ON THE ILLUMINATION OF A CLASS OF CONVEX BODIES

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Abstract. We study Boltyanski’s illumination problem (or Hadwiger’s covering problem) for the class of convex bodies in $\mathbb{R}^n$ consisting of convex hulls of a pair of compact convex sets contained in two parallel hyperplanes of $\mathbb{R}^n$. This special case of the problem is completely solved when $n = 3$.

1. Introduction

We denote by $K_n$ the family of nonempty compact convex subsets of $\mathbb{R}^n$, and by $K_m^n$ ($m \leq n$) the set of compact convex subsets of $\mathbb{R}^n$ whose affine dimension is $m$. Each member of $K_n$ is called a convex body, i.e., a convex body in $\mathbb{R}^n$ is a compact convex set having interior points.

For each $K \in K_n$, we denote by int $K$, relint $K$, bd $K$, and relbd $K$ the interior, relative interior, boundary, and relative boundary of $K$, respectively. A unit vector in $\mathbb{R}^n$ is called a direction. Suppose that $K \in K_n$, $x \in \text{relbd} K$, and $u$ is a direction. If there exists a positive number $\lambda$ such that $x + \lambda u \in \text{relint} K$, then we say that $x$ is illuminated by $u$. Let $A$ be a subset of relbd $K$ and $D$ be a set of directions. If each point in $A$ is illuminated by a direction in $D$, then we say that $A$ is illuminated by $D$. When relbd $K \neq \emptyset$, the illumination number $c(K)$ of $K$ is defined by

$$c(K) = \min \{ \text{card} \ D : \ D \text{ is a set of directions illuminating relbd } K \},$$

where card $D$ is the cardinality of $D$. It is not difficult to see that $c(K)$ is affinely invariant. Concerning the least upper bound of $c(K)$ for all $K \in K_n$, there is a long-standing conjecture (see the monographs [11], [12], and [4], and the surveys [9], [14], [3], [5] for the history, known results, and relevant references of this conjecture):

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Conjecture 1.1 (Boltyanski’s illumination conjecture). For each \( K \in \mathcal{K}_n \), we have
\[
c(K) \leq 2^n;
\]
equality holds if and only if \( K \in \mathcal{K}_n \) is a parallelotope.

Since, when \( K \in \mathcal{K}_n \), \( c(K) \) equals the least number of translates of \( \text{int} \ K \) needed to cover \( K \), Conjecture 1.1 is also called Hadwiger’s covering conjecture. A “dual” version of Conjecture 1.1 is called Bezdek’s separation conjecture, cf. [4, p. 24].

Conjecture 1.2 (Bezdek’s separation conjecture). Let \( K \in \mathcal{K}_n \) \((n \geq 3)\) and \( p \) be an arbitrary interior point of \( K \). Then there exists a collection \( \mathcal{H} \) of \( 2^n \) hyperplanes such that each exposed face of \( K \) and \( p \) can be strictly separated by at least one hyperplane in \( \mathcal{H} \). Furthermore, \( 2^n \) hyperplanes are necessary only if \( K \) is the convex hull of \( n \) line segments having linearly independent directions which intersect at the common relative interior point \( p \).

See, e.g., [2], [6], [8], and [7] for progress toward the solution of Conjecture 1.2.

Conjecture 1.1 is completely solved only when \( n = 2 \). More precisely, it is known that \( c(K) = 4 \) when \( K \in \mathcal{K}_2^2 \) is a parallelogram and \( c(K) = 3 \) holds for the rest. Even when \( n = 3 \), Conjecture 1.1 is widely open. Thus it is natural to consider how to make efficient use of the knowledge of \( c(K) \) when \( K \) is constructed from lower dimensional convex bodies.

For example, it is shown in [10] that
\[
c(M_1 \oplus M_2 \oplus \cdots \oplus M_k) \leq \prod_{i=1}^{k} c(M_i),
\]
where \( M_i \), \( i = 1, \ldots, k \), are compact convex sets and \( M_1 \oplus M_2 \oplus \cdots \oplus M_k \) is their direct sum. Moreover, when \( M_1, M_2 \in \mathcal{K}_2^2 \), the set of all possible values of \( c(M_1 \oplus M_2) \) is \( \{7, 8, 9, 12, 16\} \), cf. [1]. Also, solving Conjecture 1.1 for centrally symmetric convex bodies in \( \mathbb{R}^{n+1} \) (the case when \( n = 2 \) is solved by Lassak in [13]) would lead to the solution of Conjecture 1.1 for all convex bodies in \( \mathbb{R}^n \) if the answer to the following problem (possibly due to P. Soltan) is positive:

**Problem 1.3.** Suppose that \( T = -B \in \mathcal{K}_n \) and
\[
K = \text{conv} \left( (T \times \{1\}) \cup (B \times \{0\}) \right).
\]
Is it true that \( c(K) = c(T) + c(B) = 2c(T) \)?

In this paper we estimate the illumination number for the class \( \mathcal{A} \) of convex bodies in \( \mathcal{K}_n \) constructed from lower dimensional convex bodies, where each member of \( \mathcal{A} \) is the convex hull of two compact convex sets contained in two parallel hyperplanes of \( \mathbb{R}^n \). In Section 2 we introduce several fundamental and useful lemmas. The illumination numbers of convex bodies in \( \mathcal{A} \) will be studied in Section 3.
2. Several Lemmas

For each positive integer $m$, we put 
\[ [m] = \{ k \in \mathbb{Z}^+ : 1 \leq k \leq m \} . \]

Let $a$ and $b$ be two distinct points in $\mathbb{R}^n$. We denote by $[a, b]$ the segment connecting $a$ and $b$, and by $\langle a, b \rangle$ the line passing through $a$ and $b$. Suppose that $K \in \mathcal{K}_n$ and $[a, b] \subseteq K$. If there exists a pair of parallel supporting hyperplanes $H_1$ and $H_2$ of $K$ such that $a \in H_1$ and $b \in H_2$, then we say that $[a, b]$ is an affine diameter of $K$. In this situation, each segment contained in $K$ and parallel to $\langle a, b \rangle$ is not longer than $[a, b]$. See the excellent survey [15] for more information about affine diameters of convex bodies.

**Lemma 2.1.** Let $K \in \mathcal{K}_n$ and $[a, b]$ be an affine diameter of $K$. If $u \in \mathbb{R}^n$ is a direction illuminating $a$, then $u$ does not illuminate $b$.

**Proof.** Suppose the contrary that $u$ also illuminates $b$. Then there exists a positive number $\lambda$ such that 
\[ a + \lambda u, b + \lambda u \in \text{int } K. \]
It follows that there exists a segment contained in $K$ parallel to $\langle a, b \rangle$ that is strictly longer than $[a, b]$, a contradiction to the fact that $[a, b]$ is an affine diameter of $K$. \hfill \Box

**Lemma 2.2.** Suppose that $K \in \mathcal{K}_n$, $c \in \text{relint } K$, and $0 < \lambda_1 \leq \lambda_2 < 1$. Then
\[ (1 - \lambda_1)c + \lambda_1 K \subseteq (1 - \lambda_2)c + \lambda_2 K. \]

**Proof.** Let $z$ be an arbitrary point in $(1 - \lambda_1)c + \lambda_1 K$. There exists a point $x \in K$ such that $z = (1 - \lambda_1)c + \lambda_1 x$. We have
\begin{align*}
  z - (1 - \lambda_2)c &= z - (1 - \lambda_1)c + (\lambda_2 - \lambda_1)c \\
  &= \lambda_1 x + (\lambda_2 - \lambda_1)c \\
  &= \lambda_2 \left( \frac{\lambda_1}{\lambda_2} x + \frac{\lambda_2 - \lambda_1}{\lambda_2} c \right) \in \lambda_2 K,
\end{align*}
from which it follows that $z \in (1 - \lambda_2)c + \lambda_2 K$. \hfill \Box

**Lemma 2.3.** Suppose that $K \in \mathcal{K}_n$, $p \in \text{relint } K$, $C$ is a nonempty compact subset of $\text{relbd } K$, and $D = \{ u_i : i \in [m] \}$ is a set of directions illuminating $C$. Then there exist two numbers $\lambda > 0$ and $\gamma \in (0, 1)$ such that, for each point $x \in C$, there exists $i \in [m]$ satisfying
\[ x + \lambda u_i \in \gamma(\text{relint } K) + (1 - \gamma)p. \]

**Proof.** For each pair of $k, j \in \mathbb{Z}^+$, and each $i \in [m]$, put
\[ C_{i,j,k} = \left\{ x \in C : x + \frac{1}{j} u_i \in \left( 1 - \frac{1}{k} \right) K + \frac{1}{k} p \subset \text{relint } K \right\}. \]
It suffices to show that there exists \( j_0 \in \mathbb{Z}^+ \) satisfying
\[
C \subseteq \bigcup_{i \in [m]} C_{i,j_0,j_0^2}.
\]
Otherwise, for each \( j \in \mathbb{Z}^+ \), there exists a point
\[
x_j \in C \setminus \left( \bigcup_{i \in [m]} C_{i,j,j^2} \right).
\]
By choosing a subsequence if necessary, we may assume that \( \{x_j\}_{j \in \mathbb{Z}^+} \) converges to a point \( x_0 \in C \). Then there exist an \( i_0 \in [m] \) and a number \( \lambda > 0 \) such that
\[
x_0 + \lambda u_{i_0} \in \text{relint } K.
\]
Hence, there exists a positive number \( \delta \) such that (cf. Theorem 2.23 on p. 85 in [16] for the last equality)
\[
B(x_0 + \lambda u_{i_0}, \delta) \cap \text{aff } K \subset \text{relint } K = \bigcup_{k \in \mathbb{Z}^+} \left( \left( 1 - \frac{1}{k} \right) K + \frac{1}{k} p \right),
\]
where \( B(x_0 + \lambda u_{i_0}, \delta) \) is the closed ball centered at \( x_0 + \lambda u_{i_0} \) having radius \( \delta \). Then there exists a number \( \eta \in (0,1) \) such that
\[
B(x_0 + \lambda u_{i_0}, \delta) \cap \text{aff } K \subset (1 - \eta)K + \eta p.
\]
Let \( j_0 \) be an integer in \( \mathbb{Z}^+ \) such that
\[
\|x_{j_0} - x_0\| < \delta \text{ and that } j_0 > \max \left\{ \frac{\lambda}{\eta}, \frac{1}{\lambda} \right\}.
\]
Then
\[
x_{j_0} + \frac{1}{j_0} u_{i_0} = \left( 1 - \frac{1}{\lambda j_0} \right) x_{j_0} + \frac{1}{\lambda j_0} (x_{j_0} + \lambda u_{i_0})
\]
\[
\subseteq \left( 1 - \frac{1}{\lambda j_0} \right) K + \frac{1}{\lambda j_0} (B(x_0 + \lambda u_{i_0}, \delta) \cap \text{aff } K)
\]
\[
\subseteq \left( 1 - \frac{1}{\lambda j_0} \right) K + \frac{1}{\lambda j_0} ((1 - \eta)K + \eta p)
\]
\[
= \left( 1 - \frac{\eta}{\lambda j_0} \right) K + \frac{\eta}{\lambda j_0} p
\]
\[
\subseteq \left( 1 - \frac{1}{j_0^2} \right) K + \frac{1}{j_0^2} p,
\]
where the last step follows from Lemma 2.2, a contradiction to (2.1).
\[\square\]

**Corollary 2.4.** Let \( K \in \mathcal{K}_n^+ \) and \( m = c(K) \). Then there exist a set \( D = \{u_i : i \in [m]\} \) of \( m \) directions in \( \mathbb{R}^n \) and a number \( \lambda > 0 \) such that, for each point \( x \in \text{bd } K \), there exists \( i \in [m] \) satisfying \( x + \lambda u_i \in \text{int } K \).

We will use several fundamental results related to relative interior of convex sets. First we have:
Lemma 2.5 (cf. Theorem 2.34 on p. 92 in [16]). If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is an affine transformation and \( K \in \mathcal{K}^n \), then
\[
\text{relint} \left( T(K) \right) = T(\text{relint} \ K).
\]

From Lemma 2.5 it follows that
\[
\text{relint} \left( K + x \right) = \text{relint} \ K + x, \quad \forall K \in \mathcal{K}^n, \quad x \in \mathbb{R}^n.
\]

By Theorem 2.63 in [16, p. 109], we have
\[
\text{relbd} \left( K + x \right) = \text{relbd} \ K + x, \quad \forall K \in \mathcal{K}^n, \quad x \in \mathbb{R}^n.
\]

Finally, the following lemma will also be used in the next section.

Lemma 2.6 (cf. Theorem 2.29 on p. 89 in [16]). If \( K_1, \ldots, K_m \) are convex sets in \( \mathbb{R}^n \) and \( \mu_1, \ldots, \mu_m \) are in \( \mathbb{R} \), then
\[
\text{relint} \left( \sum_{i=1}^m \mu_i K_i \right) = \sum_{i=1}^m \mu_i \text{relint} \ K_i.
\]

3. Illuminations numbers of convex bodies in \( \mathcal{A} \)

Throughout this section \( n \) is an integer not less than 3, \( T \) and \( B \) are two sets in \( \mathcal{K}^{n-1} \) such that
\[
K = \text{conv}((T \times \{1\}) \cup (B \times \{0\})) \in \mathcal{K}^n.
\]

For each \( \eta \in [0, 1] \), put \( H_\eta = \mathbb{R}^{n-1} \times \{\eta\} \) and \( K_\eta = K \cap H_\eta \). It is clear that
- \( H_1 \) and \( H_0 \) are the two supporting hyperplanes of \( K \) parallel to \( \mathbb{R}^{n-1} \times \{0\} \),
- \( T \times \{1\}, B \times \{0\} \subset \text{bd} \ K,
- for each \( \eta \in (0, 1) \), \( H_\eta \) intersects \( \text{int} \ K \).

Lemma 3.1. For each \( \eta \in (0, 1) \), we have
1. \( K_\eta = (\eta T) \times \{\eta\} + ((1 - \eta) B) \times \{0\} \),
2. \( \text{relint} K_\eta = \text{int} \ K \cap H_\eta \).

Proof. (1). First suppose that \( x \in K \cap H_\eta \). Then there exist two positive integers \( m_1 \) and \( m_2 \), two sets of positive numbers \( \{\alpha_i : i \in [m_1]\} \) and \( \{\beta_j : j \in [m_2]\} \), and two sets of points
\[
\{p_i : i \in [m_1]\} \subset T \text{ and } \{q_j : j \in [m_2]\} \subset B
\]
such that
\[
\sum_{i \in [m_1]} \alpha_i + \sum_{j \in [m_2]} \beta_j = 1 \text{ and } x = \sum_{i \in [m_1]} \alpha_i (p_i, 1) + \sum_{j \in [m_2]} \beta_j (q_j, 0).
\]

It follows that
\[
\sum_{i \in [m_1]} \alpha_i = \eta \text{ and } \sum_{j \in [m_2]} \beta_j = (1 - \eta).
\]
Therefore,

\[ x = \eta \sum_{i \in [m_1]} \frac{\alpha_i}{\eta} (p_i, 1) + (1 - \eta) \sum_{j \in [m_2]} \frac{\beta_j}{1 - \eta} (q_j, 0) \]

\[ \in \eta(T \times \{1\}) + (1 - \eta)(B \times \{0\}) \]

\[ = (\eta T) \times \{\eta\} + ((1 - \eta)B) \times \{0\}. \]

Conversely, suppose that

\[ x \in (\eta T) \times \{\eta\} + ((1 - \eta)B) \times \{0\}. \]

Then there exist two points \( p \in T \) and \( q \in B \) such that

\[ x = (\eta p, \eta) + ((1 - \eta)q, 0) = \eta(p, 1) + (1 - \eta)(q, 0) \in K \cap H_\eta. \]

(2). Since \( \eta \in (0, 1) \), \( \text{int } K \cap H_\eta \neq \emptyset \). For each point \( x \in \text{int } K \cap H_\eta \), there exists a number \( \delta > 0 \) such that \( B(x, \delta) \subset K \). It follows that

\[ B(x, \delta) \cap \text{aff } K_\eta \subset B(x, \delta) \cap H_\eta \subset K \cap H_\eta = K_\eta. \]

Thus \( x \in \text{relint } K_\eta \).

Fix a point \( x_0 \in \text{int } K \cap K_\eta \). Then \( x_0 \in \text{relint } K_\eta \). For each point \( y \in \text{relint } K_\eta \), there exists a number \( \gamma > 0 \) such that

\[ z = y + \gamma(y - x_0) \in K_\eta. \]

It follows that

\[ y = \frac{1}{1 + \gamma} z + \gamma \frac{1}{1 + \gamma} x_0 \in \text{int } K \cap H_\eta. \]

This completes the proof. \( \square \)

**Lemma 3.2.** The boundary \( \text{bd } K \) of \( K \) can be illuminated by a set \( D \) of directions if and only if \( D \) illuminates both \( T \times \{1\} \) and \( B \times \{0\} \).

**Proof.** Clearly, if \( D \) illuminates \( \text{bd } K \) then \( D \) has to illuminate both \( T \times \{1\} \) and \( B \times \{0\} \). Conversely, suppose that \( D \) illuminates \( T \times \{1\} \) as well as \( B \times \{0\} \). Denote by \( D_T \) and \( D_B \) the subsets of \( D \) that can illuminate \( T \times \{1\} \) and \( B \times \{0\} \), respectively. Let \( (z, \eta) \) be an arbitrary point in \( \text{bd } K \setminus (T \times \{1\} \cup B \times \{0\}) \). Then \( \eta \in (0, 1) \) and, by Lemma 2.6,

\[ (z, \eta) \in K_\eta \setminus (\eta \text{relint } T \times \{1\} + (1 - \eta) \text{relint } B \times \{0\}) \]

We distinguish two cases.

**Case 1:** There exist \( x \in \text{relbd } T \) and \( y \in B \) such that \( z = \eta x + (1 - \eta)y \).

There exist a direction \((u, \gamma) \in D_T \) and \( \lambda > 0 \) such that \((x, 1) + \lambda(u, \gamma) \in \text{int } K \). Thus

\[ (z, \eta) + \eta \lambda(u, \gamma) = (\eta x + (1 - \eta)y, \eta) + \eta \lambda(u, \gamma) \]

\[ = \eta((x, 1) + \lambda(u, \gamma)) + (1 - \eta)(y, 0) \in \text{int } K. \]

**Case 2:** There exist \( x \in T \) and \( y \in \text{relbd } B \) such that \( z = \eta x + (1 - \eta)y \).

As in Case 1, there exists a direction in \( D_B \) illuminating \((z, \eta)\).

Thus \( \text{bd } K \) is illuminated by \( D \). \( \square \)
Lemma 3.3. Suppose that $T$ is not a singleton. Let $m = c(T)$, $\{v_i : i \in [m]\} \subset \mathbb{R}^{n-1}$ be a set of directions illuminating $T$, $p$ be a point in relint $T$, and $q$ be a point in relint $B$. Then there exist a positive number $\lambda$ and a number $\gamma \in (0,1)$ such that $T \times \{1\}$ can be illuminated by

$$D = \left\{ \frac{\lambda v_i + u_0}{\|\lambda v_i + u_0\|} : i \in [m] \right\},$$

where $u_0 = (1 - \gamma)(-p + q, -1)$.

Proof. By Lemma 2.3, there exist a positive number $\lambda$ and a number $\gamma \in (0,1)$ such that for each $x \in \text{relbd } T$, there exists $i \in [m]$ satisfying

$$x + \lambda v_i \in \gamma(\text{relint } T) + (1 - \gamma)p.$$

We claim that $\lambda$ and $\gamma$ have the desired property. Let $(x, 1)$ be an arbitrary point in $T \times \{1\}$. We distinguish two cases.

Case 1: $x \in \text{relbd } T$.

Without loss of generality we may assume that

$$x + \lambda v_1 \in \gamma(\text{relint } T) + (1 - \gamma)p.$$

Then

$$x, 1 + \|\lambda v_1, 0\| + u_0 \frac{(\lambda v_1, 0) + u_0}{\|\lambda v_1, 0\|} = (x, 1) + \lambda v_1, 0 + u_0 = (x + \lambda v_1, 1) + (1 - \gamma)(-p + q, -1) = (x + \lambda v_1 + (1 - \gamma)(-p + q), \gamma) \in (\gamma(\text{relint } T) + (1 - \gamma)(\text{relint } B)) \times \{\gamma\} \subset \text{int } K.$$

Case 2: $x \in \text{relint } T$.

There exist two points $y, z \in \text{relbd } T$ and a number $\eta \in (0,1)$ such that $x = \eta y + (1 - \eta)z$. Without loss of generality we may assume that

$$y + \lambda v_1 \in \gamma(\text{relint } T) + (1 - \gamma)p.$$

As in Case 1, we have

$$(y, 1) + \|\lambda v_1, 0\| + u_0 \frac{(\lambda v_1, 0) + u_0}{\|\lambda v_1, 0\|} \in \text{int } K.$$

Thus

$$x, 1 + \eta \|\lambda v_1, 0\| + u_0 \frac{(\lambda v_1, 0) + u_0}{\|\lambda v_1, 0\|} = \eta \left( (y, 1) + \|\lambda v_1, 0\| + u_0 \frac{(\lambda v_1, 0) + u_0}{\|\lambda v_1, 0\|} \right) + (1 - \eta)(z, 1) \in \text{int } K.$$

It follows that $T \times \{1\}$ can be illuminated by $D$. \qed
In a similar way we can prove the following lemma.

**Lemma 3.4.** Suppose that \( B \) is not a singleton. Let \( m = c(B), \{ v_i : i \in [m] \} \subset \mathbb{R}^{n-1} \) be a set of directions illuminating \( B \), \( p \) be a point in \( \text{relint} \ B \), and \( q \) be a point in \( \text{relint} \ T \). Then there exist a positive number \( \lambda \) and a number \( \gamma \in (0, 1) \) such that \( B \times \{ 0 \} \) can be illuminated by

\[
D = \left\{ \left( \frac{\lambda v_i, 0}{||\lambda v_i, 0||}, \gamma \right) + u : i \in [m] \right\},
\]

where \( u_0 = (1 - \gamma)(-p + q, 1) \).

**Theorem 3.5.** \( c(K) \leq c(T) + c(B) \).

**Proof.** Suppose that one of \( T \) and \( B \), say \( T \), is a singleton. Then \( K \) is a convex cone having \( B \) as base. By Theorem 4 in [17] or by Theorem 3.11 below, we have

\[
c(K) \leq 1 + c(B) = c(T) + c(B).
\]

If neither \( T \) nor \( B \) is a singleton, then the desired inequality follows from Lemma 3.2, Lemma 3.3, and Lemma 3.4. \( \square \)

In general, the estimation given in Theorem 3.5 is best possible. See the following examples.

**Example 3.6.** Let

\[
T = [(-1, 0), (1, 0)] \subset \mathbb{R}^2 \text{ and } B = [(0, -1), (0, 1)] \subset \mathbb{R}^2.
\]

Then \( K \in \mathcal{K}_3^3 \) is a tetrahedron and \( c(K) = c(T) + c(B) \).

**Example 3.7.** Let \( T = B = [0, 1] \times [0, 1] \subset \mathbb{R}^2 \). Then \( K \in \mathcal{K}_3^3 \) is a cube and \( c(K) = c(T) + c(B) \).

**Lemma 3.8.** If \( B \) is contained in a translate of \( T \), then the minimal cardinality \( m \) of a set of directions in \( \mathbb{R}^n \) that can illuminate \( \text{relbd} T \times \{ 1 \} \) equals \( c(T) \).

**Proof.** By Lemma 3.3, we only need to prove that \( m \geq c(T) \).

Let \( t \in \mathbb{R}^{n-1} \) be a point such that \( B \subset T + t \), \( p \) an arbitrary point in \( \text{int} K \), and \( D = \{(u_i, \gamma_i) : i \in [m]\} \) be a set of directions in \( \mathbb{R}^n \) illuminating \( \text{relbd} T \times \{ 1 \} \). Then it is clear that each direction in \( D \) illuminates at least one point in \( \text{relbd} T \times \{ 1 \} \). It is not difficult to see that \( \gamma_i < 0, \forall i \in [m] \). By Lemma 2.3, there exist two positive numbers \( \lambda \) and \( \gamma \in (0, 1) \) such that, for each point \( (x, 1) \in \text{relbd} T \times \{ 1 \} \), there exists \( i \in [m] \) satisfying

\[
(x, 1) + \lambda(u_i, \gamma_i) \in \gamma \text{int} K + (1 - \gamma)p.
\]

Let

\[
\eta = \max \{ (y|e_n) : y \in \gamma K + (1 - \gamma)p \},
\]

where \( (y|e_n) \) is the inner product of \( y \) and the \( n \)th member of the standard basis of \( \mathbb{R}^n \). Then \( \eta \in (0, 1) \) and \( 1 + \lambda \gamma_i \leq \eta \) holds for each \( i \in [m] \).
For each point $x \in \text{relbd} T$, there exists $i \in [m]$ such that

$$(x, 1) + \lambda(u_i, \gamma_i) \in \gamma \text{int } K + (1 - \gamma)p.$$ 

Let $\lambda_i$ be the number such that $1 + \lambda_i \gamma_i = \eta$. Clearly, $\lambda_i \in (0, 1]$.

$$(x, 1) + \lambda_i \lambda(u_i, \gamma_i) = (x + \lambda_i \lambda u_i, 1 + \lambda_i \lambda \gamma_i)$$

$$\in [(x, 1), (x, 1) + \lambda(u_i, \gamma_i)] \setminus \{(x, 1)\} \cap H_\eta$$

$$\subset \text{int } K \cap H_\eta$$

$$= (\eta \text{relint } T + (1 - \eta) \text{relint } B) \times \{\eta\}$$

$$\subset (\eta \text{relint } T + (1 - \eta)B) \times \{\eta\}$$

$$\subset (\eta \text{relint } T + (1 - \eta)(T + t)) \times \{\eta\}$$

$$\subset (\text{relint } T + (1 - \eta)t) \times \{\eta\}.$$ 

It follows that

$$x + \lambda_i \lambda \left(u_i - \frac{1 - \eta}{\lambda_i \lambda} t\right) \in \text{relint } T.$$ 

Since $x \in \text{relbd} T$,

$$v_i := u_i - \frac{1 - \eta}{\lambda_i \lambda} t \neq o.$$ 

Hence

$$\left\{ \frac{v_i}{\|v_i\|} : i \in [m] \right\}$$

is a set of directions in $\mathbb{R}^{n-1}$ illuminating relbd $T$. Therefore $c(T) \leq m$. □

In a similar way, we can prove the following lemma.

**Lemma 3.9.** If $T$ is contained in a translation of $B$, then the minimal cardinality $m$ of a set of directions in $\mathbb{R}^n$ illuminating relbd $B \times \{0\}$ equals $c(B)$.

**Corollary 3.10.** If $T$ is a translate of $B$, then $c(K) = c(T) + c(B) = 2c(T)$.

For some special cases, the estimation of $c(K)$ can be improved. In the proof of the next theorem, we shall use the following result (see (6) of Theorem 2.15 in [16, p. 79]): if $A$ and $B$ are subsets of $\mathbb{R}^n$ such that a translate of $B$ lies in aff $A$, then relint $A + B \subset \text{relint } (A + B)$.

**Theorem 3.11.** Suppose that $T + c \subseteq \text{relint } B$ holds for some point $c \in \mathbb{R}^{n-1}$. Then $c(K) = 1 + c(B)$.

**Proof.** By Lemma 3.9, the minimal cardinality $m$ of a set of directions in $\mathbb{R}^n$ illuminating relbd $B \times \{0\}$ equals $c(B)$. By Lemma 2.1, $c(K) \geq c(B) + 1$. 

Put $u = (c, -1)/2$. Next we show that each point in $K_1 = T \times \{1\}$ can be illuminated by $u/\|u\|$:

$$K_1 + u = T \times \{1\} + \frac{1}{2}(c, -1)$$

$$= \left(\frac{1}{2}T\right) \times \{1\} + \left(\frac{1}{2}T\right) \times \{0\} + \frac{1}{2}(c, 0) + \frac{1}{2}(0, -1)$$

$$= \frac{1}{2}(T \times \{1\}) + \left(\frac{1}{2}(T + c)\right) \times \{0\}$$

$$\subseteq \frac{1}{2}(T \times \{1\}) + \frac{1}{2}(\text{relint } B \times \{0\})$$

$$= \left(\frac{1}{2}T + \frac{1}{2}\text{relint } B\right) \times \left\{\frac{1}{2}\right\}$$

$$\subseteq \left(\text{relint } \left(\frac{1}{2}T + \frac{1}{2}B\right)\right) \times \left\{\frac{1}{2}\right\}$$

$$\subseteq \text{relint } K_1 \subseteq \text{int } K.$$ 

Hence each point in $K_1$ can be illuminated by $u/\|u\|$. By Lemma 3.2, $c(K) \leq 1 + c(B)$. Therefore $c(K) = c(B) + 1$. \qed

**Remark.** Both Example 3.7 and Corollary 3.10 show that the condition $T + x \subseteq \text{relint } B$ in Theorem 3.11 cannot be replaced by $T + x \subseteq B$.

It is easy to verify the following:

**Lemma 3.12.** A set $D$ of directions in $\mathbb{R}^n$ illuminates $T \times \{1\}$ (resp. $B \times \{0\}$) if and only if $D$ illuminates the set of extreme points of $T \times \{1\}$ (resp. of $B \times \{0\}$).

**Theorem 3.13.** If $n = 3$ then $c(K) \leq 8$; equality holds if and only if $K$ is a parallelepiped.

**Proof.** Since $n = 3$, $T$ and $B$ are in $K^2$. Thus $c(K) \leq c(T) + c(B) \leq 4 + 4 = 8$.

Suppose that $c(K) = 8$ holds. It follows that $c(T) = c(B) = 4$. Therefore, $T$ and $B$ are two parallelograms, and each one of them has 4 extreme points. By a suitable translation if necessary, we may assume that $B$ is centered at the origin $o$ of $\mathbb{R}^{n-1}$. Let $t$ be the center of $T$. It is clear that $(t/2, 1/2) \in \text{int } K$. To show that $K$ is a parallelepiped it suffices to show that $T - t = B$.

Otherwise $T - t \nsubseteq B$ or $B \nsubseteq T - t$. We may assume, without loss of generality, that one vertex $v$ of $T - t$ is exterior to $B$. We distinguish two cases.
**Case 1:** The segment $[o,v]$ contains a vertex $w$ of $B$.

There exists a number $\lambda \in (0,1)$ such that $w = \lambda v$. In this case we have

\[
(w,0) + \frac{1}{2}(t,1) = \lambda(v,0) + \frac{1}{2}(t,1)
\]

\[
= \frac{2\lambda}{\lambda + 1} \cdot \frac{1}{2}((w,0) + (v + t,1)) + \frac{1 - \lambda}{\lambda + 1} \cdot \frac{1}{2}(t,1)
\]

\[
\in \text{int } K.
\]

In a similar way, we can show that $(-w,0) + (t/2,1/2) \in \text{int } K$. By Lemma 3.2 and Lemma 3.12, $\text{bd } K$ can be illuminated by at most 7 directions, a contradiction.

**Case 2:** The segment $[o,v]$ contains a point $p$ which is a relative interior point of an edge $[w,w']$ of $B$.

There exist two numbers $\lambda, \eta \in (0,1)$ such that $p = \eta v = \lambda w + (1 - \lambda)w'$. By interchanging $w$ and $w'$ if necessary, we may assume that $\lambda \leq 1 - \lambda$. In this case we have

\[
\frac{1}{2}(t,1) + (p,0) = \frac{1}{2}(t,1) + \eta(v,0)
\]

\[
= \frac{1 - \eta}{1 + \eta} \cdot \frac{1}{2}(t,1) + \frac{2\eta}{1 + \eta} \cdot \frac{1}{2}((1 + \eta)(v,0) + (t,1))
\]

\[
= \frac{1 - \eta}{1 + \eta} \cdot \frac{1}{2}(t,1) + \frac{2\eta}{1 + \eta} \cdot \frac{1}{2}((p,0) + (v,0) + (t,1))
\]

\[
= \frac{1 - \eta}{1 + \eta} \cdot \frac{1}{2}(t,1) + \frac{2\eta}{1 + \eta} \cdot \frac{1}{2}((p,0) + (v + t,1))
\]

\[
\in \frac{1 - \eta}{1 + \eta} \cdot \text{int } K + \frac{2\eta}{1 + \eta} \cdot \text{int } K
\]

\[
\subseteq \text{int } K.
\]

In a similar way we can show that

\[
\frac{1}{2}(t,1) - (p,0) \in \text{int } K.
\]

Put

\[
u = \frac{1}{\frac{1}{2}(t,1) + (p - w,0)} \left| \frac{1}{2}(t,1) + (p - w,0) \right|
\]

Thus

\[
(w,0) + \left| \frac{1}{2}(t,1) + (p - w,0) \right| \cdot u = \frac{1}{2}(t,1) + (p,0) \in \text{int } K,
\]
ON THE ILLUMINATION OF A CLASS OF CONVEX BODIES

and

\[- (w', 0) + \frac{\lambda}{1 - \lambda} \left( \frac{1}{2} (t, 1) + (1 - \lambda)(w' - w, 0) \right)\]

\[= - (w', 0) + \frac{\lambda}{1 - \lambda} \cdot \frac{1}{2} (t, 1) + \lambda(w' - w, 0)\]

\[= - (p, 0) + \frac{\lambda}{1 - \lambda} \cdot \frac{1}{2} (t, 1)\]

\[\in \text{int} K;\]

i.e., \((w, 0)\) and \(-(w', 0)\) can be illuminated by the direction \(u\). Thus \(K\) can be illuminated by at most 7 directions, again a contradiction.

\[\square\]

**Remark.** Let \(A^+ \subset K^n\) be the family of convex bodies having a summand in \(A\). Then, by Theorem 34.8 on p. 266 in [11], for each \(K' \in A^+\), we have

\[c(K') \leq \max \{c(K) : K \in A\}.\]

Thus our main results also yield good estimations of \(c(K)\) for \(A^+\).

**References**


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