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EQUALITY PERFECT GRAPHS AND DIGRAPHS

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Abstract. In the graph colouring game introduced by Bodlaender [7], two players, Alice and Bob, alternately colour uncoloured vertices of a given graph G with one of k colours so that adjacent vertices receive different colours. Alice wins if every vertex is coloured at the end. The game chromatic number of G is the smallest k such that Alice has a winning strategy. In Bodlaender's original game, Alice begins. We also consider variants of this game where Bob begins or skipping turns is allowed [1] and their generalizations to digraphs [2]. By means of forbidden induced subgraphs (resp. forbidden induced subdigraphs), for several pairs (g_1, g_2) of such graph (resp. digraph) colouring games g_1 and g_2, which define game chromatic numbers χ_{g_1} and χ_{g_2}, we characterise the classes of graphs (resp. digraphs) such that, for any induced subgraph (resp. subdigraph) H, the game chromatic numbers $\chi_{g_1}(H)$ and $\chi_{g_2}(H)$ of H are equal.

1. Introduction

The following graph colouring game was introduced by Bodlaender [7]. Two players, Alice and Bob, alternately colour vertices of a given simple, undirected graph with colours from a given colour set, so that adjacent vertices receive distinct colours. If at the end of the game when no further moves are possible, all vertices are coloured, Alice wins. Otherwise, if an uncoloured vertex is surrounded by neighbours in all colours, Bob wins.

In Bodlaender's original game, which we denote by g_A, Alice moves first. It is also convenient to consider the game g_B, where Bob moves first.

The *game chromatic number* of a graph G is the smallest number of colours such that Alice has a winning strategy in the graph colouring game. We denote it by $\chi_{g_A}(G)$ or $\chi_{g_B}(G)$, depending on which game (g_A or g_B) we consider.

If we consider the question of characterising the class of graphs G with

$$\chi_{g_A}(G) = \chi_{g_B}(G),$$

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we cannot expect to obtain a class of graphs with an interesting structure, since, for any graph G, if we add a large clique C of size

$$s \geq \Delta(G) + 1 \geq \max\{\chi_{g_A}(G), \chi_{g_B}(G)\},$$

where $\Delta(G)$ denotes the maximum degree of G, then

$$\chi_{g_A}(G \cup C) = s = \chi_{g_B}(G \cup C).$$

Therefore we consider a slightly modified problem. A graph G is *equality perfect* if, for any induced subgraph H of G,

$$\chi_{g_A}(H) = \chi_{g_B}(H).$$

We will characterise the class of equality perfect graphs. It turns out that this class is equal to the class of $[B, B]$-perfect graphs that were characterised by forbidden induced subgraphs and by an explicit structural characterisation in [4].

Moreover, we consider other variants of the graph colouring game, where skipping turns is allowed for one of the players, and define equality perfectness more generally between any pair of such games. For some of these pairs, we characterise the equality perfect graphs by a set of forbidden induced subgraphs, for the other pairs we provide partial results. In some cases we obtain the class of trivially perfect graphs, which appears in several contexts.

Finally, we extend the characterisations to digraph colouring games and propose a similar problem for marking games.

2. Preliminaries

The *graph colouring game* $[X, Y]$ with parameter $X \in \{A, B\}$ and parameter $Y \in \{A, B, -\}$ is defined as follows. Given an initially uncoloured, simple, undirected, finite graph $G = (V, E)$ and a set C of colours, two players, Alice and Bob, alternately colour uncoloured vertices with a colour from C such that adjacent vertices receive distinct colours. The player denoted by $X \in \{A, B\}$ begins first, where "A" means "Alice" and "B" means Bob for short. The parameter $Y \in \{A, B, -\}$ denotes whether Alice (A), Bob (B), or none of the players ($-$) has the right to skip any number of turns, respectively. In particular, this right includes the right to skip the first turn. The game ends if no move is possible. Alice wins if every vertex is coloured at the end of the game. Otherwise there is a *surrounded* vertex, i.e., an uncoloured vertex with neighbours in every colour. In the latter case, Bob wins. Thus, for any (X, Y), the game $[X, Y]$ is a maker-breaker game, where Alice is the maker who tries to make a complete colouring whereas the breaker, Bob, tries to prevent such a situation. We remark that the standard graph colouring games g_A and g_B defined in the introduction are the special cases $g_A = [A, -]$ and $g_B = [B, -]$, respectively.

The *game chromatic number* $\chi_{[X,Y]}(G)$ of G with regard to the game $[X, Y]$ is the smallest size $|C|$ of a colour set C such that Alice has a winning strategy for the graph colouring game $[X, Y]$ played on G.

Determining good upper bounds for the maximum game chromatic numbers of several classes of graphs has received considerable attention. Among the many classes of graphs that have been investigated are forests [12], cactuses [19], outerplanar graphs [14], planar graphs [6, 15, 26], other graphs of fixed genus [15, 25], line graphs of k-degenerate graphs [8], line graphs of planar graphs and line graphs of other graphs admitting some special kind of decomposition [10], and incidence graphs of graphs admitting some very general kind of decomposition [11].

The tightness of some of these bounds (resp. lower bounds) have also been considered, e.g., for the class of forests [7], planar graphs and k-trees [22], or planar graphs with large girth [9].

It is well-known that the difference between the game chromatic numbers of a graph for different games can be arbitrarily large. A popular example which first was mentioned by Kierstead [15] is the graph $K_{n,n} - M$, which is a complete bipartite graph with bipartitions of size n in which a perfect matching M is deleted. Here Alice wins with 2 colours in the game g_B, but needs n colours to win in the game g_A (cf. Figure 1). On the other hand, if we add an isolated vertex v to $K_{n,n} - M$, the situation switches: on $(K_{n,n} - M) \cup K_1$, Alice wins with 2 colours in the game g_A, but needs n colours to win in the game g_B. This shows that this type of game is very sensitive with regard to rule changes (resp. small changes in the play graph).

(a) Alice begins (b) Bob begins

FIGURE 1. In the depictions of $K_{n,n} - M$, the vertices of the left bipartite class are denoted by v_i, those of the right bipartite class by w_i, respectively ($i = 1, \ldots, n$), where v_i and w_j are adjacent if and only if $i \neq j$. (a) In game $[A, -]$, whenever Alice colours v_i, Bob colours w_i with the same colour. For the remaining vertices new colours are needed. With less than n colours, Alice cannot win. (b) In game $[B, -]$ played with 2 colours, if Bob colours say v_1 with colour 1, then Alice colours w_1 with colour 2. All colours of the remaining vertices are fixed now, thus Alice wins with 2 colours.

Therefore it is interesting to characterise those graphs, for which the game chromatic number for a pair of games is equal for any induced subgraphs.

Formally, we define for any pairs $(X, Y), (X', Y') \in \{A, B\} \times \{A, B, -\}$ that a graph G is $([X, Y], [X', Y'])$-*equality perfect* if, for any induced subgraph H of G,

$$\chi_{[X,Y]}(H) = \chi_{[X',Y']}(H).$$

An $([A, -], [B, -])$-equality perfect graph is also called simply *equality perfect*. The class of all $([X, Y], [X', Y'])$-equality perfect graphs is denoted by

$$EP([X, Y], [X', Y']).$$

It turns out that this notion is strongly related to the notion of game-perfect graphs. For $(X, Y) \in \{A, B\} \times \{A, B, -\}$, a graph G is *game-perfect with regard to the game* $[X, Y]$ (or $[X, Y]$-*perfect* for short) if, for any induced subgraph H of G,

$$\chi_{[X,Y]}(H) = \omega(H),$$

where $\omega(H)$ denotes the size of a largest clique in H. The class of all game-perfect graphs with regard to the game $[X, Y]$ is denoted by

$$GP[X, Y].$$

Obviously, for any graph H,

$$\chi_{[B,B]}(H) \begin{Bmatrix} \geq \chi_{[A,B]}(H) \geq \chi_{[A,-]}(H) \geq \\ \geq \chi_{[B,-]}(H) \geq \chi_{[B,A]}(H) \geq \end{Bmatrix} \chi_{[A,A]}(H) \geq \chi(H), \tag{2.1}$$

where $\chi(H)$ denotes the chromatic number of H. Together with the inequality $\chi(H) \geq \omega(H)$, the inequalities (2.1) imply immediately

$$GP[B, B] \begin{Bmatrix} \subseteq GP[A, B] \subseteq GP[A, -] \subseteq \\ \subseteq GP[B, -] \subseteq GP[B, A] \subseteq \end{Bmatrix} GP[A, A] \subseteq P, \tag{2.2}$$

where P denotes the class of perfect graphs. In fact, $GP[A, B] = GP[A, -]$ and the other inclusions in (2.2) are proper [4].

Game-perfect graphs were introduced in [1]. Forbidden induced subgraph characterisations and explicit structural characterisations of the four classes $GP[B, B]$, $GP[A, B]$, $GP[A, -]$, and $GP[B, -]$ and partial characterisations of the other two classes $GP[B, A]$ and $GP[A, A]$ are known [4, 5, 17].

Before we formulate the characterisation of $[B, B]$-perfect graphs we fix some notation. By $\overline{G}$ we denote the complement of a graph G. Now, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs and $G_2' = (V_2', E_2')$ an isomorphic copy of G_2 with $V_1 \cap V_2' = \emptyset$. The *disjoint union* of the graphs G_1 and G_2 is defined as the graph $G_1 \cup G_2 := (V_1 \cup V_2', E_1 \cup E_2')$, i.e., the graph where G_1 and G_2' are disconnected. The *join* of the graphs G_1 and G_2 is defined as the graph

$$G_1 \vee G_2 := \overline{\overline{G_1} \cup \overline{G_2}},$$

i.e., the graph where G_1 and G_2' are completely connected. By P_n, C_n, and K_n we denote the path, cycle, and the complete graph with n vertices, respectively.

Theorem 2.1 ([4, Thm. 3]). *Let G be a graph. Then the following conditions are equivalent.*

(i) $G \in GP[B, B]$.
(ii) *G neither contains P_4, nor C_4, nor the* split 3-star $K_2 \vee \overline{K_3}$, *nor the* double fan $K_1 \vee (P_3 \cup P_3)$ *(see Figure 2) as an induced subgraph.*
(iii) *Every component of G is an* ear animal

$$K_1 \vee (K_{m_1} \cup \ldots \cup K_{m_k} \cup (K_h \vee (K_{e_1} \cup K_{e_2})))$$

for some $k, h, e_1, e_2, m_1, \ldots, m_k \geq 0$ (see Figure 3).

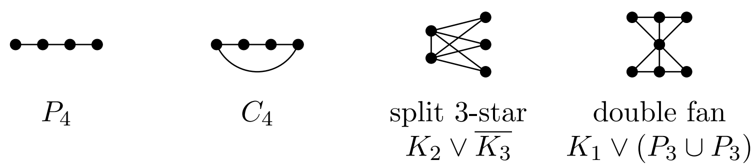


FIGURE 2. Forbidden configurations in game-perfect graphs for the game $[B, B]$.

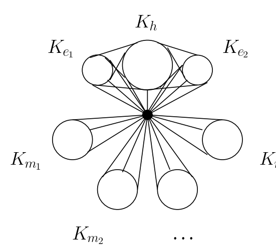


FIGURE 3. An ear animal.

A graph is *trivially perfect* if it neither contains P_4 nor C_4 as an induced subgraph. We denote the class of all trivially perfect graphs by TP. By a result of Wolk [20, 21], TP is the class of comparability graphs of forests of rooted trees. Golumbic [13] showed that TP is the class of all graphs where, for each induced subgraph, the stability number and the number of maximal cliques are equal, which motivated him to give trivially perfect graphs its name.

Since Bob wins Bodlaender's graph colouring game with two colours on P_4 and C_4,

$$GP[B, B] \overset{(2.2)}{\subseteq} GP[A, -] \subseteq TP.$$

However, there are many $[B, -]$-perfect graphs that are not trivially perfect: P_4, C_4, the bull, or the house are among the smallest examples.

Trivially perfect graphs admit a decomposition based on a tree, which relies on the following result of Wolk [21]. A *central vertex* of a graph G is a vertex adjacent to every other vertex of G.

Lemma 2.2 ([21, p. 18f]). *Every connected trivially perfect graph has a central vertex.*

Let $G = (V, E)$ be a trivially perfect graph. The *clique module decomposition forest* of G is constructed recursively from G using Lemma 2.2 as follows. In each component of G we replace the nonempty complete graph consisting of all central vertices of the component by a single vertex. The set of these central vertices is called *layer* L_0. For any $i \geq 1$, if we delete the vertices of all layers L_j, $j < i$, the remaining graph is trivially perfect by the definition of trivially perfect graphs, thus by Lemma 2.2 all of its components contain at least one central vertex. For any such component, we again replace the complete graph consisting of all central vertices of the component by a single vertex. The set of these central vertices is called *layer* L_i. Keeping the adjacencies of the original graph G, the graph obtained by this procedure is a comparability graph of a forest of rooted trees each vertex of which represents a *maximal clique module*, i.e., an inclusion-wise maximal clique K such that, for every vertex $v \in V \setminus K$, v is either completely connected or not at all connected to K. If we delete the transitive edges in this graph, we obtain a unique forest, which is the *clique module decomposition forest*. We will use this clique module decomposition in the proof of Theorem 2.3 (ii) and (iv). The clique module decomposition forest is related to the notion of elimination trees. In elimination trees each vertex of a clique module is eliminated one by one, which, in general, is not unique, since the vertices of a clique module can be chosen in an arbitrary order, whereas in the clique module decomposition the whole clique is replaced by a vertex. Elimination trees have been used for several problems, e.g., for the Cholesky factorization of sparse square matrices [16].

We can now formulate our main result in the case of undirected graphs. In (iii) and (iv) of Theorem 2.3, let $\mathcal{D}$ be the class of disconnected graphs.

Theorem 2.3. *The structure of equality perfect graphs with regard to some pairs of games can be described as follows.*

(i) $EP([A, A], [B, B]) = GP[B, B]$,
$EP([A, -], [B, -]) = GP[B, B]$,
$EP([A, B], [B, -]) = GP[B, B]$,
$EP([A, -], [B, A]) = GP[B, B]$,
$EP([A, B], [B, A]) = GP[B, B]$.

(ii) $EP([A, A], [A, -]) = TP$,
$EP([A, A], [A, B]) = TP$,
$EP([B, A], [B, B]) = TP$,
$EP([B, -], [B, B]) = TP$.

(iii) $EP([A, -], [B, B]) \cap \mathcal{D} = GP[B, B] \cap \mathcal{D}$,
$EP([A, A], [B, -]) \cap \mathcal{D} = GP[B, B] \cap \mathcal{D}$.

(iv) $EP([A, -], [A, B]) \cap \mathcal{D} = TP \cap \mathcal{D}$,
$EP([B, A], [B, -]) \cap \mathcal{D} = TP \cap \mathcal{D}$.

We remark that in Theorem 2.3 only 13 of the 15 possible pairs of games have been considered. Some partial results for the pairs $\{[B,A],[A,A]\}$ and $\{[A,B],[B,B]\}$, which are missing in Theorem 2.3, are given in Section 5.

3. Equality Perfect Graphs: Proof of Theorem 2.3

The relation between equality perfectness and game-perfectness is given by the following simple but fundamental observation, the corollary of which will be frequently used in the proof of Theorem 2.3.

Observation 3.1. *For any $X, X' \in \{A, B\}$, $Y, Y' \in \{A, B, -\}$,*

$$GP[X,Y] \cap GP[X',Y'] \subseteq EP([X,Y],[X',Y']).$$

Proof. Let $G \in GP[X,Y] \cap GP[X',Y']$ and H be an induced subgraph of G. Then

$$\chi_{[X,Y]}(H) = \omega(H) = \chi_{[X',Y']}(H),$$

thus $G \in EP([X,Y],[X',Y'])$. □

Corollary 3.2. *For any $X, X' \in \{A, B\}$, $Y, Y' \in \{A, B, -\}$,*

$$GP[B,B] \subseteq EP([X,Y],[X',Y']).$$

Proof. $GP[B,B] \overset{(2.2)}{\subseteq} GP[X,Y] \cap GP[X',Y'] \overset{\text{Obs. 3.1}}{\subseteq} EP([X,Y],[X',Y'])$. □

The proof of Theorem 2.3 uses the following three lemmata.

Lemma 3.3. *Let G be the split 3-star $K_2 \vee \overline{K_3}$ or the double fan $K_1 \vee (P_3 \cup P_3)$ and $Y \in \{A, B, -\}$. Then*

(a) $\chi_{[A,Y]}(G) = 3$,
(b) $\chi_{[B,Y]}(G) = 4$.

Lemma 3.4. *Let G be P_4 or C_4. Then*

(a) $\chi_{[B,-]}(G) = \chi_{[B,A]}(G) = \chi_{[A,A]}(G) = 2$,
(b) $\chi_{[A,-]}(G) = \chi_{[A,B]}(G) = \chi_{[B,B]}(G) = 3$.

Lemma 3.5. *Let G be $P_4 \cup K_1$, $C_4 \cup K_1$, the* 4-fan *$P_4 \vee K_1$, or the* 4-wheel *$C_4 \vee K_1$. Then*

(a) $\chi_{[A,-]}(G) = \chi_{[B,A]}(G) = \chi_{[A,A]}(G) = \omega(G)$,
(b) $\chi_{[B,-]}(G) = \chi_{[A,B]}(G) = \chi_{[B,B]}(G) = \omega(G) + 1$.

Lemma 3.3, Lemma 3.4, and Lemma 3.5 are (mainly) proved by describing explicit winning strategies with $\omega(G)$ colours for Alice in case (a) and for Bob in case (b), respectively. We first make two obvious remarks that will be implicitly used in all the proofs of the lemmata without further mentioning.

Remark 3.6. For any graph G, Bob wins any variant of the colouring game with $k < \omega(G)$ colours.

Remark 3.7. For any graph G with maximum degree Δ, Alice wins any variant of the colouring game with $k \geq \Delta + 1$ colours.

Proof of Lemma 3.3. (a) A winning strategy for Alice with 3 colours in a game, where Alice begins, played on the split 3-star, is the following. In her first move, Alice colours a vertex of degree 4 with the first colour. No matter what Bob does, with her next move Alice can create a situation in which both vertices of degree 4 are coloured. Since the other two or three vertices can be coloured in any case, she wins.

A winning strategy for Alice with 3 colours in a game, where Alice begins, played on the double fan is the following. In her first move, Alice colours the central vertex with the first colour. If Bob colours an end vertex of a P_3 in the graph of the remaining uncoloured vertices, Alice colours the other end vertex of the same P_3 with the same colour. Otherwise, she colours the central vertex of a P_3. By this strategy, the end vertices of the two P_3 will have the same colour, thus Alice wins.

(b) A winning strategy for Bob with 3 colours in a game, where Bob begins, played on the split 3-star, is the following. In his first move, Bob colours a vertex of degree 2 with the first colour. No matter what Alice does, with his next move Bob can create a situation in which two of the vertices of degree 2 are coloured differently. Since at least one of the two vertices of degree 4 cannot be coloured any more, Bob wins. This proves the inequality $\chi_{[B,Y]}(K_2 \vee \overline{K_3}) \geq 4$.

In order to prove the inverse inequality $\chi_{[B,Y]}(K_2 \vee \overline{K_3}) \leq 4$, we describe a winning strategy for Alice with 4 colours on the split 3-star: If Bob colours a vertex of degree 4, she colours the second vertex of degree 4. If Bob colours a vertex of degree 2, she colours another vertex of degree 2 with the same colour. After this pair of moves, it is impossible for a vertex of degree 4 to be surrounded by vertices of all colours.

A winning strategy for Bob with 3 colours in a game, where Bob begins, played on the double fan, is the following. In his first move, Bob colours the upper left vertex with colour 1. To prevent Bob from colouring the upper right vertex with colour 2, Alice must colour the upper right vertex with colour 1. Then Bob colours the lower left vertex with colour 2, and in the same way, Alice must colour the lower right vertex with colour 2. Then Bob colours a vertex of degree 3 with colour 3 and wins, since the central vertex cannot be coloured. This proves the inequality $\chi_{[B,Y]}(K_1 \vee (P_3 \cup P_3)) \geq 4$.

In order to prove the inverse inequality $\chi_{[B,Y]}(K_1 \vee (P_3 \cup P_3)) \leq 4$, we describe a winning strategy for Alice with 4 colours on the double fan: No matter what Bob does, in her first move Alice can create a situation such that the central vertex is coloured. After that she wins since every uncoloured vertex has degree at most 3 and there are 4 colours available. $\square$

Proof of Lemma 3.4. (a) In the games where Alice can force Bob to begin, Bob must colour a vertex of G and Alice colours a vertex at distance 2 with the same colour. Therefore only 2 colours are needed.

(b) In the games, where Bob can force Alice to begin, Alice must colour a vertex of G and Bob colours a vertex at distance 2 with a different colour. Then a third colour is needed to complete the colouring. □

Proof of Lemma 3.5. (a) In the games where Alice can force Bob to begin the colouring of the induced P_4 (resp. C_4), whenever Bob colours a vertex of the P_4 (resp. C_4), Alice colours a vertex at distance 2 with the same colour. Therefore only $\omega(G)$ colours are needed.

(b) In the games where Bob can force Alice to begin the colouring of the induced P_4 (resp. C_4), at some point, Alice must colour a vertex of the (P_4 resp. C_4) and Bob colours a vertex at distance 2 with a different colour. Then at least $\omega(G) + 1$ colours are needed to complete the colouring. □

The following key lemma states that the game chromatic number of trivially perfect graphs depends only on the player who is starting.

Lemma 3.8. *Let $G \in TP$. Then skipping turns is not an advantage for the first player, i.e.,*

$$\chi_{[X,Y]}(G) = \chi_{[X,X]}(G)$$

for any $X \in \{A, B\}$, $Y \in \{A, B, -\}$.

The idea of the proof of Lemma 3.8 consists in describing a strategy for Alice for the game $[X, B]$ under the assumption that she has a winning strategy for the game $[X, A]$, which Alice will use as a basis for her strategy. Whenever her basic strategy does not tell her what to do, she uses the clique module decomposition of G and chooses an uncoloured vertex v from a clique module from the layer L_i with the smallest i such that L_i contains an uncoloured vertex, and she colours v with any feasible colour. It can be shown that there is such a feasible colour and that this colouring does not help Bob to surround a vertex, in particular Bob cannot exploit this colouring of v to surround any other vertex later.

Proof of Lemma 3.8. Let $\mathcal{F}$ be the clique module decomposition forest of G. $\mathcal{F}$ consists of trees which are rooted in the clique modules of layer L_0. For these roots we can define the parent, children, predecessors, and successors of a clique module. For a vertex v of G, let $M(v)$ be the vertex of $\mathcal{F}$ (i.e., the clique module of G) that contains the vertex v. Furthermore, let $B(v)$ be the branch of $\mathcal{F}$ which is rooted in the clique module $M(v)$ and contains $M(v)$ and all its successors. We will use the clique module decomposition and this notation frequently in the proof.

Let $X \in \{A, B\}$. Since, by (2.1), $\chi_{[X,A]}(G) \leq \chi_{[X,-]}(G) \leq \chi_{[X,B]}(G)$, we only have to prove that if Alice has a winning strategy with k colours in the game $[X, A]$ on the trivially perfect graph G, she also has a winning strategy with at most k colours in the game $[X, B]$. Assume Alice has a winning strategy in the game $[X, A]$. Alice will use the same strategy for the game $[X, B]$ whenever this is possible. That means her basic strategy for the game $[X, A]$ tells her which vertex she should choose in the real game $[X, B]$, and, if possible, she takes the same colour for this vertex as her strategy tells her, otherwise she chooses any feasible colour. However, it might be impossible to apply her basic strategy if, by her strategy she should skip, or if by her strategy she should colour a vertex already coloured, or if her strategy does not apply since Bob skips. In all these cases she considers the layers L_i of vertices defined in the construction of the clique module decomposition forest of G and colours a vertex v in the layer L_i with the smallest label i among all layers with uncoloured vertices.

We consider the case that Alice cannot apply her basic strategy to choose a vertex and we first have to prove that

Alice has a feasible colour to perform this move.

This follows from the fact that in the imagined game $[X, A]$ the vertex v would be still uncoloured and, since Alice has a winning strategy for the imagined game, even later Alice will have a feasible colour for v in the game $[X, A]$, thus she has a feasible colour now (at the time when the move is performed) in the game $[X, A]$. Since, by Alice's strategy, all neighbours of v that have been coloured in the game $[X, B]$ but not in the imagined game $[X, A]$ lie in the same clique consisting of $M(v)$ and all its predecessor clique modules in $\mathcal{F}$, there are no two nonadjacent such neighbours with distinct colours. Note that an additional colour in the game $[X, B]$ would only be needed if there are two nonadjacent neighbours of an uncoloured vertex which are coloured by different colours in the game $[X, B]$ but will be coloured with the same colours in the game $[X, A]$. As there are no such nonadjacent neighbours of v whatsoever, since all such neighbours form a clique, there must be a feasible colour for v in the game $[X, B]$, as well.

Secondly, we have to prove that

Colouring v does not help Bob to surround a vertex.

In particular this means that Bob cannot exploit this colouring of v later to surround any other vertex. Let u be an uncoloured neighbour of v. We will prove that every neighbour z of u is adjacent to v. This will prove our second assertion since, as already argued above, only the existence of nonadjacent neighbours of an uncoloured vertex may lead to surrounding the uncoloured vertex. We distinguish several cases.

The vertex u cannot be a member of a layer L_j with $j < i$, since, when Alice colours v, every vertex in such a layer L_j is already coloured by Alice's strategy.

In case u is a member of layer L_i, then, by the definition of $\mathcal{F}$, u belongs to $M(v)$, since members of other clique modules in L_i are nonadjacent to v. By the definition of a clique module, u and v have the same neighbours.

In case u is a member of layer L_j with $j > i$, the vertex u must be in a clique module corresponding to a vertex of $B(v)$. Each neighbour z of u must lie either also in a clique module corresponding to a vertex of $B(v)$ or in a predecessor clique module of $M(v)$. In the first subcase $M(z)$ is a successor of $M(v)$. Since v is a central vertex for the graph induced by the vertices of the clique modules of $B(v)$, v is adjacent to z. In the second subcase z is a central vertex of the graph induced by the vertices of the clique modules of $B(z)$, and $M(v)$ is a vertex of $B(z)$, in particular z is adjacent to v.

This completes the discussion of the case that Alice cannot apply her strategy to choose a vertex.

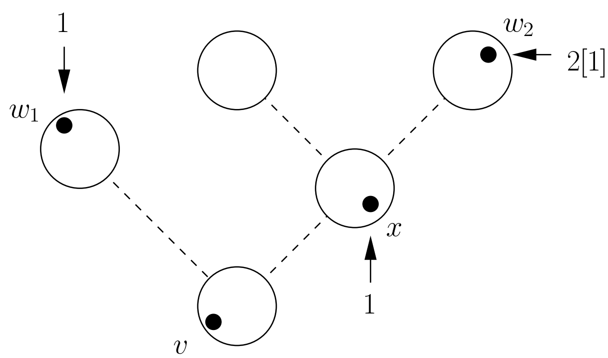


FIGURE 4. Proof of the third assertion in the proof of Lemma 3.8.

Now we consider the case that Alice is forced to use a different colour in the game $[X, B]$ than in the game $[X, A]$. As a third assertion we have to prove that

> *If Alice is forced to use a different colour in the game* $[X, B]$ *than in the game* $[X, A]$ *this does not help Bob surround a vertex.*

Assume the contrary, an uncoloured vertex v in layer L_i is going to be surrounded by this type of move, i.e., some vertex w_1 in a clique module of $B(v)$ is coloured by colour 1 and a vertex w_2 in another clique module of $B(v)$ should be coloured by colour 1 by Alice's strategy for the game $[X, A]$, but cannot be coloured by this colour in the real game $[X, B]$, thus Alice colours w_2 with colour 2. We assume that the layer L_i of vertex v is such that i is minimal, and among all vertices that have a property like w_2 corresponding to v, w_2 is chosen to be in a layer L_j with minimal j. Since w_2 cannot be coloured with colour 1, there must be a clique module on the path in $\mathcal{F}$ from $B(v)$ to $B(w_2)$ that contains a vertex x that is coloured with colour 1 in the real game $[X, B]$ (see Figure 4). Since $[X, A]$ cannot be coloured with colour 1 in the imagined game $[X, A]$, there are only two reasons how that can happen: either (I) x also had to be coloured in a different colour than by Alice's strategy for game $[X, A]$ or (II) the vertex x was chosen in some move because Alice could not apply her basic strategy.

Case (I) contradicts the minimality of the layers of v and w_2.

Case (II), which is shown in Figure 4, implies that when x was chosen, vertex v was already coloured since it is in a lower layer than x, which contradicts the fact that v is not coloured.

Therefore our assumption is wrong and the lemma is proven. $\square$

After these preparations we prove our main result.

Proof of Theorem 2.3. (i) Let EP be one of the five classes

$$\begin{array}{ll} EP([A,A],[B,B]), & EP([A,-],[B,-]), \\ EP([A,B],[B,-]), & EP([A,-],[B,A]), \text{or} \\ EP([A,B],[B,A]), & \end{array}$$

see Figure 5 (a).

By Corollary 3.2, $GP[B,B] \subseteq EP$.

For the other implication, let $G \in EP$. By Lemma 3.3, G neither contains the split 3-star nor the double fan as an induced subgraph. By Lemma 3.4, G neither contains P_4 nor C_4 as an induced subgraph. This implies by Theorem 2.1 that $G \in GP[B,B]$.

(ii) Let EP be one of the four classes $EP([A,A],[A,-])$, $EP([A,A],[A,B])$, $EP([B,A],[B,B])$, or $EP([B,-],[B,B])$, see Figure 5(b).

Let $G \in TP$ be a trivially perfect graph and H be a subgraph of G. Then H is trivially perfect. By Lemma 3.8, for any $X \in \{A,B\}$, $Y \in \{A,B,-\}$,

$$\chi_{[X,Y]}(H) = \chi_{[X,X]}(H),$$

i.e., $G \in EP([X,X],[X,Y])$. In particular, $G \in EP$.

For the other implication, let $G \in EP$. By Lemma 3.4, G neither contains P_4 nor C_4 as an induced subgraph. Thus, by definition, $G \in TP$.

(iii) Let EP be one of the classes $EP([A,-],[B,B])$ or $EP([A,A],[B,-])$, see Figure 5(c).

In line with (i), by Corollary 3.2, $GP[B,B] \subseteq EP$. Thus

$$GP[B,B] \cap \mathcal{D} \subseteq EP \cap \mathcal{D}.$$

For the other implication, let $G \in EP \cap \mathcal{D}$. By Lemma 3.3, G neither contains the split 3-star nor the double fan as an induced subgraph. By Lemma 3.5, G neither contains $P_4 \cup K_1$ nor $C_4 \cup K_1$ as an induced subgraph. Since, by assumption, G has at least two components, this means that none of the components of G contains a P_4 or a C_4. This implies by Theorem 2.1 that $G \in GP[B,B]$. Thus $G \in GP[B,B] \cap \mathcal{D}$.

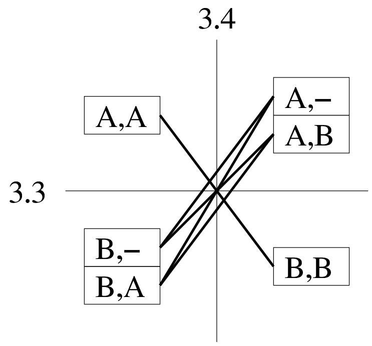
(iv) Let EP be one of the classes $EP([A,-],[A,B])$ or $EP([B,A],[B,-])$, see Figure 5(d).

Let $G \in TP$ be a trivially perfect graph and H be a subgraph of G. Then H is trivially perfect. By Lemma 3.8, for any $X \in \{A,B\}$, $Y, Y' \in \{A,B,-\}$,

$$\chi_{[X,Y]}(H) = \chi_{[X,X]}(H) = \chi_{[X,Y']}(H),$$

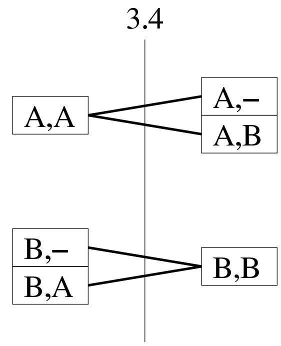
i.e., $G \in EP([X,Y],[X,Y'])$. In particular, $G \in EP$. Thus

$$TP \cap \mathcal{D} \subseteq EP \cap \mathcal{D}.$$



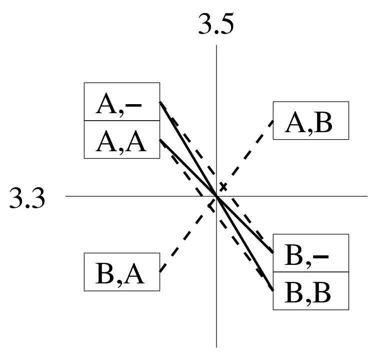
(a)

The pairs of games separated by Lemma 3.3 and Lemma 3.4 in (i) of Theorem 2.3.



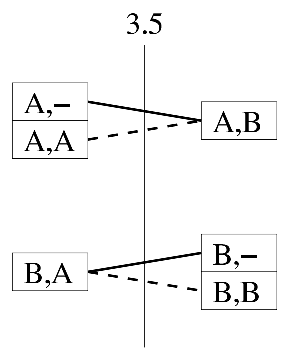
(b)

The pairs of games (with the same player beginning) separated by Lemma 3.4 in (ii) of Theorem 2.3.



(c)

The pairs of games separated by Lemma 3.3 and Lemma 3.5 in (iii) of Theorem 2.3. Pairs with dashed lines are already contained in (i).



(d)

The pairs of games (with the same player beginning) separated by Lemma 3.5 in (iv) of Theorem 2.3. Pairs with dashed lines are already contained in (ii).

FIGURE 5.

For the other implication, let $G \in EP \cap \mathcal{D}$. By Lemma 3.5, G neither contains $P_4 \cup K_1$ nor $C_4 \cup K_1$ as an induced subgraph. Since, by assumption, G has at least two components, this means that none of the components of G

contains P_4 or C_4. Therefore every component of G is trivially perfect. Thus $G \in TP \cap \mathcal{D}$. □

From Theorem 2.3, using Theorem 2.1, we immediately get the following characterisation of equality perfect graphs for some pairs of games by sets of forbidden induced subgraphs.

Corollary 3.9. *A graph is equality perfect*

(i) *with regard to a pair of games as in Theorem 2.3 (i) if and only if it contains no induced P_4, C_4, split 3-star, or double fan;*
(ii) *with regard to a pair of games as in Theorem 2.3 (ii) if and only if it contains no induced P_4 or C_4.*

4. Equality Perfect Digraphs

In this section we consider digraphs without loops and multiple arcs, however, pairs of antiparallel arcs (v, w) and (w, v) are allowed. If we regard such pairs as undirected edges, a natural generalization of graph colouring to digraphs is given by the dichromatic number introduced by Neumann-Lara [18]. The *dichromatic number* of a digraph D is the smallest number of induced acyclic digraphs that cover all vertices of D.

Two digraph colouring games motivated by this notion were proposed. In the *strong digraph colouring game* $s[X, Y]$ introduced in [2], Alice and Bob alternately colour uncoloured vertices of a given digraph D with a colour c from a given colour set such that c is different from the colours of the previously coloured in-neighbours. In the *weak digraph colouring game* $w[X, Y]$ introduced by Yang and Zhu [23], Alice and Bob alternately colour uncoloured vertices of a given digraph D with a colour from a given colour set such that no monochromatic directed cycles are created. In both types of games, $X \in \{A, B\}$ denotes the player who begins, $Y \in \{A, B, -\}$ is the player who is allowed to skip (if $Y \neq -$), and Alice wins if every vertex is coloured at the end. The smallest number of colours such that Alice has a winning strategy in the game $s[X, Y]$ and $w[X, Y]$ is the *game chromatic number* $\chi_{s[X,Y]}$ and $\chi_{w[X,Y]}$ of D, respectively. Note that both the strong and the weak game chromatic number of a symmetric digraph S are equal to the game chromatic number of its underlying graph G_S. In the following, we identify G_S with S.

We define the classes $EPs([X, Y], [X', Y'])$ of *strongly equality perfect digraphs* and $EPw([X, Y], [X', Y'])$ of *weakly equality perfect digraphs* with regard to the pair $([X, Y], [X', Y'])$ with $X, X' \in \{A, B\}$, $Y, Y' \in \{A, B, -\}$ as the classes of those digraphs for which, for any induced subdigraph H,

$$\chi_{s[X,Y]}(H) = \chi_{s[X',Y']}(H) \quad \text{or} \quad \chi_{w[X,Y]}(H) = \chi_{w[X',Y']}(H),$$

respectively. Notions of *game-perfect digraphs* have been introduced in [3]. Observation 3.1 can be generalized with these notions to digraphs.

Lemma 4.1. *Let $Y \in \{A, B, -\}$ and $\vec{P}_2$ be the directed path on two vertices. Then*

(a) $\chi_{s[A,Y]}(\vec{P}_2) = 1$,
(b) $\chi_{s[B,Y]}(\vec{P}_2) = 2$.

Proof. (a) Playing with one colour, Alice colours the terminal vertex of the single arc and wins, since the other vertex can be coloured with the same colour.

(b) Playing with one colour, Bob colours the initial vertex of the single arc and wins, since the other vertex cannot be coloured with the same colour. □

Corollary 4.2. *For any $Y, Y' \in \{A, B, -\}$, the class $EPs([A, Y], [B, Y'])$ is contained in the class of undirected graphs (symmetric digraphs).*

Proof. Let $D \in EPs([A, Y], [B, Y'])$. By Lemma 4.1, D is symmetric. □

Theorem 4.3. *The structure of strongly equality perfect digraphs with regard to some pairs of games can be described as follows.*

(i) $EPs([A, A], [B, B]) = EP([A, A], [B, B]) = GP[B, B]$,
$EPs([A, -], [B, -]) = EP([A, -], [B, -]) = GP[B, B]$,
$EPs([A, B], [B, -]) = EP([A, B], [B, -]) = GP[B, B]$,
$EPs([A, -], [B, A]) = EP([A, -], [B, A]) = GP[B, B]$,
$EPs([A, B], [B, A]) = EP([A, B], [B, A]) = GP[B, B]$.

(ii) $EPs([A, A], [B, -]) = EP([A, A], [B, -])$,
$EPs([A, -], [B, B]) = EP([A, -], [B, B])$,
$EPs([A, A], [B, A]) = EP([A, A], [B, A])$,
$EPs([A, B], [B, B]) = EP([A, B], [B, B])$.

Proof. Combining Corollary 4.2 with Theorem 2.3 (i) we obtain the statements of Theorem 4.3 (i). The statements of Theorem 4.3 (ii) are immediate from Corollary 4.2. □

Corollary 4.4. *A digraph is strongly equality perfect with regard to a pair of games as in Theorem 4.3 (i) if and only if it contains no induced P_4, C_4, split 3-star, double fan, or directed path $\vec{P}_2$.*

5. Final remarks and Open Problems

For the classes not mentioned in Theorem 2.3 (i) or (ii), or in Theorem 4.3 (i), partial lists of minimal forbidden induced subgraphs (resp. partial lists of minimal forbidden induced subdigraphs) result from our work. However, we do not know whether these lists are complete.

Problem 5.1. *Let $\mathcal{C}$ be the class of connected graphs. Characterise the classes*

$$EP([A, -], [B, B]) \cap \mathcal{C}, \quad EP([B, -], [A, A]) \cap \mathcal{C}.$$

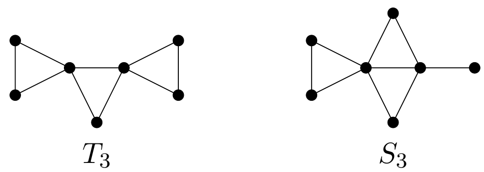


FIGURE 6. The triple triangle T_3 and the triple sword S_3.

Lemma 5.2. *Let* $Y \in \{A, B, -\}$*,* T_3 *be the* triple triangle*, and* S_3 *be the* triple sword *which are depicted in Figure 6.*

(a) $\chi_{[A,Y]}(T_3) = \chi_{[A,Y]}(S_3) = 3.$
(b) $\chi_{[B,Y]}(T_3) = \chi_{[B,Y]}(S_3) = 4.$

Proof. (a) We describe a winning strategy for Alice for the game $[A, Y]$ played on the graphs T_3 or S_3 with 3 colours. Both graphs have two adjacent vertices of degree at least 4, which we call *middle vertices*, all other vertices are of degree at most 2, thus can be coloured in any case. In her first move Alice colours a middle vertex with colour 1. No matter what Bob does, by her next move Alice can achieve a situation where both middle vertices are coloured, therefore she wins.

(b) We describe a winning strategy for Bob for the game $[B, Y]$ played on the triple triangle T_3 with 3 colours. In his first move, Bob colours the bottom vertex with colour 1. By symmetry, we distinguish only three cases.

- If Alice colours a vertex v of degree 4 with colour 2, Bob colours a vertex at distance 2 from v with colour 3 and wins.
- If Alice colours a vertex w of degree 2 with colour 1 or 2 instead, Bob colours a vertex at distance 3 from w with the other colour 2 or 1, respectively. Now the neighbour z with degree 4 of the vertex coloured with 2 is threatened: if one of its two uncoloured neighbours is coloured with colour 3, then Bob will win. Alice can only prevent him from colouring one of them unless she colours z. But if she colours z, necessarily with colour 3, Bob colours the uncoloured vertex at distance 2 from z with colour 2, which results in a win for him, since the second vertex of degree 4 is surrounded.
- If Alice skips, Bob colours a vertex w of degree 2 with colour 2. In the same way as in the second case, Alice is forced to colour the neighbour z of w with degree 4 with colour 3, which enables Bob to win if he colours a vertex at distance 2 from z.

We describe a winning strategy for Bob for the game $[B, Y]$ played on the triple sword S_3 with 3 colours, which is similar to the previous strategy. In his first move, Bob colours the top vertex of degree 2 with colour 1. To prevent Bob from winning by colouring the bottom vertex of degree 2 with another colour, Alice must colour this bottom vertex with colour 1. But then Bob colours another vertex of degree 2 with colour 2. Now the vertex of degree 5 is threatened: To prevent a situation in which it cannot be coloured, Alice must colour it with colour 3. Then Bob colours the vertex of degree 1 with colour 2 and wins. □

From Lemma 3.3, Lemma 3.5, and Lemma 5.2 we know that the split 3-star, the double fan, $P_4 \cup K_1$, $C_4 \cup K_1$, the 4-fan $P_4 \vee K_1$, the 4-wheel $C_4 \vee K_1$, the triple triangle T_3, and the triple sword S_3 are forbidden induced subgraphs for the classes occurring in Problem 5.1.

Problem 5.3. *Let $\mathcal{C}$ be the class of connected graphs. Characterise the classes*

$$EP([A,-],[A,B]) \cap \mathcal{C}, \quad EP([B,A],[B,-]) \cap \mathcal{C}.$$

From Lemma 3.5 we know that $P_4 \cup K_1$, $C_4 \cup K_1$, the 4-fan $P_4 \vee K_1$, and the 4-wheel $C_4 \vee K_1$ are forbidden induced subgraphs for the classes occurring in Problem 5.3.

Problem 5.4. *Characterise the graph classes $EP([A,B],[B,B])$ as well as $EP([B,A],[A,A])$.*

From Lemma 3.3 and Lemma 5.2 we know that the split 3-star, the double fan, the triple triangle T_3, and the triple sword S_3 are forbidden induced subgraphs for the classes $EP([B,A],[A,A])$ and $EP([A,B],[B,B])$ occurring in Problem 5.4. Both classes seem to have a rich and rather complicated structure. We notice that the T_3 and S_3 also occur as two of the minimal forbidden induced subgraphs for $[B,A]$-perfect and $[B,-]$-perfect graphs [5, 17].

For $Y \in \{A,B,-\}$, let

$$\overline{Y} := \begin{cases} B & \text{if } Y = A, \\ - & \text{if } Y = -, \\ A & \text{if } Y = B. \end{cases}$$

Our results encourage us to formulate the following, surprisingly unintuitive conjecture:

Conjecture 5.5 (Duality Conjecture)**.** *Let*

$$(X,Y),(X',Y') \in \{A,B\} \times \{A,B,-\}.$$

Then

$$EP([X,Y],[X',Y']) = EP\left(\left[\,\overline{X},\overline{Y}\,\right],\left[\,\overline{X'},\overline{Y'}\,\right]\right).$$

Problem 5.6. *Characterise the classes*

$$EPs([A,B],[A,-]) \text{ and } EPs([B,A],[B,-]).$$

Problem 5.7. *Characterise the classes*

$$EPs([A,A],[A,-]), \quad EPs([A,A],[A,B]),$$
$$EPs([B,A],[B,B]), \quad EPs([B,-],[B,B]).$$

Lemma 5.8. *Let sg_1 be one of the games $s[A,A]$, $s[B,A]$, or $s[B,-]$ and sg_2 be one of the games $s[A,-]$, $s[A,B]$, or $s[B,B]$. Let D be one of the digraphs depicted in Figure 7. Then*

(a) $\chi_{sg_1}(D) = 2$.
(b) $\chi_{sg_2}(D) = 3$.

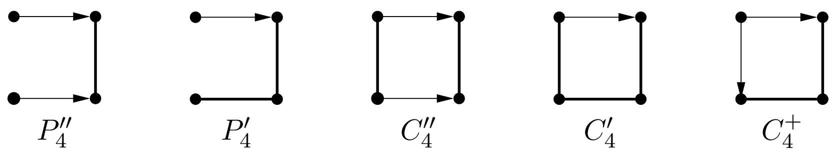


FIGURE 7. Some digraphs based on P_4 or C_4. In the figure, an undirected edge vw represents the two antiparallel arcs (v,w) and (w,v).

Proof. (a) In line with the proof of Lemma 3.4, we describe a winning strategy for Alice with 2 colours for a game where she can force Bob to begin. No matter which vertex Bob colours in his first move, Alice colours the vertex at distance 2 from it with the same colour and wins.

(b) In line with the proof of Lemma 3.4, we describe a winning strategy for Bob with 2 colours for a game where he can force Alice to begin. If Alice colours a vertex, Bob colours the vertex at distance 2 from it with the other colour and wins. □

The pairs of games occurring in Problem 5.7 have been characterised in the case of undirected graphs in Theorem 2.3 (ii). From Lemma 5.8 we know that the digraphs depicted in Figure 7 belong to the list of forbidden induced nongraphical digraphs.

We have not investigated the classes of weakly equality perfect digraphs, which seem to be rich classes and an interesting subject of further research.

Problem 5.9. *Characterise the classes*

$$EPw([X,Y],[X',Y'])$$

for any $X, X' \in \{A,B\}$, $Y, Y' \in \{A,B,-\}$.

Many results concerning game chromatic numbers were obtained through a marking game introduced by Zhu [24], which defines the *game colouring number* $\mathrm{col}_A(G)$ and $\mathrm{col}_B(G)$ of a graph G depending on whether Alice (A) or Bob (B) begins the game. We define a graph to be *marking equality perfect* if, for any induced subgraph, $\mathrm{col}_A(H) = \mathrm{col}_B(H)$.

Problem 5.10. *Characterise the class of marking equality perfect graphs by means of forbidden induced subgraphs.*

It is easy to see that for the *diamond* $K_2 \vee \overline{K_2}$ the game colouring number is 3 if Alice begins and 4 if Bob begins (cf. [4]), thus the diamond is one of the forbidden induced subgraphs in marking equality perfect graphs.

References

1. S.D. Andres, *Game-perfect graphs*, Math. Methods Oper. Res. **69** (2009), 235–250.
2. ______, *Lightness of digraphs in surfaces and directed game chromatic number*, Discrete Math. **309** (2009), 3594–3579.
3. ______, *Game-perfect digraphs*, Math. Methods Oper. Res. **76** (2012), 321–341.
4. ______, *On characterizing game-perfect graphs by forbidden induced subgraphs*, Contrib. Discrete Math. **7** (2012), 21–34.
5. S.D. Andres and E. Lock, *Characterising and recognising game-perfect graphs*, Discrete Math. Theor. Comput. Sci. **21** (2019), no. 1, Paper No. 6, 39pp.
6. T. Bartnicki, J. Grytczuk, H.A. Kierstead, and X. Zhu, *The map-coloring game*, Amer. Math. Monthly **114** (2007), 793–803.
7. H.L. Bodlaender, *On the complexity of some coloring games*, Internat. J. Found. Comput. Sci. **2** (1991), 133–147.
8. L. Cai and X. Zhu, *Game chromatic index of k-degenerate graphs*, J. Graph Theory **36** (2001), 144–155.
9. C. Charpentier, *The coloring game on planar graphs with large girth, by a result on sparse cactuses*, Discrete Math. **340** (2017), 1069–1073.
10. C. Charpentier, B. Effantin, and G. Paris, *On the game coloring index of F^+-decomposable graphs*, Discrete Appl. Math. **236** (2018), 73–83.
11. C. Charpentier and É. Sopena, *The incidence game chromatic number of (a, d)-decomposable graphs*, J. Discrete Algorithms **31** (2015), 14–25.
12. U. Faigle, W. Kern, H. Kierstead, and W.T. Trotter, *On the game chromatic number of some classes of graphs*, Ars Combin. **35** (1993), 143–150.
13. M.C. Golumbic, *Trivially perfect graphs*, Discrete Math. **24** (1978), 105–107.
14. D.J. Guan and X. Zhu, *Game chromatic number of outerplanar graphs*, J. Graph Theory **30** (1999), 67–70.
15. H.A. Kierstead, *A simple competitive graph coloring algorithm*, J. Combin. Theory Ser. B **78** (2000), 57–68.
16. J.W.H. Liu, *The role of elimination trees in sparse factorization*, SIAM J. Matrix Anal. Appl. **11** (1990), 134–172.
17. E. Lock, *The structure of g_B-perfect graphs*, Bachelor's thesis, FernUniversität in Hagen, 2016.
18. V. Neumann-Lara, *The dichromatic number of a digraph*, J. Combin. Theory Ser. B **33** (1982), 265–270.
19. E. Sidorowicz, *The game chromatic number and the game colouring number of cactuses*, Inform. Process. Lett. **102** (2007), 147–151.
20. E.S. Wolk, *The comparability graph of a tree*, Proc. Amer. Math. Soc. **13** (1962), 789–795.
21. ______, *A note on "the comparability graph of a tree"*, Proc. Amer. Math. Soc. **16** (1965), 17–20.
22. J. Wu and X. Zhu, *Lower bounds for the game colouring number of partial k-trees and planar graphs*, Discrete Math. **308** (2008), 2637–2642.
23. D. Yang and X. Zhu, *Game colouring directed graphs*, Electron. J. Combin. **17** (2010), no. 1, Research Paper 11, 19pp.
24. X. Zhu, *The game coloring number of planar graphs*, J. Combin. Theory Ser. B **75** (1999), 245–258.
25. ______, *The game coloring number of pseudo partial k-trees*, Discrete Math. **215** (2000), 245–262.
26. ______, *Refined activation strategy for the marking game*, J. Combin. Theory Ser. B **98** (2008), 1–18.

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