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COMBINATORIAL RESULTS FOR ORDER-PRESERVING PARTIAL INJECTIVE CONTRACTION MAPPINGS

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Abstract. Let $\mathcal{I}_n$ be the symmetric inverse semigroup on $X_n = \{1, 2, \ldots, n\}$. Let $\mathcal{OCI}_n$ be the subsemigroup of $\mathcal{I}_n$ consisting of all order-preserving injective partial contraction mappings, and let $\mathcal{ODCI}_n$ be the subsemigroup of $\mathcal{I}_n$ consisting of all order-preserving and order-decreasing injective partial contraction mappings of X_n. In this paper, we investigate the cardinalities of some equivalences on $\mathcal{OCI}_n$ and $\mathcal{ODCI}_n$ which lead naturally to obtaining the order of these semigroups. Then, we relate the formulae obtained to Fibonacci numbers. Similar results about $\mathcal{ORCI}_n$, the semigroup of order-preserving or order-reversing injective partial contraction mappings, are deduced.

1. Introduction and Preliminaries

Let $X_n = \{1, 2, \ldots, n\}$ and $\mathcal{I}_n$ be the partial one-to-one transformation semigroup on X_n under composition of mappings. Then $\mathcal{I}_n$ is an *inverse* semigroup, that is, for all $\alpha \in \mathcal{I}_n$ there exists a unique $\alpha' \in \mathcal{I}_n$ such that $\alpha = \alpha\alpha'\alpha$ and $\alpha' = \alpha'\alpha\alpha'$. The importance of $\mathcal{I}_n$, commonly known as the *symmetric inverse semigroup or monoid*, to inverse semigroup theory is similar to the importance of the symmetric group $\mathcal{S}_n$ to group theory. Every finite inverse semigroup S is embeddable in $\mathcal{I}_n$, an analogue to Cayley's theorem for finite groups. Thus, just as the study of symmetric, alternating, and dihedral groups has significantly contributed to group theory, the study of various subsemigroups of $\mathcal{I}_n$ lead to significant contributions to the theory of semigroups. For instance, see Borwein et al. [6], Fernandes [8], Fernandes et al. [9], Garba [10], Laradji and Umar [14], and Umar [19, 20].

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We shall denote the domain of $\alpha \in \mathcal{I}_n$ by $\operatorname{Dom}\alpha$. A transformation $\alpha \in \mathcal{I}_n$ is said to be *order-preserving* if for all x, y in $\operatorname{Dom}\alpha$, $x \leq y$ implies $x\alpha \leq y\alpha$. A transformation $\alpha \in \mathcal{I}_n$ is said to be *order-reversing* if for all x, y in $\operatorname{Dom}\alpha$, $x \leq y$ implies $x\alpha \geq y\alpha$. A transformation $\alpha \in \mathcal{I}_n$ is an *isometry* if it is *distance-preserving*, i.e, for all x, y in $\operatorname{Dom}\alpha$, $|x - y| = |x\alpha - y\alpha|$. A transformation $\alpha \in \mathcal{I}_n$ is a *contraction* if for all x, y in $\operatorname{Dom}\alpha$, $|x\alpha - y\alpha| \leq |x - y|$. A transformation is said to be *order-decreasing* if for all x in $\operatorname{Dom}\alpha$, $x\alpha \leq x$.

Analogous to Al-Kharousi et al. [2, 3], we investigate the combinatorial properties of $\mathcal{OCI}_n$ and $\mathcal{ODCI}_n$, thereby complementing the results in Al-Kharousi et al. [1] which deals primarily with the algebraic and rank properties of $\mathcal{OCI}_n$. In the present section, we introduce basic definitions and terminology, and we quote some elementary results from Al-Kharousi et al. [1, 2] that will be needed subsequently. In section 2, we thoroughly investigate the combinatorial properties of $\mathcal{OCI}_n$, the semigroup of order-preserving partial injective contraction mappings of the finite chain X_n. We obtain a formula for $F(n; p)$, the number of transformations with height p. Then we refine this formula obtaining $F(n; p, m)$, the number of transformations with height p and m fixed points.

We recall that the Fibonacci sequence, denoted by F_n, is defined recursively such that each number is the sum of the two preceding ones, starting from 0 and 1. The order of $\mathcal{OCI}_n$ is obtained and expressed in terms of two consecutive Fibonacci numbers, and it is shown that this is Sequence A094864 in The On-Line Encyclopedia of Integer Sequences [18]. In section 3, we study the combinatorial properties of $\mathcal{ODCI}_n$, the semigroup of order-preserving and order-decreasing partial injective contraction mappings of the finite chain X_n. The investigations go along the lines of the ones in section 2 in the case of $\mathcal{OCI}_n$. The order of $\mathcal{ODCI}_n$, as a function of n, is shown to be Sequence A001519 in The On-Line Encyclopedia of Integer Sequences [18]. In section 4, we study combinatorial properties of $\mathcal{ORCI}_n$, the semigroup of order-preserving or order-reversing partial injective contraction mappings of the finite chain X_n.

For standard concepts in semigroup and symmetric inverse semigroup theory, we refer to Howie [13] and Lipscomb [15].

We define the set of all partial injective contractions of X_n as

$$\mathcal{CI}_n = \{\alpha \in \mathcal{I}_n : \forall x, y \in \operatorname{Dom}\alpha,\ |\ x\alpha - y\alpha\ | \leq |\ x - y\ |\}.$$

We define the set of all order-preserving partial injective contractions of X_n as

$$\mathcal{OCI}_n = \{\alpha \in \mathcal{CI}_n : \forall x, y \in \operatorname{Dom}\alpha,\ x \leq y \implies x\alpha \leq y\alpha\}.$$

We define the set of all order-preserving or order-reversing partial injective contractions of X_n as

$$\begin{aligned}\mathcal{ORCI}_n = \{\alpha \in \mathcal{CI}_n : \forall x, y \in \operatorname{Dom}\alpha,\ x \leq y \implies x\alpha \leq y\alpha,\\ \text{or } \forall x, y \in \operatorname{Dom}\alpha,\ x \leq y \implies x\alpha \geq y\alpha\}.\end{aligned}$$

We define the set of all order-preserving and order-decreasing injective contractions of X_n as

$$\mathcal{ODCI}_n = \{\alpha \in \mathcal{OCI}_n : \forall x \in \operatorname{Dom}\alpha,\ x\alpha \leq x\}.$$

From these definitions, we can deduce the following.

Lemma 1.1. *$\mathcal{CI}_n$, $\mathcal{OCI}_n$, $\mathcal{ORCI}_n$ and $\mathcal{ODCI}_n$ are subsemigroups of $\mathcal{I}_n$.* □

Let α be an arbitrary element in $\mathcal{I}_n$. The *height* or *rank* of α is defined as $h(\alpha) = |\operatorname{Im}\alpha|$. The *right waist* of α is defined as $w^+(\alpha) = \max(\operatorname{Im}\alpha)$. Similarly, the *left waist* of α is defined as $w^-(\alpha) = \min(\operatorname{Im}\alpha)$. The *right shoulder* of α is defined as $\varpi^+(\alpha) = \max(\operatorname{Dom}\alpha)$. Similarly, the *left shoulder* of α is defined as $\varpi^-(\alpha) = \min(\operatorname{Dom}\alpha)$. The *fix* of α is denoted by $f(\alpha)$, and defined as $f(\alpha) = |F(\alpha)|$, where $F(\alpha) = \{x \in X_n : x\alpha = x\}$.

For a given transformation semigroup S, we define

$$F(n;p) = |\{\alpha \in S :\ h(\alpha) = p\}|.$$

The *gap* of an ordered tuple $\mathbf{a} = (a_1, a_2, \ldots, a_p)$ is the ordered $(p-1)$-tuple

$$g(\mathbf{a}) = (a_2 - a_1, a_3 - a_2, \ldots, a_p - a_{p-1})\,.$$

Accordingly, for

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ a_1\alpha & a_2\alpha & \cdots & a_p\alpha \end{pmatrix}$$

with $1 \leq a_1 < a_2 < \cdots < a_p \leq n$, let the gap of the domain of α be

$$g(\operatorname{Dom}\alpha) = (a_2 - a_1, a_3 - a_2, \ldots, a_p - a_{p-1})\,,$$

and let the gap of the image of α be

$$g(\operatorname{Im}\alpha) = (a_2\alpha - a_1\alpha, a_3\alpha - a_2\alpha, \ldots, a_p\alpha - a_{p-1}\alpha)\,.$$

For example, if

$$\alpha = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 6 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix} \in \mathcal{OCI}_6,$$

then
$g(\operatorname{Dom}\alpha) = (2,2)$, $g(\operatorname{Im}\alpha) = (2,1)$, $g(\operatorname{Dom}\beta) = (1,1,2)$ and
$g(\operatorname{Im}\beta) = (1,1,1)$. For

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ a_1\alpha & a_2\alpha & \cdots & a_p\alpha \end{pmatrix}$$

with $1 \leq a_1 < a_2 < \cdots < a_p \leq n$, let $d_i = a_{i+1}\alpha - a_i\alpha$ for $i = 1, 2, \ldots, p-1$. Then the *gap of the image set* of α is the ordered $(p-1)$-tuple $(d_1, d_2, \ldots, d_{p-1})$. Similarly, we let $t_i = a_{i+1} - a_i$ for $i = 1, 2, \ldots, p-1$.

Then the *gap of the domain set* of α is the ordered $(p-1)$-tuple $(t_1, t_2, \ldots, t_{p-1})$. With these definitions, it is clear that

$$p-1 \leq \sum_{i=1}^{p-1} |d_i| \leq n-1 \quad \text{and} \quad p-1 \leq \sum_{i=1}^{p-1} t_i \leq n-1.$$

Here we state some well-known identities which we will use in the proofs of the results in the following sections. We adopt the convention that $\binom{n}{k} = 0$ if $k > n$ or if k or n are negative.

Lemma 1.2.

(i) *The number of compositions of n into p positive parts is $\binom{n-1}{p-1}$, (See* [17, p. 151]*).*

(ii) *Let $d(n,p)$ be the number of distinct ordered p-tuples: $(r_1, r_2, \ldots, r_p)$ with $r_i \geq 0$ for $1 \leq i \leq p$ and $\sum_{i=1}^{p} r_i = n$. Then, $d(n,p)$ is the number of compositions of n into p nonnegative parts. That is,*

$$d(n,p) = \binom{n+p-1}{p-1},$$

(See [7, p. 589]*).*

The Vandermonde convolution identity states that for integers m, n, r,

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r},$$

(See [17, p. 8]). Using this identity we can deduce the following result.

Lemma 1.3. *For $r, s, t \in \mathbb{N}$ such that $r > s$,*

$$\sum_{i=0}^{r-s} \binom{r-i}{s}\binom{i+t}{t} = \binom{r+t+1}{s+t+1},$$

(See [17, 3(b), p. 8]*).*

The following result can be proved by straightforward induction.

Lemma 1.4. *For nonnegative integers n and r,*

$$\sum_{j=r}^{n} \binom{j}{r} = \binom{n+1}{r+1}.$$

2. Combinatorial results for $\mathcal{OCI}_n$

Let $A = \{(a_1, a_2, \ldots, a_m) | a_i \in \{1, 2, \ldots, n\} \text{ for } 1 \leq i \leq m\}$ be a set of ordered tuples with fixed length m. Define a relation $\mathbf{R}$ on the set A as follows: For $a = (a_1, a_2, \ldots, a_m)$ in A and $b = (b_1, b_2, \ldots, b_m)$ in A,

$$a \ \mathbf{R} \ b \Leftrightarrow |a_i| \leq |b_i|, \qquad \text{for all } 1 \leq i \leq m.$$

Lemma 2.1. *A transformation α is a contraction if and only if $g(\operatorname{Im}\alpha) \ \mathbf{R} \ g(\operatorname{Dom}\alpha)$.*

Proof. The result follows from the definition of a contraction. □

For a tuple $\mathbf{r} = (r_1, r_2, \ldots, r_p)$ with $1 \leq r_1 < r_2 < \cdots < r_p \leq n$, let

$$a_{\mathbf{r}} = \{\alpha \in \mathcal{OCI}_n | \operatorname{Im} \alpha = \mathbf{r}\}.$$

Let $g(\mathbf{r}) = (d_1, d_2, \ldots, d_{p-1})$ and $\sum d_i = q$. Then

Lemma 2.2.

$$|a_{\mathbf{r}}| = \binom{n-q+p-1}{p}.$$

Proof. The number of maps in $\mathcal{OCI}_n$ with image $\mathbf{r}$ is equal to the number of possible domain sets which we obtain by summing over the number of possible expansions of the image set by inserting i extra spaces, $\binom{i+p-2}{p-2}$, multiplied by the number of possible transversal shifts, $n-q-i$.

$$\begin{aligned} |a_{\mathbf{r}}| &= \sum_{i=0}^{n-1-q} (n-q-i)\binom{i+p-2}{p-2} \\ &= \sum_{i=0}^{n-1-q} \binom{n-q-i}{1}\binom{i+p-2}{p-2} \\ &= \binom{n-q+p-1}{p} \qquad \text{(by Lemma 1.3).} \end{aligned}$$

□

Corollary 2.3. *Let* $\mathbf{r} = (r_1, r_2, \ldots, r_p)$ *with* $1 \leq r_1 < r_2 < \cdots < r_p \leq n$ *and* $g(\mathbf{r}) = (d_1, d_2, \ldots, d_{p-1})$. *Let* $\mathbf{s} = (s_1, s_2, \ldots, s_p)$ *with* $1 \leq s_1 < s_2 < \cdots < s_p \leq n$ *and* $g(\mathbf{s}) = (d'_1, d'_2, \ldots, d'_{p-1})$. *If* $\sum_{i=1}^{p-1} d_i = \sum_{i=1}^{p-1} d'_i$ *then* $|a_{\mathbf{r}}| = |a_{\mathbf{s}}|$.

Proof. The result follows from Lemma 2.2, since $|a_{\mathbf{r}}|$ depends only on $\sum d_i = q$ and p. □

Theorem 2.4. *Let* $S = \mathcal{OCI}_n$. *Then for* $p \geq 1$,

$$F(n;p) = n\binom{n+p-1}{2p-1} + (1-p)\binom{n+p}{2p}.$$

Proof. From the results above, we can count $F(n;p)$ by summing over possible images and their possible domains as follows.

$$\begin{aligned} &F(n;p) \\ &= \sum_{q=p-1}^{n-1} \binom{q-1}{p-2}(n-q)\binom{n-q+p-1}{p} \\ &= \sum_{i=0}^{n-p} \binom{p-2+i}{p-2}[(n-p+1)-i]\binom{n-i}{p} \qquad (i = q-p+1) \end{aligned}$$

$$
\begin{aligned}
&= (n-p+1)\binom{n+p-1}{2p-1} - \sum_{i=0}^{n-p} i\binom{n-i}{p}\binom{p+i-2}{p-2} \quad \text{(by Lemma 1.3)} \\
&= (n-p+1)\binom{n+p-1}{2p-1} - \sum_{i=1}^{n-p} (p-1)\binom{n-i}{p}\frac{(p+i-2)\cdots(i+1)i}{(p-1)!} \\
&= (n-p+1)\binom{n+p-1}{2p-1} - \sum_{i=1}^{n-p} (p-1)\binom{n-i}{p}\binom{p+i-2}{p-1} \\
&= (n-p+1)\binom{n+p-1}{2p-1} - (p-1)\binom{n+p-1}{2p} \qquad \text{(by Lemma 1.3)} \\
&= n\binom{n+p-1}{2p-1} + (1-p)\binom{n+p-1}{2p-1} + (1-p)\binom{n+p-1}{2p} \\
&= n\binom{n+p-1}{2p-1} + (1-p)\binom{n+p}{2p}.
\end{aligned}
$$

□

The height, $h(\alpha)$, the right and left shoulder, $\varpi^+(\alpha)$ and $\varpi^-(\alpha)$, the right and left waist, $w^+(\alpha)$ and $w^-(\alpha)$, and the fix, $f(\alpha)$, are defined in section 1. To compute $F(n;p,m)$, the number of maps in $\mathcal{OCI}_n$ with height p and m fixed points, we introduce further notation.

Let

$$f^-(\alpha) = \min\{x \in \operatorname{Dom}\alpha : x\alpha = x\}$$

and

$$f^+(\alpha) = \max\{x \in \operatorname{Dom}\alpha : x\alpha = x\}.$$

Let

$$h^-(\alpha) = |\operatorname{Dom}\alpha \cap \{1, 2, \ldots, f^-(\alpha) - 1\}|$$

and

$$h^+(\alpha) = |\operatorname{Dom}\alpha \cap \{f^+(\alpha) + 1, f^+(\alpha) + 2, \ldots, n\}|.$$

Lemma 2.5. *Let $\alpha \in \mathcal{OCI}_n$. The set of fixed points of α is convex with respect to* $\operatorname{Dom}\alpha$. *That is, if $x \in \operatorname{Dom}\alpha$ such that $f^-(\alpha) \leq x \leq f^+(\alpha)$, then $x\alpha = x$.*

Proof. The result follows from the contraction property, as shown by Adeshola and Umar [4, Lemma 1.1]. □

For a semigroup S, we define

$$F(n;l^-,l^+,\lambda^-,\lambda^+,m,m^-,m^+,p,p^-,p^+) =$$
$$= |\{\alpha \in S : (\varpi^-(\alpha) = l^-) \wedge (\varpi^+(\alpha) = l^+) \wedge (w^-(\alpha) = \lambda^-) \wedge$$
$$\wedge (w^+(\alpha) = \lambda^+) \wedge (f(\alpha) = m) \wedge (f^-(\alpha) = m^-) \wedge (f^+(\alpha) = m^+) \wedge$$
$$\wedge (h(\alpha) = p) \wedge (h^-(\alpha) = p^-) \wedge (h^+(\alpha) = p^+)\}|.$$

Theorem 2.6. *Let $S = \mathcal{OCI}_n$. If $m = p$, then $F(n;p,m) = \binom{n}{m}$. If $m < p$, then*

$$F(n;p,m) = (p-m-1)\binom{n+p-m-2}{2p-m} + 2\binom{n+p-m-1}{2p-m}.$$

Proof. The case $m = p$ is clear. For $m < p$ and $m \neq 0$, as in the proof of Lemma 2.2, the maps can be counted by considering the number of possibilities of obtaining a domain set by expanding the image set. Due to the contraction condition and since there are fixed points, right or left shifts of the domain set are not possible.

Along the lines of Al-Kharousi et al., [1], a transformation $\alpha \in \mathcal{OCI}_n$ can be split into three parts, an increasing part, followed by a fixed part, and then a decreasing part. If a mapping has all three parts, then the set of fixed points is nonempty and occurs in the middle. In this case the number of spaces available for expansion on the left of m^- is $\lambda^- - l^-$. One of these spaces necessarily needs to be inserted just before m^-. Thus effectively we need to take into account all the possibilities of separating $\lambda^- - l^- - 1$ objects with $p^- - 1$ separators. These $p^- - 1$ separators correspond to the elements in the image which are between the smallest one and the smallest of the fixed points. We need to multiply the number of possibilities of distributing these $p^- - 1$ image points by the number of possibilities of expanding the image set, and then take the sum over all possible λ^-. Using Lemma 1.3, we obtain

$$\sum_{\lambda^- = l^- + 1}^{m^- - p^-} \binom{m^- - \lambda^- - 1}{p^- - 1}\binom{\lambda^- - l^- - 1 + p^- - 1}{p^- - 1} = \binom{m^- - l^- + p^- - 2}{2p^- - 1}.$$

Since $p^+ = p - m - p^-$, a similar calculation allows one to determine the number of possibilities of expanding the image set on the right of the fixed points as

$$\binom{l^+ - m^+ + (p - m - p^-) - 2}{2(p - m - p^-) - 1}.$$

For given m^- and m^+, the number of possibilities to distribute the fixed points is

$$\binom{m^+ - m^- - 1}{m - 2}.$$

The total number of transformations in $\mathcal{OCI}_n$ with the three parts (an increasing, a fixed, and a decreasing part) is obtained by multiplying these three expressions and summing over all possible l^+, l^-,m^+, m^- and p^-. Lemmas 1.4 and 1.3 will be applied repeatedly.

$$\sum_{p^-=1}^{p-m-1} \sum_{m^-=p^-+2}^{n-(p-p^-)} \sum_{m^+=m^-+m-1}^{n-(p-m-p^-)-1} \sum_{l^-=1}^{m^--p^--1} \sum_{l^+=m^++(p-m-p^-)+1}^{n}$$

$$\binom{m^- - l^- + p^- - 2}{2p^- - 1} \binom{l^+ - m^+ + (p-m-p^-) - 2}{2(p-m-p^-) - 1} \binom{m^+ - m^- - 1}{m-2}$$

$$= \sum_{p^-=1}^{p-m-1} \sum_{m^-=p^-+2}^{n-(p-p^-)} \sum_{m^+=m^-+m-1}^{n-(p-m-p^-)-1} \binom{m^+ - m^- - 1}{m-2}$$

$$\sum_{l^-=1}^{m^--p^--1} \binom{m^- - l^- + p^- - 2}{2p^- - 1}$$

$$\sum_{l^+=m^++(p-m-p^-)+1}^{n} \binom{l^+ - m^+ + (p-m-p^-) - 2}{2(p-m-p^-) - 1}$$

$$= \sum_{p^-=1}^{p-m-1} \sum_{m^-=p^-+2}^{n-(p-p^-)} \sum_{m^+=m^-+m-1}^{n-(p-m-p^-)-1}$$

$$\binom{m^+ - m^- - 1}{m-2} \binom{m^- - 1 + p^- - 1}{2p^-} \binom{n - m^+ + (p-m-p^-) - 1}{2(p-m-p^-)}$$

$$= \sum_{p^-=1}^{p-m-1} \sum_{m^-=p^-+2}^{n-(p-p^-)} \binom{m^- + p^- - 2}{2p^-}$$

$$\sum_{m^+=m^-+m-1}^{n-(p-m-p^-)-1} \binom{m^+ - m^- - 1}{m-2} \binom{n - m^+ + (p-m-p^-) - 1}{2(p-m-p^-)}$$

$$= \sum_{p^-=1}^{p-m-1} \sum_{m^-=p^-+2}^{n-(p-p^-)} \binom{m^- + p^- - 2}{2p^-} \binom{n - m^- + (p-m-p^-) - 1}{2(p-p^-) - m - 1}$$

$$= \sum_{p^-=1}^{p-m-1} \binom{n+p-m-2}{2p-m} = (p-m-1)\binom{n+p-m-2}{2p-m}.$$

Now we consider the cases where the fixed points are at the beginning or the end. Here we do the case where they are at the beginning, the other

case is similar. In the case under consideration, $p^- = 0$. Using Lemmas 1.4 and 1.3, we obtain the total number of transformations in $\mathcal{OCI}_n$ with a fixed part followed by a decreasing part.

$$\sum_{m^+=m}^{n-(p-m)-1} \sum_{l^+=m^++(p-m)+1}^{n} \binom{m^+-1}{m-1}\binom{l^+-m^++(p-m)-2}{2(p-m)-1}$$

$$= \sum_{m^+=m}^{n-(p-m)-1} \binom{m^+-1}{m-1} \sum_{l^+=m^++(p-m)+1}^{n} \binom{l^+-m^++(p-m)-2}{2(p-m)-1}$$

$$= \sum_{m^+=m}^{n-(p-m)-1} \binom{m^+-1}{m-1}\binom{n-m^++(p-m)-1}{2(p-m)} = \binom{n+p-m-1}{2p-m}.$$

Finally, we need to consider the case where there are no fixed points. Again we do this by expanding the image set in order to obtain the domain set. If all points in the image which come before the $(i+1)$th point are shifted to the left by inserting $j+1$ spaces, we get the following number of possibilities

$$\binom{j+(i-1)}{i-1}.$$

If all the elements in the image which come after the ith point are shifted to the right by inserting $k+1$ spaces, this gives the following number of possibilities

$$\binom{k+(p-i-1)}{p-i-1}.$$

Given λ^- and λ^+, the number of possible images is

$$\binom{\lambda^+-\lambda^--1}{p-2}.$$

Summing over all possible λ^- and λ^+, using Lemmas 1.4 and 1.3, we obtain

$$\sum_{i=1}^{p-1} \sum_{\lambda^+=p+1}^{n-1} \sum_{\lambda^-=2}^{\lambda^+-p+1} \sum_{j=0}^{\lambda^--2} \sum_{k=0}^{n-\lambda^+-1}$$

$$\binom{\lambda^+-\lambda^--1}{p-2}\binom{j+(i-1)}{i-1}\binom{k+(p-i-1)}{p-i-1}$$

$$= \sum_{i=1}^{p-1} \sum_{\lambda^+=p+1}^{n-1} \sum_{\lambda^-=2}^{\lambda^+-p+1} \binom{\lambda^+-\lambda^--1}{p-2}$$

$$\sum_{j=0}^{\lambda^--2} \binom{j+(i-1)}{i-1} \sum_{k=0}^{n-\lambda^+-1} \binom{k+(p-i-1)}{p-i-1}$$

$$= \sum_{i=1}^{p-1} \sum_{\lambda^+=p+1}^{n-1} \sum_{\lambda^-=2}^{\lambda^+-p+1} \binom{\lambda^+ - \lambda^- - 1}{p-2} \binom{\lambda^- + i - 2}{i} \binom{n - \lambda^+ + p - i - 1}{p-i}$$

$$= \sum_{i=1}^{p-1} \sum_{\lambda^+=p+1}^{n-1} \binom{\lambda^+ + i - 2}{i+p-1} \binom{n - \lambda^+ + p - i - 1}{p-i}$$

$$= \sum_{i=1}^{p-1} \binom{n+p-2}{2p} = (p-1)\binom{n+p-2}{2p}.$$

Here also there are additional cases to consider, namely when all elements in the image are shifted to the left to obtain the domain, or all elements in the image are shifted to the right to obtain the domain. We will do the case where all elements in the image are shifted to the right and double the number. Using Lemmas 1.4 and 1.3, we obtain

$$\sum_{\lambda^+=p}^{n-1} \sum_{\lambda^-=1}^{\lambda^+-p+1} \sum_{k=0}^{n-\lambda^+-1} \binom{\lambda^+ - \lambda^- - 1}{p-2} \binom{k+p-1}{p-1}$$

$$= \sum_{\lambda^+=p}^{n-1} \sum_{\lambda^-=1}^{\lambda^+-p+1} \binom{\lambda^+ - \lambda^- - 1}{p-2} \sum_{k=0}^{n-\lambda^+-1} \binom{k+p-1}{p-1}$$

$$= \sum_{\lambda^+=p}^{n-1} \binom{\lambda^+ - 1}{p-1} \binom{n - \lambda^+ + p - 1}{p} = \binom{n+p-1}{2p}.$$

When $p = 1$ and $m = 0$, then it is clear that $F(n; p, m) = n(n-1)$.

The result of the theorem is obtained by taking the sum of all cases. □

Remark: $F(n; p)$ of $\mathcal{OCI}_n$ can be obtained from the expression $F(n; p, m)$ in Theorem 2.6 by summing over all possible values of m.

It turns out that the order of $\mathcal{OCI}_n$ can be related to Fibonacci numbers. We need the following proposition to show the relation.

Proposition 2.8. *The alternating Fibonacci numbers can be evaluated by the following formulae.*

(i)

$$F_{2n+1} = \sum_{p \geq 0} \binom{n+p}{2p}.$$

This is Sequence A001519 in The On-Line Encyclopedia of Integer Sequences [18] *and satisfies the recurrence relation*

$$a_n = 3a_{n-1} - a_{n-2},$$

with $a_0 = 1 = F_1$ *and* $a_1 = 2 = F_3$.

(ii)

$$F_{2n} = \sum_{p\geq 0} \binom{n+p-1}{2p-1}.$$

This is Sequence A001906 in The On-Line Encyclopedia of Integer Sequences [18] *and satisfies the recurrence relation*

$$a_n = 3a_{n-1} - a_{n-2},$$

with $a_0 = 0 = F_0$ *and* $a_1 = 1 = F_2$.

□

The order of $\mathcal{OCI}_n$ can be determined using the formula for $F(n;p)$ obtained in Theorem 2.4.

Theorem 2.9. *The order of* $\mathcal{OCI}_n$*, as a function of* n*, is equal to*

$$h_n = \frac{3n-1}{5}F_{2n} - \frac{n-5}{5}F_{2n+1}.$$

This is Sequence A094864 in The On-Line Encyclopedia of Integer Sequences [18] *and satisfies the recurrence relation*

$$h_n = 6h_{n-1} - 11h_{n-2} + 6h_{n-3} - h_{n-4},$$

with $h_0 = 1$, $h_1 = 2$, $h_2 = 6$, $h_3 = 18$.

Proof. We use the formula in Theorem 2.4 and sum over all possible p. For $p = 0$, the order is 1, since the empty map is the only element with height 0.

$$\begin{aligned}
h_n &= |\mathcal{OCI}_n| \\
&= 1 + \sum_{p=1}^{n} \left(n\binom{n+p-1}{2p-1} + (1-p)\binom{n+p}{2p} \right) \\
&= 1 + n\sum_{p=1}^{n} \binom{n+p-1}{2p-1} + \sum_{p=1}^{n} \binom{n+p}{2p} - \sum_{p=1}^{n} p\binom{n+p}{2p} \\
&= 1 + n\sum_{p\geq 0} \binom{n+p-1}{2p-1} + \left(\sum_{p\geq 0} \binom{n+p}{2p} - 1 \right) - \sum_{p=1}^{n} p\binom{n+p}{2p} \\
&= n\sum_{p\geq 0} \binom{n+p-1}{2p-1} + \sum_{p\geq 0} \binom{n+p}{2p} - \sum_{p=1}^{n} p\binom{n+p}{2p} \\
&= nF_{2n} + F_{2n+1} - \underbrace{\sum_{p=1}^{n} p\binom{n+p}{2p}}_{b_n}.
\end{aligned}$$

We obtain a recurrence relation on the sequence b_n as follows.

$$
\begin{aligned}
b_n &= \sum_{p=1}^{n} p\binom{n+p}{2p} = \sum_{p=1}^{n} \frac{2p}{2}\frac{(n+p)!}{(2p)!(n-p)!} \\
&= \sum_{p=1}^{n} \frac{1}{2}\frac{(n+p)!}{(2p-1)!(n-p)!} = \sum_{p=1}^{n} \frac{(n+p)}{2}\frac{(n+p-1)!}{(2p-1)!(n-p)!} \\
&= \frac{n}{2}\sum_{p=1}^{n} \frac{(n+p-1)!}{(2p-1)!(n-p)!} + \sum_{p=1}^{n} \frac{p}{2}\frac{(n+p-1)!}{(2p-1)!(n-p)!} \\
&= \frac{n}{2}\sum_{p=1}^{n} \binom{n+p-1}{2p-1} + \frac{1}{4}\sum_{p=1}^{n}(2p-1+1)\frac{(n+p-1)!}{(2p-1)!(n-p)!} \\
&= \frac{n}{2}F_{2n} + \frac{1}{4}\sum_{p=1}^{n}(2p-1)\frac{(n+p-1)!}{(2p-1)!(n-p)!} + \frac{1}{4}\sum_{p=1}^{n}\frac{(n+p-1)!}{(2p-1)!(n-p)!} \\
&= \frac{n}{2}F_{2n} + \frac{1}{4}\sum_{p=1}^{n}\frac{(n+p-1)!}{(2p-2)!(n-p)!} + \frac{1}{4}\sum_{p=1}^{n}\binom{n+p-1}{2p-1} \\
&= \frac{n}{2}F_{2n} + \frac{1}{4}\sum_{p=1}^{n}(n+p-1)\frac{(n+p-2)!}{(2p-2)!(n-p)!} + \frac{1}{4}F_{2n} \\
&= \frac{2n+1}{4}F_{2n} + \frac{1}{4}\sum_{p=1}^{n}(n+p-1)\binom{n+p-2}{2p-2} \\
&= \frac{2n+1}{4}F_{2n} + \frac{n}{4}\sum_{p=1}^{n}\binom{n+p-2}{2p-2} + \frac{1}{4}\sum_{p=1}^{n}(p-1)\binom{n+p-2}{2p-2} \\
&= \frac{2n+1}{4}F_{2n} + \frac{n}{4}\sum_{p=0}^{n-1}\binom{n-1+p}{2p} + \frac{1}{4}\sum_{p=0}^{n-1}p\binom{n-1+p}{2p} \\
&= \frac{2n+1}{4}F_{2n} + \frac{n}{4}F_{2n-1} + \frac{1}{4}b_{n-1}.
\end{aligned}
$$

This recurrence relation can be used to eliminate b_n from the above formula for h_n as follows.

$$
\begin{aligned}
h_n &= nF_{2n} + F_{2n+1} - b_n, \\
h_{n-1} &= (n-1)F_{2n-2} + F_{2n-1} - b_{n-1}.
\end{aligned}
$$

Thus we can express h_n in terms of h_{n-1}, and after some simplifications which merely involve using the property $F_n = F_{n-1} + F_{n-2}$, we obtain

$$h_n = \frac{1}{4}h_{n-1} + \frac{n+2}{4}F_{2n} + \frac{1}{2}F_{2n+1}.$$

Since $h_0 = 1$, $h_1 = 2$, $h_2 = 6$, $h_3 = 18$, it is straightforward to verify that h_n is Sequence A094864 in The On-Line Encyclopedia of Integer Sequences [18] which was studied by Barcucci et al. [5] and Rinaldi and Rogers [16]. A closed formula for this sequence is

$$h_n = \frac{3n-1}{5}F_{2n} - \frac{n-5}{5}F_{2n+1}.$$

□

3. Combinatorial results for $\mathcal{ODCI}_n$

In this section, we will obtain results analogous to the ones obtained in section 2 for the semigroup $\mathcal{ODCI}_n$. For a semigroup S, we define

$$F(n;k^-,k^+,l^+,p) = |\{\alpha \in S : (w^-(\alpha) = k^-) \wedge (w^+(\alpha) = k^+) \wedge \\ \wedge (\varpi^+(\alpha) = l^+) \wedge (h(\alpha) = p)\}|.$$

Lemma 3.1. *Let $S = \mathcal{ODCI}_n$, then*

$$F(n;k^-,k^+,l^+,p) = \binom{l^+ - k^+ + p - 1}{p-1}\binom{k^+ - k^- - 1}{p-2}.$$

Proof. The number of possible images is $\binom{k^+-k^--1}{p-2}$. This number needs to be multiplied by the number of possible pre-images, which depends on k^+ and l^+. Because of the decreasing property, there is only one direction in which the image set can be expanded to obtain the domain set, namely to the right. There are $l^+ - k^+$ extra spaces and $p-1$ separators, so the number of possibilities to expand the domain set is

$$\binom{l^+ - k^+ + p - 1}{p-1}.$$

□

This allows us to find the number of maps in $\mathcal{ODCI}_n$ of height p.

Theorem 3.2. *Let $S = \mathcal{ODCI}_n$. Then*

$$F(n;p) = \binom{n+p}{2p}.$$

Proof. Using the expression from Lemma 3.1, we obtain

$$F(n;k^+,l^+,p) = \sum_{k^-=1}^{k^+-p+1}\binom{l^+ - k^+ + p - 1}{p-1}\binom{k^+ - k^- - 1}{p-2}$$

$$= \binom{l^+ - k^+ + p - 1}{p-1} \sum_{k^-=1}^{k^+-p+1} \binom{k^+ - k^- - 1}{p-2}$$

$$= \binom{l^+ - k^+ + p - 1}{p-1} \binom{k^+ - 1}{p-1} \qquad \text{(by Lemma 1.4).}$$

Summing over all possible right waists and right shoulders, using Lemma 1.3, we obtain

$$F(n;p) = \sum_{l^+=p}^{n} \sum_{k^+=p}^{l^+} F(n;k^+,l^+,p)$$

$$= \sum_{l^+=p}^{n} \sum_{k^+=p}^{l^+} \binom{l^+ - k^+ + p - 1}{p-1} \binom{k^+ - 1}{p-1}$$

$$= \sum_{l^+=p}^{n} \binom{l^+ + p - 1}{2p-1} = \binom{n+p}{2p}.$$

□

We can extend these results to compute $F(n;p,m)$ as follows.

Lemma 3.3. *Let $S = \mathcal{ODCI}_n$. For $m < p$,*

$$F(n;k^-,k^+,l^+,m,p) = \binom{l^+ - k^+ + p - m - 2}{p-m-1} \binom{k^+ - k^- - 1}{p-2}.$$

Proof. The proof is analogous to the one of Lemma 3.1. If there are m fixed points, because of the contraction and decreasing properties, these are the first m. There are $l^+ - k^+$ extra spaces and $p - m - 1$ separators. One of these spaces is used to ensure that there are no more than m fixed points, i.e., it is inserted after the last of the fixed points. Thus the number of possibilities to expand the domain set is

$$\binom{l^+ - k^+ + p - m - 2}{p-m-1}.$$

□

This allows us to find the number of maps in $\mathcal{ODCI}_n$ of height p with m fixed points.

Theorem 3.4. *Let $S = \mathcal{ODCI}_n$. If $m = p$, then $F(n;p,m) = \binom{n}{p}$. If $m < p$, then*

$$F(n;p,m) = \binom{n+p-m-1}{2p-m}.$$

Proof. The case $m = p$ is clear. For $m < p$, we repeat the procedure in the proof of Theorem 3.2, using the expression from Lemma 3.3. Note that because $m \neq p$, the case $k^+ = l^+$ is excluded. Thus we get

$$F(n; k^+, l^+, p, m) = \sum_{k^-=1}^{k^+-p+1} \binom{l^+ - k^+ + p - m - 2}{p - m - 1} \binom{k^+ - k^- - 1}{p - 2}$$

$$= \binom{l^+ - k^+ + p - m - 2}{p - m - 1} \binom{k^+ - 1}{p - 1} \qquad \text{(by Lemma 1.4).}$$

Then we sum over all possible right waists and right shoulders. Using Lemmas 1.3 and 1.4, we obtain

$$F(n; p, m) = \sum_{l^+=p+1}^{n} \sum_{k^+=p}^{l^+-1} F(n; k^+, l^+, m, p)$$

$$= \sum_{l^+=p+1}^{n} \sum_{k^+=p}^{l^+-1} \binom{l^+ - k^+ + p - m - 2}{p - m - 1} \binom{k^+ - 1}{p - 1}$$

$$= \sum_{l^+=p+1}^{n} \binom{l^+ + p - m - 2}{2p - m - 1} = \binom{n + p - m - 1}{2p - m}.$$

□

Remark: $F(n; p)$ of $\mathcal{ODCI}_n$ can be obtained from the expression $F(n; p, m)$ in Theorem 3.4 by summing over all possible values of m.

The order of $\mathcal{ODCI}_n$ can be determined using the formula for $F(n; p)$ obtained in Theorem 3.2. It turns out that the order, as a function of n, can be expressed in terms of a single Fibonacci number.

Theorem 3.6. *$|\mathcal{ODCI}_n| = F_{2n+1}$, where F_n is the nth Fibonacci number.*

Proof. The result follows from Theorem 3.2 and Proposition 2.8. □

4. Combinatorial results for $\mathcal{ORCI}_n$

Let $\mathcal{OCI}_n^+$ be the set of order-reversing contraction mappings of a finite chain X_n, defined as

$$\mathcal{OCI}_n^+ = \{\alpha \in \mathcal{CI}_n : \forall x, y \in \operatorname{Dom}\alpha,\ x \leq y \implies x\alpha \geq y\alpha\}.$$

Lemma 4.1. *There is a bijection between $\mathcal{OCI}_n$ and $\mathcal{OCI}_n^+$.*

Proof. Let $\alpha \in \mathcal{OCI}_n$. If $h(\alpha) = 1$, then $\alpha \in \mathcal{OCI}_n^+$. Let $h(\alpha) > 1$ and

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ a_1\alpha & a_2\alpha & \cdots & a_p\alpha \end{pmatrix},$$

where $1 \leq a_1 < a_2 < \cdots < a_p \leq n$. By the definition of $\mathcal{OCI}_n$,

$$1 \leq a_1\alpha < a_2\alpha < \cdots < a_p\alpha \leq n.$$

Let $g(\mathrm{Dom}\,\alpha) = (t_1, t_2, \ldots, t_{p-1})$, and $g(\mathrm{Im}\,\alpha) = (d_1, d_2, \ldots, d_{p-1})$. Then we have

$$\alpha = \begin{pmatrix} a_1 & a_1 + t_1 & \cdots & a_1 + t_1 + \cdots + t_{p-1} \\ a_1\alpha & a_1\alpha + d_1 & \cdots & a_1\alpha + d_1 + \cdots + d_{p-1} \end{pmatrix},$$

and by the contraction property, $d_i \leq t_i$ for all i.

We define a function $\theta : \mathcal{OCI}_n \to \mathcal{OCI}_n^+$, with $\theta(\alpha) = \alpha'$ given by

$$\begin{aligned} \alpha' &= \begin{pmatrix} a_1 & a_1 + t_{p-1} & \cdots & a_1 + t_{p-1} + \cdots + t_1 \\ a_p\alpha & a_p\alpha - d_{p-1} & \cdots & a_p\alpha - d_{p-1} - d_{p-2} - \cdots - d_1 \end{pmatrix} \\ &= \begin{pmatrix} a_1 & a_1 + t_{p-1} & \cdots \\ a_1\alpha + d_1 + \cdots + d_{p-1} & a_1\alpha + d_1 + \cdots + d_{p-2} & \cdots \\ & \cdots & a_1 + t_{p-1} + \cdots + t_1 \\ & \cdots & a_1\alpha \end{pmatrix}. \end{aligned}$$

The function θ is well defined, and since $d_i \geq 0$ for all i, α' is order reversing. The gap of the domain of α' is equal to the reverse of the gap of the domain of α,

$$g(\mathrm{Dom}\,\alpha') = (t_{p-1}, t_{p-2}, \ldots, t_1) = g(\mathrm{Dom}\,\alpha)^R.$$

The gap of the image of α' is equal to minus the reverse of the gap of the image of α,

$$g(\mathrm{Im}\,\alpha') = (-d_{p-1}, -d_{p-2}, \ldots, -d_1) = -g(\mathrm{Im}\,\alpha)^R.$$

Because for all i with $1 \leq i \leq p-1$, we have $1 \leq d_i \leq t_i$, it follows that $\alpha' \in \mathcal{OCI}_n^+$. □

Remark: Note that $\mathcal{ORCI}_n = \mathcal{OCI}_n \cup \mathcal{OCI}_n^+$ and $\mathcal{OCI}_n \cap \mathcal{OCI}_n^+ = \{\alpha \in \mathcal{OCI}_n : \ h(\alpha) \leq 1\}$.

The following result follows from Theorem 2.4, Lemma 4.1, and Remark 4.2.

Theorem 4.3. *Let $S = \mathcal{ORCI}_n$. If $p = 1$, then $F(n; p) = n^2$. For $p > 1$,*

$$F(n; p) = 2n\binom{n+p-1}{2p-1} + (2-2p)\binom{n+p}{2p}.$$

□

We determine the number of maps with m fixed points, for a given height p.

Theorem 4.4. *Let $S = \mathcal{ORCI}_n$. If $m = p$, then $F(n; p, m) = \binom{n}{m}$. If $m = 1 < p$, then*

$$F(n; p, m_1) = 2(p-2)\binom{n+p-3}{2p-1} + 4\binom{n+p-2}{2p-1} - n.$$

If $1 < m < p$, then

$$F(n; p, m) = (p-m-1)\binom{n+p-m-2}{2p-m} + 2\binom{n+p-m-1}{2p-m}.$$

Proof. From Lemma 4.1 and Remark 4.2, it follows that the number of maps in $\mathcal{OCI}_n^+$ with one fixed point is equal to the number of maps with one fixed point in $\mathcal{OCI}_n$. In order to find the number of maps with one fixed point in $\mathcal{ORCI}_n$, we add the number for $\mathcal{OCI}_n$ and the number for $\mathcal{OCI}_n^+$, and subtract the number of partial identities of height one, as otherwise they would be counted twice. Using $m = 1$ in Theorem 2.6, this gives

$$F(n; p, 1) = 2(p-2)\binom{n+p-3}{2p-1} + 4\binom{n+p-2}{2p-1} - n.$$

In $\mathcal{OCI}_n^+$, there are clearly no maps with more than one fixed point. Thus for $m \geq 2$, the number of maps with m fixed points in $\mathcal{ORCI}_n$ is equal to the number of maps with m fixed points in $\mathcal{OCI}_n$. From Theorem 2.6, we get

$$F(n; p, m) = (p-m-1)\binom{n+p-m-2}{2p-m} + 2\binom{n+p-m-1}{2p-m}.$$

□

We will calculate the order of $\mathcal{ORCI}_n$ and find an expression for the order in terms of two consecutive Fibonacci numbers. In light of Remark 4.1, we can use results from section 2 to significantly reduce the amount of calculations.

Theorem 4.5. *The order of $\mathcal{ORCI}_n$, as a function of n, is given by*

$$|\mathcal{ORCI}_n| = \frac{6n-2}{5}F_{2n} - \frac{2n-10}{5}F_{2n+1} - 1 - n^2.$$

Proof. From Theorem 2.9, we know that the order of $\mathcal{OCI}_n$ is $\frac{3n-4}{5}F_{2n} - \frac{4n-10}{5}F_{2n-1}$. This was obtained by summing over $F(n; p)$ for $p \geq 1$ and adding $F(n; p_0) = 1$ for the empty map. In $\mathcal{ORCI}_n$, according to Remark 4.1, the case $p = 1$ needs to be handled separately as well. Using the expression from Theorem 4.3, this gives

$$\begin{aligned}
|\mathcal{ORCI}_n| &= 1 + n^2 + \sum_{p=2}^{n}\left(2n\binom{n+p-1}{2p-1} + (2-2p)\binom{n+p}{2p}\right) \\
&= 1 + n^2 + \sum_{p=1}^{n}\left(2n\binom{n+p-1}{2p-1} + (2-2p)\binom{n+p}{2p}\right) - 2n\binom{n}{1} \\
&= 1 + n^2 + 2\left(|\mathcal{OCI}_n| - 1\right) - 2n^2 \\
&= 1 + n^2 + 2\frac{3n-1}{5}F_{2n} - 2\frac{n-5}{5}F_{2n+1} - 2 - 2n^2 \\
&= \frac{6n-2}{5}F_{2n} - \frac{2n-10}{5}F_{2n+1} - 1 - n^2.
\end{aligned}$$

□

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References

1. F. Al-Kharousi, G. U. Garba, M. J. Ibrahim, A. T. Imam, and A. Umar, *On the semigroup of finite order-preserving partial injective contraction mappings,* (*Submitted*).
2. F. Al-Kharousi, R. Kehinde, and A. Umar, *Combinatorial results for certain semigroups of partial isometries of a finite chain,* Australas. J. Combin. **58** (2014), 365–375.
3. F. Al-Kharousi, R. Kehinde, and A. Umar, *Combinatorial results for certain semigroups of order-decreasing partial isometries of a finite chain,* Journal of Algebra, Number Theory: Advances and Applications. **19**(1) (2018), 29–49.
4. A. D. Adeshola, and A. Umar, *Combinatorial results for certain semigroups of order-preserving full contraction mappings of a finite chain,* JCMCC. **106** (2017), 34–49.
5. E. Barcucci, R. Pinzani, and R. Sprugnoli, *Directed column-convex polyominoes by recurrence relations,* Lecture Notes in Comput. Sci. **668** (1993), 282–298.
6. D. Borwein, S. Rankin, and L. Renner, *Enumeration of injective partial transformations,* Discrete Math. **73** (1989), 291–296.
7. S. Epp, *Discrete Mathematics with Applications, 4th Edition,* Brooks/Cole, Boston, MA, 2011.
8. V. H. Fernandes, *The monoid of all injective orientation-preserving partial transformations on a finite chain,* Comm. Algebra **28** (2000), 3401–3426.
9. V. H. Fernandes, G. M. S. Gomes, and M. M. Jesus, *The cardinal and idempotent number of various monoids of transformations on a finite chain,* Bull. Malays. Math. Sci. Soc. **34** (2011), 79–85.
10. G. U. Garba, *Nilpotents in semigroups of partial one-to-one order-preserving mappings,* Semigroup Forum **48** (1994), 37–49.
11. H. W. Gould, *Combinatorial identities. A standardized set of tables listing 500 binomial coefficient summations*, Morgantown, W.Va., 1972.
12. R. P. Grimaldi, *Fibonacci and Catalan numbers. An introduction,* John Wiley & Sons, 2012.
13. J. M. Howie, *Fundamentals of Semigroup Theory,* London Mathematical Society Monographs. New series, **12**. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
14. A. Laradji and A. Umar, *Combinatorial results for the symmetric inverse semigroup,* Semigroup Forum **75** (2007), 221–236.
15. S. Lipscomb, *Symmetric Inverse Semigroups*, Mathematical Surveys and Monographs, **46**. American Mathematical Society, Providence, R. I., 1996.
16. S. Rinaldi and D. G. Rogers, *How the odd terms in the Fibonacci sequence stack up,* Math. Gaz., **90** (2006), 431–442.
17. J. Riordan, *Combinatorial Identities, reprint of the 1968 original,* Robert E. Krieger Publishing Co., Huntington, N.Y., 1979.
18. N .J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically at `https://oeis.org`.
19. A. Umar, *On the semigroups of partial one-to-one order-decreasing finite transformations,* Proc. Roy. Soc. Edinburgh, Sect. A, **123** (1993), 355–363.
20. ———, *Some combinatorial problems in the theory of symmetric inverse semigroups,* Algebra Discrete Math. **9** (2010), 115–126.

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