DECOMPOSITION OF COMPLETE TRIPARTITE GRAPHS INTO CYCLES AND PATHS OF LENGTH THREE

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Abstract. Let $C_k$ and $P_k$ denote a cycle and a path on $k$ vertices, respectively. In this paper, we obtain necessary and sufficient conditions for the decomposition of $K_{r,s,t}$ into $p$ copies of $C_3$ and $q$ copies of $P_4$ for all possible values of $p, q \geq 0$.

1. Introduction

We consider only finite undirected simple graphs. Let $K_{n_1,n_2,...,n_r}$ denote a complete $r$-partite graph with part sizes $n_1, n_2, \ldots, n_r$, where each $n_i > 0$ is an integer. A partition of a graph $G$ into edge disjoint subgraphs $G_1, G_2, G_3, \ldots, G_n$ such that their union gives $G$ is called a decomposition of $G$. Let $C_k$ and $P_k$ respectively denote a cycle and a path on $k$ vertices. They are also called a $k$-cycle and $k$-path, respectively. The problem of finding necessary and sufficient conditions to decompose complete $n$-partite graphs into $k$-cycles has been considered for many values of $n$ and $k$. The case $n=2$ was completely solved by Sotteau [13]. Smith [12] proved that the necessary conditions for the decomposition of complete equipartite graphs into cycles of length $2p$ (where $p \geq 3$ is a prime) are also sufficient. In the case of complete tripartite graphs, Cavenagh [5] has shown that $K_{m,m,m}$ can be decomposed into $k$-cycles if and only if $k \leq 3m$ and $k$ divides $3m^2$. Billington [2] gave necessary and sufficient conditions for the existence of a decomposition of any complete tripartite graph into specified number of $3$-cycles and $4$-cycles. Mahmoodian and Mirzakhani [10] proved the existence of a $C_5$-decomposition of $K_{r,s,t}$ whenever the necessary conditions are satisfied and two of the partite sets have equal size, except when $r = s = 0 \pmod{5}$ and $t \neq 0 \pmod{5}$. The authors of [1, 3, 6, 7] also studied this problem. Billington et al. [4] gave necessary and sufficient conditions for the path and cycle decomposition of complete equipartite graphs with 3 and 5 parts. Priyadharsini and Muthusamy [11] gave necessary and sufficient conditions for the existence of $(G_n, H_n)$-decomposition of $\lambda K_n$ and $\lambda K_{n,n}$, where $G_n$, $H_n$,
Given $H_n \in \{C_n, P_n, S_{n-1}\}$. Jeevadoss and Muthusamy [8] gave necessary and sufficient conditions for the existence of $\{P_{k+1}, C_k\}_{p,q}$-decomposition of $K_{m,n}$ and $K_n$, when $m \geq k/2$, $n \geq [(k+1)/2]$ for $k \equiv 0 \pmod{4}$ and when $m$, $n \geq 2k$ for $k \equiv 2 \pmod{4}$.

In this paper we give necessary and sufficient conditions for decomposing $K_{r,s,t}$ with $r \leq s \leq t$ into $p$ copies of $C_3$ and $q$ copies of $P_4$ for all possible values of $p, q \geq 0$. Definitions and notation not defined here can be referred to in [9].

**Lemma 1.1** ([7]). Let $r, s, t$ be integers such that $r \leq s \leq t$. A Latin rectangle of order $r \times s$ based on $t$ elements is equivalent to the existence of $rs$ edge-disjoint triangles sitting inside the complete tripartite graph $K_{r,s,t}$.

The triangle $(i, j, k)$ in the 3-partite graph $K_{r,s,t}$ is the subgraph of $K_{r,s,t}$ induced by the $i$th vertex of part 1, $j$th vertex of part 2, and $k$th vertex of part 3.

**Definition 1.2** ([7]). Consider a rectangular array of order $r \times s$ with entries from the set $T = \{1, 2, \ldots, t\}$. If each element of $T$ appears at most once in each row and at most once in each column, we call such an array a Latin rectangle of order $r \times s$ on $t$ elements.

**Definition 1.3** ([7]). Let $r, s, t$ be integers such that $r \leq s \leq t$. A Latin representation of the complete tripartite graph $K_{r,s,t}$ is a Latin rectangle of order $r \times s$ on $t$ elements, together with a set of $t - s$ elements at the end of each row and a set of $t - r$ elements at the bottom of every column so that each element from the set $T = \{1, 2, 3, \ldots, t\}$ occurs once in each of the $r$ rows and once in each of the $s$ columns.

**Remark:** To construct a Latin representation of the complete tripartite graph $K_{r,s,t}$, we first take a Latin rectangle of order $r \times s$ on $t$ elements. We then adjoin to the end of each row a set of remaining elements from the set $\{1, 2, 3, \ldots, t\}$ not already used in that row and to the bottom of each column we adjoin a set of remaining elements from the set $\{1, 2, 3, \ldots, t\}$ not already used in that column as in Figure 1.

Each entry $k$ of the set appended at the end of the $i$th row represents an edge from the $i$th element of the partite set of size $r$ to the element $k$ of the partite set of size $t$. Similarly, each entry $k$ of the set appended at the bottom of the $j$th column represents an edge from the $j$th element of the partite set of size $s$ to the element $k$ of the partite set of size $t$. So a Latin representation of $K_{r,s,t}$ is in fact equivalent to a decomposition of $K_{r,s,t}$ into $rs$ triangles and $rK_{1,t-s} + sK_{1,t-r}$.

Here we define trade to be a set of elements in the Latin representation, corresponding to a set of triangles and edges in $K_{r,s,t}$ which are $P_4$-decomposable. We define relabelling of the elements of a trade to be a bijection $\phi$ from the set of elements of $T = \{1, 2, \ldots, t\}$ onto itself. Thus every occurrence of $i \in T$ in the trade is replaced by $\phi(i)$. The relabelling of
Two copies of $C_3$, with a common vertex, is equivalent to two copies of $P_4$. Let $(a_1, b_1, c_1), (a_1, b_2, c_2)$ be two copies of $C_3$ with a common vertex $a_1$; then it can be written as two copies of $P_4$, $P(c_1, a_1, b_2, c_2), P(c_2, a_1, b_1, c_1)$. In general, $n$ copies of $C_3$ with a common vertex is equivalent to $n$ copies of $P_4$. Let $(a_1, b_1, c_1), (a_1, b_2, c_2), \ldots, (a_1, b_{(n-1)}, c_{(n-1)}), (a_1, b_n, c_n)$ be $n$ copies of $C_3$ with a common vertex $a_1$; then it can be written $n$ copies of $P_4$ as $P(c_1, a_1, b_2, c_2), P(c_2, a_1, b_3, c_3), \ldots, P(c_{(n-1)}, a_1, b_n, c_n), P(c_n, a_1, b_1, c_1)$.

**Construction 1.6.** Here we define two types of trades, in the first type we use elements from outside the Latin rectangle which are $P_4$-decomposable. The trades of first type are $T_1, T_2, T_3, T_4$, as shown in Figure 2 from the elements outside the Latin rectangle in which each copy of trades in $K_{r,s,t}$ are all edge-disjoint and $P_4$-decomposable.

The trade $T_1$ can be obtained from the newly adjoined elements on the right side of the Latin rectangle which can be decomposed into three copies of $P_4$ as follows: $P(c_{(s+1)}, a_i, c_{(s+2)}, a_j), P(a_i, c_{(s+3)}, a_k, c_{(s+5)}), P(c_{(s+3)}, a_j, c_{(s+4)}, a_k)$. Similarly, by relabelling we can obtain the trade $T_1$ from the newly adjoined elements on the bottom of the Latin rectangle.

The trade $T_2$ can be obtained from the newly adjoined elements on the right side of the Latin rectangle which can be decomposed into two copies of $P_4$ as $P(c_{(s+1)}, a_i, c_{(s+2)}, a_j), P(a_j, c_{(s+3)}, a_k, c_{(s+4)}).$ Similarly, by relabelling we can obtain the trade $T_2$ from the newly adjoined elements on the bottom of the Latin rectangle.

![Figure 1](image-url)

The elements in a trade does not change the structure of the corresponding set of edges in $K_{r,s,t}$.

\begin{table}
\begin{tabular}{|c|c|c|c|}
\hline
1 & 2 & \cdots & s \\
\hline
2 & 3 & \cdots & s + 1 \\
\hline
\cdots & \cdots & \cdots & \cdots \\
\hline
r & r + 1 & \cdots & r + s - 1 \\
\hline
r + 1 & r + 2 & \cdots & r + s \\
\hline
\cdots & \cdots & \cdots & \cdots \\
\hline
t & 1 & \cdots & s - 1 \\
\hline
\end{tabular}
\end{table}
The trade $T_3$ can be obtained from the newly adjoined elements on the right side and the bottom of the Latin rectangle which can be decomposed into three copies of $P_4$ as $P(c_{r+3}, b_i, c_{r+4}, a_i)$, $P(c_{r+4}, b_j, c_{s+5}, a_j)$, $P(c_{r+5}, b_k, c_{r+6}, a_k)$, where $s+4$, $s+5$, $s+6$ in the right side of the Latin rectangle are equivalent to $r+4$, $r+5$, $r+6$ respectively.

The trade $T_4$ can be obtained from the newly adjoined elements on the right side and the bottom of the Latin rectangle which can be decomposed into four copies of $P_4$ as $P(c_{r+1}, b_i, c_{r+2}, b_j)$, $P(b_i, c_{s+3}, b_j, c_{s+4})$, $P(c_{s+3}, b_k, c_{s+4}, a_i)$, $P(b_k, c_{s+5}, a_i, c_{s+3})$ where $s+3$, $s+4$, $s+5$ in the right side of the Latin rectangle are equivalent to $r+3$, $r+4$, $r+5$ respectively.

In the second type, the elements from both inside and outside of the Latin rectangle are used. We use these two types of suitable trades until all the edges in $K_{r,s,t}$ are used.

2. Necessary conditions

**Theorem 2.1.** If the complete tripartite graph $K_{r,s,t}$, where $r \leq s \leq t$, has a decomposition into $p$ copies of $C_3$ and $q$ copies of $P_4$, then the following holds:

(i) $3(rst + tr)$,

(ii) $q \neq 1$.

**Proof.** By a counting argument, we get the required condition (i). We prove (ii) by a contradiction. Suppose that $q = 1$. Then the end vertices of the only path $P_4$ have odd degree in $(K_{r,s,t} - E(P_4))$. Therefore the resulting graph $(K_{r,s,t} - E(P_4))$ cannot be decomposed into $C_3$, a contradiction. Hence $q \neq 1$. \qed
Corollary 2.2. If the complete tripartite graph $K_{r,s,t}$ can be decomposed into $pC_3$ and $qP_4$, where $r \leq s \leq t$, then $r$, $s$, and $t$ must satisfy one of the following:

(a) any two of $r$, $s$, $t$ are congruent to 0 (mod 3),
(b) all of $r$, $s$, $t$ are congruent to 1 (mod 3),
(c) all of $r$, $s$, $t$ are congruent to 2 (mod 3).

Proof. The proof follows from the fact that the number of edges of $K_{r,s,t}$ is divisible by 3.

3. Sufficient conditions

Lemma 3.1. The graph $K_{3,3,3}$ can be decomposed into $p$ copies of $C_3$ and $q$ copies of $P_4$, where $0 \leq p \leq 9$ and $0 \leq q \leq 9$, $q \neq 1$.

Proof. Form a Latin square of order $3 \times 3$ on 3 elements as shown in Figure 3. By Lemma 1.1, we have nine edge-disjoint 3-cycles as follows:

\[(a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3),
(a_2, b_3, c_1), (a_3, b_1, c_3), (a_3, b_2, c_1), (a_3, b_3, c_2)\]

In fact, this gives the required decomposition when $p = 9$, $q = 0$. The required decomposition for the other choices of $p$ and $q$ can be obtained by using Construction 1.5.

Lemma 3.2. The graph $K_{3,3,4}$ can be decomposed into $pC_3$ and $qP_4$, where $0 \leq p \leq 7$ and $4 \leq q \leq 11$.

Proof. We form a Latin rectangle of order $3 \times 3$ on 4 elements. By Lemma 1.1, we have nine copies of $C_3$ as follows:

\[(a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3),
(a_2, b_3, c_4), (a_3, b_1, c_3), (a_3, b_2, c_4), (a_3, b_3, c_1)\]

The newly added element to the right side of each row in the Latin rectangle represents a single edge which cannot be decomposed into $P_4$. Similarly the newly added element to the bottom of each column of the Latin rectangle represents a single edge which cannot be decomposed into $P_4$. Here we use trades of the second type to get required number of copies of $P_4$. The single
edges outside the Latin rectangle along with the two copies of $C_3$ indicated by bold letters in Figure 4 give four copies of $P_4$:

$$P(a_1, c_4, b_2, a_3), P(b_1, c_4, a_3, b_3), P(a_2, c_1, a_3, c_2), P(b_2, c_1, b_3, c_2).$$

Also, when $p = 6, q = 5$, we have six copies of $C_3$:

$$(a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3), (a_2, b_3, c_1)$$

and five copies of $P_4$:

$$P(c_3, b_3, c_1, a_2), P(c_1, a_3, b_1, c_3), P(c_3, a_3, b_2, c_4),$$
$$P(a_1, c_4, b_2, c_1), P(b_1, c_4, a_3, b_3).$$

The other choices of $p$ and $q$ can be obtained by using Construction 1.5. Hence the graph $K_{3,3,4}$ has the desired decomposition. □

**Theorem 3.3.** If $r \equiv 0 \pmod{3}, s \equiv 0 \pmod{3},$ and for any $t$, then the complete tripartite graph $K_{r,s,t}$, $r \leq s \leq t$, can be decomposed into $p$ copies of $C_3$ and $q$ copies of $P_4$, where $q \neq 1$.

**Proof.** The proof is separated into three cases.

**Case 1:** $r \equiv 0 \pmod{3}, s \equiv 0 \pmod{3}, t \equiv 0 \pmod{3}$.

The Latin rectangle of order $r \times s$ on $t$ elements give $rs$ triangles. The other choices of $p$ and $q$ can be obtained by Construction 1.5. Now the newly added elements to the right side of the Latin rectangle form $(r/3)[(t-s)/3]$ copies of $3 \times 3$ arrays each representing the trade $T_1$. Similarly the newly added elements to the bottom of the Latin rectangle form $(s/3)[(t-r)/3]$ copies of $3 \times 3$ arrays each representing the trade $T_1$. By Construction 1.6, the copies of trade $T_1$ are all edge-disjoint and $P_4$-decomposable. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$r \left( \frac{t-s}{3} \right) + s \left( \frac{t-r}{3} \right) \leq q \leq rs + r \left( \frac{t-s}{3} \right) + s \left( \frac{t-r}{3} \right).$$

**Case 2:** $r \equiv 0 \pmod{3}, s \equiv 0 \pmod{3}, t \equiv 1 \pmod{3}$.

For the graph $K_{r,r,r+1}$, the newly added elements to the right side of Latin rectangle form an $r \times 1$ array which cannot be decomposed into $P_4$. Similarly the newly added elements to the bottom of Latin rectangle form a $1 \times s$ array which cannot be decomposed into $P_4$. Therefore we
use trades of the second type to obtain the required number of copies of $P_4$. The single edges on the both side of the Latin rectangle along with $2r/3$ copies of $C_3$ give $4r/3$ copies of $P_4$. These $4r/3$ copies of $P_4$ and the remaining $(r^2 - (2r/3))$ copies of $C_3$ give the maximum number of copies of $P_4$ by Construction 1.5. Hence the graph has the required decomposition, where $0 \leq p \leq (r^2 - \frac{2r}{3})$,

$$\frac{4r}{3} \leq q \leq r^2 + \frac{2r}{3}.$$ 

For the graph $K_{r,s,s+1}$, the newly added elements to the right side of the Latin rectangle is an $r \times 1$ array. The newly added elements to the bottom of the Latin rectangle form $(s/3)[[(t - r) - 4)/3]$ copies of $3 \times 3$ arrays which represents the trade $T_1$ and the remaining elements form $2s/3$ copies of $2 \times 3$ arrays which represents the trade $T_2$. The elements of $r/3$ copies of $3 \times 1$ array in the right side of the Latin rectangle along with the elements of $r/3$ copies of $2 \times 3$ array at the bottom of the Latin rectangle which contain the same elements of a $3 \times 1$ array form the trade $T_3$. By Construction 1.6, the edge-disjoint copies of $T_1$, $T_2$, $T_3$ are $P_4$-decomposable. The remaining possible choices of $p$ and $q$ can be obtained by using Construction 1.5. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$s \left[ \frac{(t-r)-4}{3} + \frac{4s}{3} - \frac{2r}{3} + r \right] \leq q$$

For $t > s + 1$, the newly added elements to the right side of the Latin rectangle form $(r/3)[[(t - s) - 4)/3]$ copies of $3 \times 3$ arrays which represents the trade $T_1$ and the remaining elements form $2r/3$ copies of $3 \times 2$ arrays which represents the trade $T_2$. The newly added elements to the bottom of the Latin rectangle form $(s/3)[[(t - r) - 4)/3]$ copies of $3 \times 3$ arrays which represents the trade $T_1$ and $2s/3$ copies of $2 \times 3$ arrays which represents the trade $T_2$. By Construction 1.6, the edge-disjoint copies of $T_1$, $T_2$ are $P_4$-decomposable. The remaining possible choices of $p$ and $q$ can be obtained by using Construction 1.5. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$s \left[ \frac{(t-r)-4}{3} + r \left[ \frac{(t-s)-4}{3} + 4 \left( \frac{r}{3} + \frac{s}{3} \right) \right] \right] \leq q$$

Case 3: $r \equiv 0 \pmod{3}, s \equiv 0 \pmod{3}, t \equiv 2 \pmod{3}$.

In this graph, the newly added elements to the right side of Latin rectangle form $(r/3)[[(t - s) - 2)/3]$ copies of $3 \times 3$ arrays which represents the trade $T_1$ and the remaining elements form $r/3$ copies of $3 \times 2$ arrays
which represents the trade $T_2$. Similarly the newly added elements to the bottom of the Latin rectangle form $(s/3)[((t - r) - 2)/3]$ copies of $3 \times 3$ arrays which represents the trade $T_1$ and the remaining elements form $s/3$ copies of $2 \times 3$ arrays which represents the trade $T_2$. Hence we have the required decomposition, where $0 \leq p \leq rs$, 

$$s \left[\frac{(t - r) - 2}{3}\right] + r \left[\frac{(t - s) - 2}{3}\right] + \frac{2(r + s)}{3} \leq q$$

$$\leq s \left[\frac{(t - r) - 2}{3}\right] + r \left[\frac{(t - s) - 2}{3}\right] + \frac{2(r + s)}{3} + rs.$$  

\[\square\]

**Theorem 3.4.** If $r \equiv 0 \pmod{3}$, $s \equiv 1 \pmod{3}$, and $t \equiv 0 \pmod{3}$, then the complete tripartite graph $K_{r,s,t}$, $r < s < t$, can be decomposed into $pC_3$ and $qP_4$, where $q \neq 1$.

**Proof.** The Latin rectangle of order $r \times s$ on $t$ elements form $rs$ triangles. The other choices of $p$ and $q$ can be obtained by Construction 1.5. The newly added elements to the right side of the Latin rectangle form $(r/3)[((t - s) - 2)/3]$ copies of $3 \times 3$ arrays each representing the trade $T_1$ and the remaining elements form $r/3$ copies of $3 \times 2$ arrays which represents the trade $T_2$. Similarly the newly added elements to the bottom of the Latin rectangle form $[(t - r)/3][(s - 4)/3]$ copies of $3 \times 3$ arrays each representing the trade $T_1$ and the remaining elements form $[2(t - r)/3]$ copies of $3 \times 2$ arrays which represents the trade $T_2$. By Construction 1.6, all the trades are edge-disjoint and $P_4$-decomposable. Hence we have the required decomposition, where $0 \leq p \leq rs$,

$$r \left[\frac{(t - s) - 2}{3}\right] + \frac{4(t - r)}{3} + \frac{2r}{3} + \frac{(t - r)(s - 4)}{3} \leq q$$

$$\leq rs + r \left[\frac{(t - s) - 2}{3}\right] + \frac{4(t - r)}{3} + \frac{2r}{3} + \frac{(t - r)(s - 4)}{3}.$$  

\[\square\]

**Theorem 3.5.** If $r \equiv 0 \pmod{3}$, $s \equiv 2 \pmod{3}$, and $t \equiv 0 \pmod{3}$, then the complete tripartite graph $K_{r,s,t}$, $r < s < t$, can be decomposed into $pC_3$ and $qP_4$, where $q \neq 1$.

**Proof.** We consider $s = r + 2$, then the newly added elements to the right side of the Latin rectangle is an $r \times 1$ array which cannot be decomposed into $P_4$. Therefore, we use trades of the second type. These $r$ single edges along with $2r/3$ triangles give $5r/3$ copies of $P_4$. Now the newly added elements to the bottom of the Latin rectangle have $(s - 2)/3$ copies of $3 \times 3$ arrays and one copy of a $3 \times 2$ array. Hence we have the required decomposition, where $0 \leq p \leq r(s - (2/3))$,

$$\frac{5r}{3} + s \leq q \leq \frac{5r}{3} + s + r \left(\frac{s - 2}{3}\right).$$
For \( t = s + 1 \), the newly added elements to the right side of the Latin rectangle form \( r/3 \) copies of \( 3 \times 1 \) arrays which cannot be decomposed into copies of \( P_4 \). Now the elements of \( r/3 \) copies of \( 3 \times 1 \) arrays in the right side of the Latin rectangle along with the elements of \( r/3 \) copies of \( 2 \times 3 \) arrays in the bottom of the Latin rectangle which contain the same elements of a \( 3 \times 1 \) array form the trade \( T_3 \). The newly added elements to the bottom of the Latin rectangle form \( \left[(t-r) - 6\right]/3\] copies of \( 3 \times 3 \) arrays, \( (t-r)/3 \) copies of \( 3 \times 2 \) arrays and \( 3(s-2)/3 \) copies of \( 2 \times 3 \) arrays. Therefore we get \( \left[(t-r) - 6\right]/3\] copies of the trade \( T_1 \), \( (t-r)/3 \), \( 3(s-2)/3 \), and \( (3s-6-r)/3 \) copies of \( T_2 \) in which all are edge-disjoint. Hence we have the required decomposition, where \( 0 \leq p \leq rs \) and \( q \neq 1 \),

\[
(s-2) \left[ \frac{(t-r) - 6}{3} \right] + r + \frac{2(t-r)}{3} + 2(s-2) + \frac{2(3s-6-r)}{3} \leq q
\]

\[
\leq (s-2) \left[ \frac{(t-r) - 6}{3} \right] + r + \frac{2(t-r)}{3} + 2(s-2) + \frac{2(3s-6-r)}{3} + rs.
\]

Now for \( t > s + 1 \), the newly added elements to the right side of the Latin rectangle form \( r/3\left[(t-s) - 4\right]/3\) copies of \( 3 \times 3 \) arrays and \( 2r/3 \) copies of \( 3 \times 2 \) arrays which represents the trade \( T_1 \) and \( T_2 \) respectively. The newly added elements to the bottom of the Latin rectangle form \( \left[(t-r)/3\right]/\left[(s-2)/3\right] \) copies of \( 3 \times 3 \) arrays and \( \left[(t-r)/3\right] \) copies of \( 3 \times 2 \) arrays which represents the trade \( T_1 \) and \( T_2 \) respectively. Hence we have the required decomposition, where \( 0 \leq p \leq rs \) and \( q \neq 1 \),

\[
\frac{4r}{3} + \frac{2(t-r)}{3} + \frac{(t-r)(s-2)}{3} + r \left[ \frac{(t-s) - 4}{3} \right] \leq q
\]

\[
\leq \frac{4r}{3} + \frac{2(t-r)}{3} + \frac{(t-r)(s-2)}{3} + r \left[ \frac{(t-s) - 4}{3} \right] + rs.
\]

\[\square\]

**Theorem 3.6.** If \( r \equiv 1 \pmod{3}, s \equiv 1 \pmod{3}, \) and \( t \equiv 1 \pmod{3}, \) then the complete tripartite graph \( K_{r,s,t}, r \leq s \leq t, \) can be decomposed into \( pC_3 \) and \( qP_4, \) where \( q \neq 1. \)

**Proof.** We consider three cases:

**Case 1:** \( r > 1. \)

In this case the Latin rectangle of order \( r \times s \) on \( t \) elements give \( rs \) triangles. The newly added elements to the right side of the Latin rectangle form \( \left[(t-s)/3\right]/\left[(r-4)/3\right] \) copies of \( 3 \times 3 \) arrays which represents the trade \( T_1 \) and the remaining elements form \( 2(t-s)/3 \) copies of \( 2 \times 3 \) arrays which represents the trade \( T_2. \) The newly added elements to the bottom of the Latin rectangle form \( \left[(t-r)/3\right]/\left[(s-4)/3\right] \) copies of \( 3 \times 3 \) arrays each representing the trade \( T_1 \) and the remaining elements form \( 2(t-r)/3 \) copies of \( 3 \times 2 \) arrays which represents the trade \( T_2. \) The other
Case 2: \( r = 1, s = 1 \).

We have one copy of \( C_3 \) and \((t - 1)/3\) copies of \( K_{2,3} \) which can be decomposed into two copies of \( P_4 \). Therefore in this case we get \( p = 1 \) and \( q = 2(t - 1)/3 \).

Case 3: \( r = 1, s > 1 \).

In this case we have \( p = s \). The newly added elements to the bottom of the Latin rectangle form \( [(t - 1)/3][(s - 4)/3] \) copies of \( 3 \times 3 \) arrays which represents the trade \( T_1 \) and the remaining elements form \( 2(t - 1)/3 \) copies of \( 3 \times 2 \) arrays which represents the trade \( T_2 \). The newly added elements to the right side of the Latin rectangle form \((t - s)/3\) copies of \( 1 \times 3 \) arrays which cannot be decomposed into copies of \( P_4 \). The elements of \((t - s)/3\) copies of \( 1 \times 3 \) arrays in the right side of the Latin rectangle along with the elements of \((t - s)/3\) copies of \( 3 \times 3 \) arrays in the bottom of the Latin rectangle which contain the same elements of \( 1 \times 3 \) arrays form the trade \( T_4 \). Therefore we get \( ([t - 1]/3][(s - 4)/3] - [(t - s)/3] \) copies of \( 3 \times 3 \) arrays which represents the trade \( T_1 \). Hence we have the required decomposition, where \( 0 \leq p \leq rs \),

\[
\frac{4(t - s)}{3} + \left[ \frac{(t - 1)(s - 4) - 3(t - s)}{3} \right] + \frac{4(t - 1)}{3} \leq q
\]

\[
\leq s + \frac{4(t - s)}{3} + \left[ \frac{(t - 1)(s - 4) - 3(t - s)}{3} \right] + \frac{4(t - 1)}{3}.
\]

\( \square \)

**Theorem 3.7.** If \( r \equiv 1 \pmod{3}, s \equiv 0 \pmod{3}, \) and \( t \equiv 0 \pmod{3} \), then the complete tripartite graph \( K_{r,s,t} \), \( r < s \leq t \), can be decomposed into \( pC_3 \) and \( qP_4 \), where \( q \neq 1 \).

**Proof.** We consider two cases:

**Case 1:** \( r = 1 \).

The Latin rectangle of order \( 1 \times s \) on \( t \) elements give \( s \) triangles. The newly added elements to the bottom of the Latin rectangle have \((s/3)[(t - 3)/3] \) copies of \( 3 \times 3 \) arrays and \( s/3 \) copies of \( 2 \times 3 \) arrays.

The newly added elements to the right side of the Latin rectangle have \((t - s)/3\) copies of \( 1 \times 3 \) arrays. The elements of \((t - s)/3\) copies of \( 1 \times 3 \) arrays on the right side of the Latin rectangle along with the elements of \((t - s)/3\) copies of \( 3 \times 3 \) array in the bottom of the Latin rectangle which contain the same elements of a \( 1 \times 3 \) array form the trade \( T_4 \). Therefore we have \((s/3)[(t - 3)/3] - [(t - s)/3] \) copies of \( T_1 \), \((t - s)/3\) copies of \( T_4 \)
and \(s/3\) copies of \(T_2\). Hence we have the required decomposition, where \(0 \leq p \leq s\),
\[
\left\lfloor \frac{s(t - 3) - 3(t - s)}{3} \right\rfloor + \frac{2s}{3} + \frac{4(t - s)}{3} \leq q \leq s + \left\lfloor \frac{s(t - 3) - 3(t - s)}{3} \right\rfloor + \frac{2s}{3} + \frac{4(t - s)}{3}.
\]

**Case 2: \(r > 1\).**

The Latin rectangle of order \(r \times s\) on \(t\) elements give \(rs\) triangles. The newly added elements to the right side of the Latin rectangle form \((t - s)(r - 4)/9\) copies of \(3 \times 3\) arrays and \(2(t - s)/3\) copies of \(2 \times 3\) arrays. The newly added elements to the bottom of the Latin rectangle form \([s(t - r - 2)/9]\) copies of \(3 \times 3\) arrays and \(s/3\) copies of \(2 \times 3\) arrays. Therefore each copy of \(3 \times 3\) arrays and \(2 \times 3\) arrays representing the trades \(T_1\) and \(T_2\) respectively. Hence we have the required decomposition, where \(0 \leq p \leq rs\),
\[
\frac{s(t - r - 2)}{3} + \frac{(t - s)(r - 4)}{3} + \frac{4(t - s)}{3} + \frac{2s}{3} \leq q \leq rs + \frac{s(t - r - 2)}{3} + \frac{(t - s)(r - 4)}{3} + \frac{4(t - s)}{3} + \frac{2s}{3}.
\]

**Proof.** The Latin rectangle of order \(r \times s\) on \(t\) elements give \(rs\) triangles. The other choices of \(p\) and \(q\) can be obtained by using Construction 1.5. The newly added elements to the right side of the Latin rectangle form \([(t - s)/3][(r - 2)/3]\) copies of \(3 \times 3\) arrays and \((t - s)/3\) copies of \(2 \times 3\) arrays each representing the trades \(T_1\) and \(T_2\) respectively. Similarly the newly added elements to the bottom of the Latin rectangle form \([(t - r)/3][(s - 2)/3]\) copies of \(3 \times 3\) arrays and \((t - r)/3\) copies of \(3 \times 2\) arrays each representing the trades \(T_1\) and \(T_2\) respectively. Hence we have the required decomposition, where \(0 \leq p \leq rs\),
\[
\frac{(t - s)(r - 2)}{3} + \frac{2(t - s)}{3} + \frac{(t - r)(s - 2)}{3} + \frac{2(t - r)}{3} \leq q \leq \frac{(t - s)(r - 2)}{3} + \frac{2(t - s)}{3} + \frac{(t - r)(s - 2)}{3} + \frac{2(t - r)}{3} + rs.
\]

**Theorem 3.8.** If \(r \equiv 2 \pmod{3}\), \(s \equiv 2 \pmod{3}\), and \(t \equiv 2 \pmod{3}\), then the complete tripartite graph \(K_{r,s,t}\), \(r \leq s \leq t\), can be decomposed into \(pC_3\) and \(qP_4\), \(q \neq 1\).

**Proof.**
Proof. We consider two cases:

**Case 1:** $s = t$.

The Latin rectangle of order $r \times s$ on $s$ elements give $rs$ triangles. The newly added elements to the bottom of the Latin rectangle is a $1 \times (s/3)$ array which cannot be decomposed into $P_4$. Here we use trades of the second type to decompose $P_4$. The edges in the bottom of the Latin rectangle along with $2s/3$ triangles give $s$ copies of $P_4$. Therefore we get $0 \leq p \leq (rs - (2s/3))$ and $s \leq q \leq (rs + s/3)$.

**Case 2:** $s < t$.

The newly added elements to the right side of the Latin rectangle form $[(t-s)/3][(r-2)/3]$ copies of $3 \times 3$ arrays and the remaining elements form $[(t-s)/3]$ copies of $2 \times 3$ arrays. Similarly the newly added elements to the bottom of the Latin rectangle form $[(t-r-4)/3][s/3]$ copies of $3 \times 3$ arrays and the remaining elements form $2s/3$ copies of $2 \times 3$ arrays. Therefore each copy of $3 \times 3$ arrays and $2 \times 3$ arrays represent the trades $T_1$ and $T_2$, respectively. The other choices of $p$ and $q$ can be obtained by using Construction 1.5. Hence we get the required decomposition, where

$$\frac{(t-s)(r-2)}{3} + \frac{2(t-s)}{3} + \frac{s(t-r-4)}{3} + \frac{4s}{3} \leq q$$

$$\leq \frac{(t-s)(r-2)}{3} + \frac{2(t-s)}{3} + \frac{s(t-r-4)}{3} + \frac{4s}{3} + rs.$$

\[\square\]

4. Conclusion

**Main Theorem.** Let $p$ and $q$ be nonnegative integers and let $r$, $s$, $t$ be positive integers. There exists a decomposition of $K_{r,s,t}$, $r \leq s \leq t$, into $pC_3$ and $qP_4$ if and only if $3(p+q) = rs + st + tr$, $q \neq 1$, where $r$, $s$, $t$ satisfy the following conditions:

(a) any two of $r$, $s$, $t$ are congruent to 0 (mod 3),
(b) all of $r$, $s$, $t$ are congruent to 1 (mod 3),
(c) all of $r$, $s$, $t$ are congruent to 2 (mod 3).

**Proof.** This follows from the Theorems 2.1, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, and 3.9. \[\square\]

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