $d_{T_j}(u) \in \{k, k+1\}$ for any vertex $u \in U_j$. By Lemma 2.1, G has k CISTs. Hence, G is a *minimal graphs* for k CISTs.

□

Based on propositions 2.1 and 2.2, we characterized all *minimal graphs* for k CISTs.

Theorem 2.3. *Let G be a connected graph with n vertices. Then G is a minimal graph for $k(k \ge 2)$ CISTs if and only if G is complete graph with $2k$ vertices or a graph $\widehat{G}$ with k-part vertex set $(V_1, V_2, \ldots, V_k)$ and $|V_i| \ge 2$ such that satisfies the following conditions:*

1. *for $i \in [k]$, the induced subgraph $G[V_i]$ is a path;*
2. *for any $i \neq j$, every connected component H of $B(V_i, V_j)$ is unicyclic graph.*

Remark: The Proposition 2.1 is a special case of Theorem 2.3.

Finally, we obtain the number of CISTs in complete $t(t \ge 2)$-partite graphs $K_{n_t, n_{t-1}, \ldots, n_1}$. In fact, we prove the Theorem 2.4 by a different method with Kung-Jui Pai's [13] and the Theorem 2.5 generalizes the main results of Kung-Jui Pai's [13].

Theorem 2.4. *Let G be complete bipartite graph K_{n_2, n_1} for all $n_2 \ge n_1 \ge 4$. Then G has $\lfloor \frac{n_1}{2} \rfloor$ CISTs.*

Theorem 2.5. *Let G be complete $t(t \ge 3)$-partite graph $K_{n_t, n_{t-1}, \ldots, n_1}$ with $n_t \ge n_{t-1} \ge \cdots \ge n_1$. Then G has $\lfloor \frac{n_{t-1} + n_{t-2}}{2} \rfloor$ CISTs.*

3. Proof of Theorem 2.3

Proof. If G is a complete graph K_{2k} and let G has k-partite vertex set $(V_1, V_2, \ldots, V_k)$ and $|V_i| = 2(i \in [k])$, then $G[V_i] \cong P_2$ and $B(V_i, V_j) \cong C_4$. Thus, the vertex set $(V_1, V_2, \ldots, V_k)$ of G is a CIST-partition. By Lemma 2.6, G has k CISTs and we have

$$e(G) = k + \frac{k(k+1)}{2}4 = k + 2k(k-1) = k(n-1).$$

So, G is a *minimal graphs* for k CISTs.

If $G \cong \widehat{G}$ and $\widehat{G}$ has k-partite vertex set $(V_1, V_2, \ldots, V_k)$ such that $G[V_i](i \in [k])$ is a path and every connected component of $B(V_i, V_j)(j \neq i)$ is unicyclic graph, then

$$\begin{aligned} e(G[V_i]) &= |V_i| - 1, \\ e(B(V_i, V_j)) &= |V_i| + |V_j|. \end{aligned}$$

We first compute $e(G)$, then

$$\begin{aligned} e(G) &= \sum_{i=1}^{k} e(G[V_i]) + \sum_{j \neq i} e(B(V_i, V_j)) \\ &= |V_1| - 1 + |V_1| + |V_2| + |V_2| - 1 + |V_2| + |V_3| + |V_1| + |V_3| + |V_3| \\ &\quad - 1 + |V_3| + |V_4| + \cdots + |V_k| - 1 + |V_k| + |V_1| \\ &= n - k + |V_1| + |V_2| \\ &\quad + |V_1| + |V_3| + |V_2| + |V_3| \\ &\quad + |V_1| + |V_4| + |V_2| + |V_4| + |V_3| + |V_4| \\ &\quad \ldots \\ &\quad + |V_1| + |V_k| + |V_2| + |V_k| + |V_3| + |V_k| + \cdots + |V_{k-1}| + |V_k| \\ &= n - k + |V_1|(k-1) + |V_2|(k-1) + \cdots + |V_k|(k-1) \\ &= n - k + n(k-1) \\ &= k(n-1). \end{aligned}$$

Thus, G is a *minimal graphs* for k CISTs.

On the other hand, let G be *minimal graphs* for k CISTs. By the definition of *minimal graphs* for k CISTs, G has k CISTs. By Lemma 2.6, G has k-partite vertex set $(V_1, V_2, \ldots, V_k)$ such that every induced subgraph $G[V_i](i \in [k])$ is a connected and every connected component H of

$B(V_i, V_j)(j \neq i)$ has no tree component. Therefore,

$$\begin{aligned} e(G) &= \sum_{i=1}^{k} e(G[V_i]) + \sum_{j\neq i} e(B(V_i, V_j)) \\ &\geq \sum_{i=1}^{k} (|V_i| - 1) + \sum_{j\neq i} (|V_i| + |V_j|) \\ &= k(n-1). \end{aligned}$$

Since $e(G) = k(n-1)$, then this equation is true only if $e(G[V_i]) = |V_i| - 1$ and $e(B(V_i, V_j)) = |V_i| + |V_j|$.

Hence, in this case, we have $G[V_i]$ is a path and $B[V_i, V_j]$ is an unicyclic graph. If $|V_i| = 2$, then $G \cong K_{2k}$ and $G[V_i]$ is a path P_2 and $B(V_i, V_j)$ is a cycle C_4. If $|V_i| \geq 3$, then $G \cong \widehat{G}$ such that $G[V_i]$ is a path $P_m (m \geq 3)$ and $B[V_i, V_j]$ is a unicyclic graph.

□

4. Proof of Theorem 2.4

Proof. Let G be a complete bipartite graph K_{n_2,n_1} with $|V_i| = n_i, i \in [2]$. Let

$$V_1 = \{u_i | i \in [n_1]\}, \qquad V_2 = \{v_i | i \in [n_2]\}.$$

where $n_2 \geq n_1 \geq 4$. We divide the $V(G)$ into $W_1, W_2, \ldots, W_{\lfloor \frac{n_1}{2} \rfloor}$ as follows. If $n_1 \equiv 0 \pmod 2$, then let

$$\begin{aligned} W_i &= \{u_{2i-1}, v_{2i-1}, u_{2i}, v_{2i}\}, \qquad 1 \leq i \leq \frac{n_1}{2} - 1, \\ W_{\frac{n_1}{2}} &= \{u_{n_1-1}, u_{n_1}, v_{n_1-1}, v_{n_1}, \ldots, v_{n_2}\}. \end{aligned}$$

If $n_1 \equiv 1 \pmod 2$, then let

$$\begin{aligned} W_i &= \{u_{2i-1}, v_{2i-1}, u_{2i}, v_{2i}\}, \qquad 1 \leq i \leq \lfloor \frac{n_1}{2} \rfloor - 1, \\ W_i &= \{u_{2i-1}, v_{2i-1}, u_{2i}, v_{2i}, u_{n_1}, v_{n_1}, \ldots, v_{n_2}\}, \qquad i = \lfloor \frac{n_1}{2} \rfloor. \end{aligned}$$

If $1 \leq i \leq \lfloor \frac{n_1}{2} \rfloor - 1$, then every vertex u_{2i-1}, u_{2i} in W_i is adjacent to v_{2i-1}, v_{2i} in W_i. Thus, $G[W_i]$ is a cycle. If $i = \lfloor \frac{n_1}{2} \rfloor$, then every vertex u_{2i-1}, u_{2i} (or $u_{2i-1}, u_{2i}, u_{n_1}$) in W_i is adjacent to each vertex $\{v_{2i-1}, v_{2i}, v_{n_1+1}, \ldots, v_{n_2}\}$ (or $\{v_{2i-1}, v_{2i}, v_{n_1}, v_{n_1+1}, \ldots, v_{n_2}\}$) in W_i, respectively. So, $G[W_i]$ is connected graph.

For $1 \leq i \neq j \leq \lfloor \frac{n_1}{2} \rfloor - 1$, since every vertex u_{2i-1}, u_{2i} in W_i is adjacent to each vertex v_{2j-1}, v_{2j} in W_j and therefore $d_{B(W_i,W_j)}(x) \geq 2$ for any vertex $x \in W_i$. For $1 \leq i \leq \lfloor \frac{n_1}{2} \rfloor - 1, j = \lfloor \frac{n_1}{2} \rfloor$, every vertex u_{2i-1}, u_{2i} in

W_i is adjacent to each vertex in $V_2 \cap W_j$ and every vertex v_{2i-1}, v_{2i} in W_i is adjacent to each vertex in $V_1 \cap W_j$. Hence, $d_{B(W_i,W_j)}(x) \geq 2$ for any vertex $x \in W_i$. So, $B(W_i, W_j)$ has no tree component. By Lemma 2.6, G has $\lfloor \frac{n_1}{2} \rfloor$ CISTs. □

5. Proof of Theorem 2.5

Proof. Let G be a complete $t(t \geq 3)$-partite graph $K_{n_t,n_{t-1},\ldots,n_1}$ with $|V_i| = n_i, i \in [t]$, where $n_t \geq n_{t-1} \geq \cdots \geq n_1$. Let

$$V_j = \{u_i^j | i = 1, 2, \ldots, n_j\}, \qquad j \in [t].$$

We divide the $V(G)$ into $W_1, W_2, \ldots, W_{\lfloor \frac{n_{t-1}+n_{t-2}}{2} \rfloor}$ as follows.
If $n_{t-1} + n_{t-2} \equiv 0 \pmod 2$, let

$$\begin{aligned}
W_i &= \{u_i^j | 1 \leq j \leq t\}, \qquad 1 \leq i \leq n_1.\\
W_i &= \{u_i^j | 2 \leq j \leq t\}, \qquad n_1 + 1 \leq i \leq n_2.\\
W_i &= \{u_i^j | 3 \leq j \leq t\}, \qquad n_2 + 1 \leq i \leq n_3.\\
&\cdots\\
W_{n_{t-2}} &= \{u_{n_{t-2}}^j | t-2 \leq j \leq t\}.\\
W_{n_{t-2}+l} &= \{u_{n_{t-2}+(2l-1)}^{t-1}, u_{n_{t-2}+2l}^{t-1}, u_{n_{t-2}+(2l-1)}^{t}, u_{n_{t-2}+2l}^{t}\},\\
&\qquad\qquad 1 \leq l \leq \frac{n_{t-1}-n_{t-2}}{2} - 1.\\
W_{n_{t-2}+l} &= \{u_{n_{t-2}+(2l-1)}^{t-1}, u_{n_{t-2}+2l}^{t-1}, u_{n_{t-2}+(2l-1)}^{t},\\
&\qquad u_{n_{t-2}+2l}^{t}, u_{n_{t-2}+2l+1}^{t}, \ldots, u_{n_t}^{t}\}, \qquad l = \frac{n_{t-1}-n_{t-2}}{2}.
\end{aligned}$$

If $n_{t-1} + n_{t-2} \equiv 1 \pmod 2$, let

$$\begin{aligned}
W_i &= \{u_i^j | 1 \leq j \leq t\}, \qquad 1 \leq i \leq n_1.\\
W_i &= \{u_i^j | 2 \leq j \leq t\}, \qquad n_1 + 1 \leq i \leq n_2.\\
W_i &= \{u_i^j | 3 \leq j \leq t\}, \qquad n_2 + 1 \leq i \leq n_3.\\
&\cdots
\end{aligned}$$

$W_{n_{t-2}} = \{u^j_{n_{t-2}} | t-2 \leq j \leq t\}$.

$W_{n_{t-2}+l} = \{u^{t-1}_{n_{t-2}+(2l-1)}, u^{t-1}_{n_{t-2}+2l}, u^t_{n_{t-2}+(2l-1)}, u^t_{n_{t-2}+2l}\},$

$$1 \leq l \leq \lfloor \frac{n_{t-1} - n_{t-2}}{2} \rfloor - 1.$$

$W_{n_{t-2}+l} = \{u^{t-1}_{n_{t-2}+(2l-1)}, u^{t-1}_{n_{t-2}+2l}, u^{t-1}_{n_{t-1}}, u^t_{n_{t-2}+(2l-1)},$

$$u^t_{n_{t-2}+2l}, u^t_{n_{t-1}}, \ldots, u^t_{n_t}\}, l = \lfloor \frac{n_{t-1} - n_{t-2}}{2} \rfloor.$$

For $1 \leq i \leq n_{t-2}$, every vertex u^j_i in W_i is adjacent to $u^l_i \in W_i (l \neq j)$. Thus, $G[W_i]$ is a complete graph. For $n_{t-2}+1 \leq i \leq \lfloor \frac{n_{t-1}+n_{t-2}}{2} \rfloor - 1$, every vertex $u^{t-1}_{n_{t-2}+(2l-1)}, u^{t-1}_{n_{t-2}+2l}$ in W_i is adjacent to each vertex $u^t_{n_{t-2}+(2l-1)}$, $u^t_{n_{t-2}+2l}$ in W_i. So, $G[W_i]$ is a cycle. For $i = \lfloor \frac{n_{t-1}+n_{t-2}}{2} \rfloor$, every vertex in $W_i \cap V_{t-1}$ is adjacent to each vertex of $W_i \cap V_t$. So, $G[W_i]$ is connected graph.

If $1 \leq i \neq j \leq n_{t-2}$, then $|W_i| \geq 3$ and $d_{B(W_i,W_j)}(x) \geq 2$ for $x \in W_i$. Thus, $B(W_i, W_j)$ has no tree component. If $n_{t-2}+1 \leq i \neq j \leq \lfloor \frac{n_{t-1}+n_{t-2}}{2} \rfloor$, then $d_{B[W_i,W_j]}(x) \geq 2$ for $x \in W_i$. In addition, we get $d_{B[W_i,W_j]}(x) \geq 2$ for $1 \leq i \leq n_{t-2}$ and $n_{t-2} + 1 \leq j \leq \lfloor \frac{n_{t-1}+n_{t-2}}{2} \rfloor$. So, $B(W_i, W_j)$ has no tree component. By Lemma 2.6, G has $\lfloor \frac{n_{t-1}+n_{t-2}}{2} \rfloor$ CISTs. □

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