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BINDING NUMBER, MINIMUM DEGREE AND (g, f)-FACTORS OF GRAPHS

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ABSTRACT. Let a and b be integers with $2 \le a < b$, and let G be a graph of order n with $n \ge \frac{(a+b-1)^2}{a+1}$ and the minimum degree $\delta(G) \ge 1+\frac{(b-2)n}{a+b-1}$. Let g and f be nonnegative integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$ for each $x \in V(G)$. We prove that if the binding number $\mathrm{bind}(G) \ge 1 + \frac{b-2}{a+1}$, then G has a (g, f)-factor.

1. INTRODUCTION

In this paper, we consider only finite, simple, undirected graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we denote by $\deg_G(x)$ the degree of x in G, and by $N_G(x)$ the set of vertices adjacent to x in G; thus $\deg_G(x) = |N_G(x)|$. For $X \subseteq V(G)$, we let $N_G(X) = \bigcup_{x\in X} N_G(x)$. The minimum degree of G is denoted by $\delta(G)$. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S; $G-S = G[V(G)-S]$. A vertex set S of G is called an independent set if $G[S]$ has no edges. The binding number $\mathrm{bind}(G)$ of G is defined by

$$\mathrm{bind}(G) = \min\left\{\frac{|N_G(X)|}{|X|} \mid \emptyset \ne X \subseteq V(G), N_G(X) \ne V(G)\right\},$$

which was introduced in 1973 by Woodall [7] (prior to that, it was called the "melting-point" of a graph, for example, in [1]).

Let f and g be nonnegative integer-valued functions defined on $V(G)$. A spanning subgraph F of G is called a (g, f)-factor if it satisfies $g(x) \le \deg_F(x) \le f(x)$ for every $x \in V(G)$. Other notation and terminology are the same as those in [3].

The following results are already known on the binding number for the existence of k-factors, f-factors and $[a, b]$-factors.

Theorem A (Tokushige [6]). *Let k be an integer with $k \ge 2$, and let G be a graph of order n with $n > 4k + 1 - 4\sqrt{k+2}$ and kn even. Suppose that*

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$\delta(G) \neq \lfloor \frac{(k-1)n+2k-3}{2k-1} \rfloor$ *or* $kn \equiv -1, 0, 1, \dots, k-1 \pmod{2k-1}$. *Then* G *has a* k*-factor if*

$$\mathrm{bind}(G) \geq 2 - \frac{1}{k}.$$

Theorem B (Kano and Tokushige [4]). *Let* a *and* b *be integers with* $1 \leq a \leq b$ *and* $2 \leq b$, *and let* G *be a connected graph of order* n *with* $n \geq (a+b)^2/a$ *and* $\sum_{x \in V(G)} f(x) \equiv 0 \pmod 2$. *Let* f *be a nonnegative integer-valued function defined on* $V(G)$ *such that* $a \leq f(x) \leq b$ *for each* $x \in V(G)$. *If one of the following three conditions is satisfied, then* G *has an* f*-factor.*

(1) $\mathrm{bind}(G) > \frac{(a+b-1)(n-1)}{an-(a+b)+3}$;
(2) $\delta(G) > \frac{bn-2}{a+b}$;
(3) $\delta(G) \geq \frac{(b-1)n+a+b-2}{a+b-1}$ *and for every nonempty independent subset* X *of* $V(G)$,

$$|N_G(X)| \geq \frac{(b-1)n+|X|-1}{a+b-1}.$$

Theorem C (Chen [2]). *Let* a *and* b *be integers with* $2 \leq a < b$, *and let* G *be a graph of order* n *with* $n \geq b+3a$. *Then* G *has an* $[a,b]$*-factor if*

$$\mathrm{bind}(G) \geq 1 + \frac{a-1}{b} \text{ and } \delta(G) \geq 1 + \frac{(a-1)n}{a+b-1}.$$

In this paper, we prove the following theorem on the existence of (g,f)-factors of graphs, which is an extension of Theorems B and C for large graphs.

Theorem 1.1. *Let* a *and* b *be integers with* $2 \leq a < b$, *and let* G *be a graph of order* n *with* $n \geq (a+b-1)^2/(a+1)$. *Let* g *and* f *be nonnegative integer-valued functions defined on* $V(G)$ *such that* $a \leq g(x) < f(x) \leq b$ *for each* $x \in V(G)$. *Then* G *has a* (g,f)*-factor if*

$$\mathrm{bind}(G) \geq 1 + \frac{b-2}{a+1} \text{ and } \delta(G) \geq 1 + \frac{(b-2)n}{a+b-1}.$$

Unfortunately, the author does not know whether the lower bound of the minimum degree in Theorem 1.1 is best possible or not. Hence, we pose the following conjecture.

Conjecture 1.2. *Let* a *and* b *be integers with* $2 \leq a < b$, *and let* G *be a graph of sufficiently large order* n. *Let* g *and* f *be nonnegative integer-valued functions defined on* $V(G)$ *such that* $a \leq g(x) < f(x) \leq b$ *for each* $x \in V(G)$. *Then* G *has a* (g,f)*-factor if*

$$\mathrm{bind}(G) \geq 1 + \frac{b-2}{a+1} \text{ and } \delta(G) \geq \frac{(b-2)n+a+1}{a+b-1}.$$

2. Proof of Theorem 1.1

In our proof, we use the following theorem, which is a special case of the (g, f)-factor theorem due to Lovász.

Lemma D (Lovász [5]). *Let G be a graph, and let g and f be nonnegative integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$ for each $x \in V(G)$. Then G has a (g, f)-factor if and only if*

$$\sum_{x\in S} f(x) + \sum_{y\in T}(\deg_{G-S}(y) - g(y)) \ge 0$$

for all disjoint subsets S and T of $V(G)$.

Let a, b, f, g, G, and n be as in Theorem 1.1. Suppose that G satisfies the assumption of Theorem 1.1 but has no (g, f)-factor. Then by Lemma D, there exist disjoint subsets S and T of $V(G)$ such that

$$\sum_{x\in S} f(x) + \sum_{y\in T}(\deg_{G-S}(y) - g(y)) < 0. \tag{2.1}$$

We choose such subsets S and T so that $|T|$ is as small as possible.

If $T = \emptyset$, then by (2.1) we have $\sum_{x\in S} f(x) \le -1$, which is a contradiction. Also, suppose that there exists a vertex $y_0 \in T$ such that $\deg_{G-S}(y_0) \ge g(y_0)$. Then the subsets S and $T - \{y_0\}$ satisfy (2.1), which contradicts the choice of T. Thus $T \ne \emptyset$ and $\deg_{G-S}(y) < g(y) \le b - 1$ for each $y \in T$.

Since $T \ne \emptyset$, we can define

$$h = \min\{\deg_{G-S}(y) \mid y \in T\}.$$

Note that $0 \le h \le b - 2$.

We divide the proof into two cases.

Case 1: $h = 0$.

Let $T_0 = \{y \in T \mid \deg_{G-S}(y) = 0\}$, and hence T_0 is an independent subset of G and $T_0 \ne \emptyset$. Then the following claim holds.

Claim 2.1. *We have* $|N_G(T_0)| \ge \frac{(b-2)n+(a+1)|T_0|}{a+b-1}$.

Proof. Let $Z = V(G) - N_G(T_0)$. Then $T_0 \subseteq Z$ and $N_G(Z) \subseteq V(G) - T_0$, and hence it follows from the binding number condition of the theorem that

$$\begin{aligned} n - |T_0| \ge |N_G(Z)| &\ge \left(1 + \frac{b-2}{a+1}\right)|Z| \\ &= \frac{a+b-1}{a+1}(n - |N_G(T_0)|), \end{aligned}$$

that is,

$$|N_G(T_0)| \ge \frac{(b-2)n + (a+1)|T_0|}{a+b-1}.$$

□

On the other hand, by (2.1) we have

$$\begin{aligned}
0 &> \sum_{x\in S} f(x) + \sum_{y\in T} \deg_{G-S}(y) - \sum_{y\in T} g(y)\\
&\geq (a+1)|S| + \sum_{y\in T} \deg_{G-S}(y) - (b-1)|T|\\
&\geq (a+1)|S| + |T| - |T_0| - (b-1)|T|\\
&\geq (a+1)|S| - |T_0| - (b-2)(n-|S|)\\
&= (a+b-1)|S| - |T_0| - (b-2)n,
\end{aligned}$$

that is,

$$|N_G(T_0)| \leq |S| < \frac{(b-2)n + |T_0|}{a+b-1},$$

which contradicts Claim 2.1.

Case 2: $1 \leq h \leq b-2$.

It follows from (2.1) that

$$\begin{aligned}
0 &> \sum_{x\in S} f(x) + \sum_{y\in T} \deg_{G-S}(y) - \sum_{y\in T} g(y)\\
&\geq (a+1)|S| + \sum_{y\in T} \deg_{G-S}(y) - (b-1)|T|\\
&\geq (a+1)|S| + (h-b+1)|T|\\
&\geq (a+1)|S| + (h-b+1)(n-|S|)\\
&= (a+b-h)|S| - (b-1-h)n.
\end{aligned}$$

Hence

$$|S| < \frac{(b-1-h)n}{a+b-h}.$$

Thus for any $y \in T$ with $\deg_{G-S}(y) = h$, we have

$$\deg_G(y) \leq \deg_{G-S}(y) + |S| < h + \frac{(b-1-h)n}{a+b-h}.$$

Let

$$f(h) = h + \frac{(b-1-h)n}{a+b-h} = h + n - \frac{(a+1)n}{a+b-h}.$$

Since $n \geq (a+b-1)^2/(a+1)$ and $h \geq 1$, we obtain

$$f'(h) = 1 - \frac{(a+1)n}{(a+b-h)^2} \leq 1 - \frac{(a+1)n}{(a+b-1)^2} \leq 0.$$

Thus $f(h)$ takes the maximum value at $h = 1$.

Consequently, it follows from the minimum degree condition of the theorem that

$$1 + \frac{(b-2)n}{a+b-1} \leq \delta(G) \leq \deg_G(y) < f(h) \leq f(1) = 1 + \frac{(b-2)n}{a+b-1},$$

a contradiction.

This completes the proof of Theorem 1.1.

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