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DISTINGUISHING NUMBER AND DISTINGUISHING INDEX OF NEIGHBOURHOOD CORONA OF TWO GRAPHS

SAEID ALIKHANI AND SAMANEH SOLTANI

Abstract. The distinguishing number (index) $D(G)$ $(D'(G))$ of a graph G is the least integer d such that G has an vertex labelling (edge labelling) with d labels that is preserved only by a trivial automorphism. The neighbourhood corona of two graphs G_1 and G_2 is denoted by $G_1 \star G_2$ and is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and joining the neighbours of the ith vertex of G_1 to every vertex in the ith copy of G_2. In this paper we describe the automorphisms of the graph $G_1 \star G_2$. Using results on automorphisms, we study the distinguishing number and the distinguishing index of $G_1 \star G_2$. We obtain upper bounds for $D(G_1 \star G_2)$ and $D'(G_1 \star G_2)$.

1. Introduction

Let $G = (V, E)$ be a simple graph with n vertices. Throughout this paper, we consider only simple graphs. The set of all automorphisms of G, with the operation of composition of permutations, is a permutation group on V and is denoted by $Aut(G)$. A labelling of G, $\phi : V \to \{1, 2, \ldots, r\}$, is r-distinguishing, if no non-trivial automorphism of G preserves all of the vertex labels. In other words, ϕ is r-distinguishing if for every non-trivial $\sigma \in Aut(G)$, there exists x in V such that $\phi(x) \neq \phi(x\sigma)$. The distinguishing number of a graph G has defined by Albertson and Collins [1] and is the minimum number r such that G has a labelling that is r-distinguishing. Similar to this definition, Kalinowski and Pilśniak [6] have defined the distinguishing index $D'(G)$ of G which is the least integer d such that G has an edge colouring with d colours that is preserved only by a trivial automorphism. These indices have been developed in a number of papers on this subject (see, for example, [2, 7, 9]).

We use the following notation: The set of vertices adjacent in G to a vertex of a vertex subset $W \subseteq V$ is the open neighborhood $N_G(W)$ of W.

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Corresponding author: Saeid Alikhani.

The closed neighborhood $G[W]$ also includes all vertices of W itself. In the case of a singleton set $W = \{v\}$, we write $N_G(v)$ and $N_G[v]$ instead of $N_G(\{v\})$ and $N_G[\{v\}]$, respectively. We omit the subscript when the graph G is clear from the context. The complement of $N[v]$ in $V(G)$ is denoted by $\overline{N[v]}$. We denote the degree of a vertex v in graph G by $d_G(v)$ and the distance between two vertices u and w in graph G, by $\mathrm{dist}_G(u, w)$. The corona of two graphs G and H which denoted by $G \circ H$ is defined in [4] and there have been some results on the corona of two graphs [3]. In [2] we have studied the distinguishing number and the distinguishing index of the corona of two graphs. In this paper we consider another variation of the corona of two graphs and study its distinguishing number and distinguishing index. Given simple graphs G_1 and G_2, the neighbourhood corona of G_1 and G_2, denoted by $G_1 \star G_2$ and is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining the neighbours of the ith vertex of G_1 to every vertex in the ith copy of G_2 ([5]). Figure 1 shows $P_4 \star P_3$, where P_n is the path of order n. Liu and Zhou in [8] determined the adjacency spectrum of $G_1 \star G_2$ for arbitrary G_1 and G_2 and the Laplacian spectrum and signless Laplacian spectrum of $G_1 \star G_2$ for regular G_1 and arbitrary G_2, in terms of the corresponding spectrum of G_1 and G_2. Also Gopalapillai in [5] has studied the eigenvalues and spectrum of $G_1 \star G_2$, when G_2 is regular.

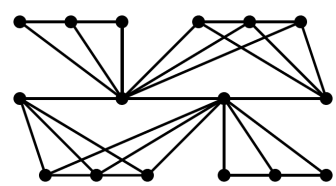

FIGURE 1. The neighbourhood corona of $P_4 \star P_3$.

In this paper, we consider the neighbourhood corona of two graphs and discuss their distinguishing number and index. In the next section, we give a complete description of the automorphisms of the neighbourhood corona of two arbitrary graphs. In Section 3, we study the distinguishing number and the distinguishing index of the neighbourhood corona of two graphs.

2. Description of automorphisms of $G_1 \star G_2$

In this section, we consider the neighbourhood corona of two graphs and describe its automorphisms. Let G_i have order n_i and size m_i ($i = 1, 2$). The neighbourhood corona $G_1 \star G_2$ of G_1 and G_2 has $n_1 + n_1 n_2$ vertices and $m_1(2n_2 + 1) + n_1 m_2$ edges and when $G_2 = K_1$, the graph $G_1 \star G_2$ is the splitting graph which has defined in [10].

Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. For $i = 1, 2, \ldots, n_1$, let $u_1^i, u_2^i, \ldots, u_{n_2}^i$ denote the vertices of the ith copy of G_2, with the understanding that u_j^i is the copy of u_j for each j. It is clear that the degrees of the vertices of $G_1 \star G_2$ are:

$$d_{G_1 \star G_2}(v_i) = (n_2+1)d_{G_1}(v_i),\ i = 1, 2, \ldots, n_1, \tag{2.1}$$

and

$$d_{G_1 \star G_2}(u_j^i) = d_{G_2}(u_j) + d_{G_1}(v_i),\ i = 1, 2, \ldots, n_1,\ j = 1, 2, \ldots, n_2. \tag{2.2}$$

Now we want to know how an automorphism of $G_1 \star G_2$ acts on the vertices G_1 and the vertices of copies G_2. First we state and prove the following lemma.

Lemma 2.1. *Let G_1 and G_2 be two connected graphs such that $G_1 \neq K_1$ and f be an automorphism of $G_1 \star G_2$ such that $f(v_i) = u_j^k$ for some $i, k = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$. Then $d_{G_1}(v_k) > d_{G_1}(v_i)$.*

Proof. Since $f(v_i) = u_j^k$, $d_{G_1 \star G_2}(v_i) = d_{G_1 \star G_2}(u_j^k)$. By (2.1) and (2.2) we have $(n_2+1)d_{G_1}(v_i) = d_{G_2}(u_j) + d_{G_1}(v_k)$. By contradiction, suppose that $d_{G_1}(v_k) \leqslant d_{G_1}(v_i)$. Hence $(n_2+1)d_{G_1}(v_i) \leqslant d_{G_2}(u_j) + d_{G_1}(v_i)$, and so $n_2 d_{G_1}(v_i) \leqslant d_{G_2}(u_j)$. This contradiction forces us to conclude that $d_{G_1}(v_k) > d_{G_1}(v_i)$. □

By Lemma 2.1 we can prove the following corollary:

Corollary 2.2. *Let G_1 be a connected graph such that $G_1 \neq K_1$ and f be an arbitrary automorphism of $G_1 \star G_2$.*

(i) *If v is the vertex of G_1 with the maximum degree in G_1, then $f(v) \in G_1$.*
(ii) *If G_1 is a regular graph, then the restriction of f to G_1 is an automorphism of G_1.*

We shall obtain some results for the automorphisms of $G_1 \star G_2$.

Lemma 2.3. *Let G_1 and G_2 be two connected graphs of orders n_1 and n_2, respectively, and $n_1 > 1$. Suppose that f is an automorphism of $G_1 \star G_2$ such that the restriction of f to G_1 is an automorphism of G_1, and also f maps the copies of G_2 to each other. Then there exist an automorphism g of G_1 and the automorphisms $h_1, \ldots, h_{n_1}$ of G_2 such that $f(G_2^i) = (h_i(G_2))^k$, where $v_k = g(v_i)$ and $i, k = 1, \ldots, n_1$.*

Proof. Let f be an automorphism of $G_1 \star G_2$ such that the restriction of f to G_1 is an automorphism of G_1, and also f maps the copies of G_2 to each other. Let f maps the ith copy of G_2, G_2^i, to the j_ith copy of G_2, $G_2^{j_i}$, where $i, j_i = 1, \ldots, n_1$, such that for the fixed numbers i and j_i we have $f(u_k^i) = u_{k'}^{j_i}$, where $k, k' = 1, \ldots, n_2$. Then we define the automorphism h_i on G_2 such that $h_i(u_k) = u_{k'}$. To complete the proof we need to show that the map g on $V(G_1)$ such that $g(v_i) = v_{j_i}$ is an automorphism of G_1, where $i, j_i = 1, \ldots, n_1$. Without loss of generality we can assume that the vertices $v_1$and v_2 are adjacent, and show that v_{j_1} and v_{j_2} are adjacent.

Since the vertices v_1 and v_2 are adjacent, the vertex v_1 is adjacent to each vertex of G_2^2 (we show this concept by $v_1 \sim G_2^2$). Hence $f(v_1) \sim (h_2(G_2))^{j_2}$, and so $f(v_1) \sim v_{j_2}$ and $v_1 \sim f^{-1}(v_{j_2})$, hence $f^{-1}(v_{j_2}) \sim G_2^1$, and finally we have $v_{j_2} \sim G_2^{j_1}$. With a similar argument we can conclude that $f(v_2) \sim v_{j_1}$, so $v_2 \sim f^{-1}(v_{j_1})$, hence $f^{-1}(v_{j_1}) \sim G_2^2$, and thus $v_{j_1} \sim G_2^{j_2}$ (see Figure 2). On the other hand, since f maps G_2^1 to $(h_2(G_2))^{j_1}$, we

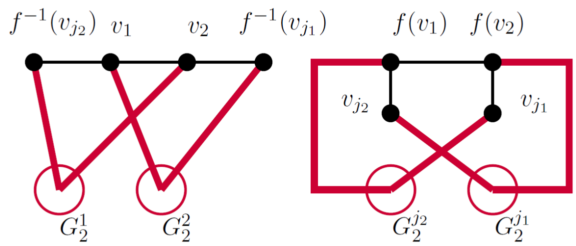

FIGURE 2. A piece of neighbourhood corona of G_1 and G_2 in the proof of Lemma 2.3.

have $d_{G_1\star G_2}(u_k^1) = d_{G_1\star G_2}((h_2(u_k))^{j_1})$. We deduce from (2.1), (2.2), and $d_{G_2}(u_k) = d_{G_2}(h_2(u_k))$, that $d_{G_1}(v_1) = d_{G_1}(v_{j_1})$. Similarly, $d_{G_1}(v_2) = d_{G_1}(v_{j_2})$. Since the restriction of f to G_1 is an automorphism of G_1, we have that $d_{G_1}(v_1) = d_{G_1}(f(v_1))$ and $d_{G_1}(v_2) = d_{G_1}(f(v_2))$. Then

$$d_{G_1}(v_1) = d_{G_1}(v_{j_1}) = d_{G_1}(f(v_1)),\ d_{G_1}(v_2) = d_{G_1}(v_{j_2}) = d_{G_1}(f(v_2)). \tag{2.3}$$

In regard to (2.3) and Figure 2, there exists vertices $v_{j_1 1}$ and $v_{j_2 1}$ adjacent to vertices v_{j_1} and v_{j_2}, respectively. Thus the vertices $v_{j_1 1}$ and $v_{j_2 1}$ are adjacent to $G_2^{j_1}$ and $G_2^{j_2}$, respectively, and so $f^{-1}(v_{j_1 1}) \sim G_2^1$ and $f^{-1}(v_{j_2 1}) \sim G_2^2$. Hence $f^{-1}(v_{j_1 1}) \sim v_1$ and $f^{-1}(v_{j_2 1}) \sim v_2$. Since $v_{j_1} \sim v_{j_1 1}$ and $v_{j_2} \sim v_{j_2 1}$, we have $f^{-1}(v_{j_1}) \sim f^{-1}(v_{j_1 1})$ and $f^{-1}(v_{j_2}) \sim f^{-1}(v_{j_2 1})$ (see Figure 3). Note that, for every vertex $x \in N_G(v_{j_2})$, we have $x \sim G_2^{j_2}$. So we see

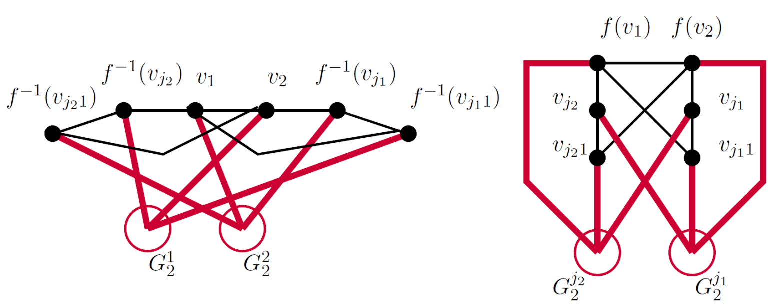

FIGURE 3. A piece of $G_1 \star G_2$ in the proof of Lemma 2.3.

that $f^{-1}(x) \sim G_2^2$, and so $f^{-1}(x) \sim v_2$ (a similar argument satisfies for each vertex in $N_G(v_{j_1})$). In regard to Figure 3 and (2.3), we need the other vertex adjacent to v_{j_1}, call it x. If x has been chosen among the nonadjacent vertices to $G_2^{j_2}$ that has been shown in Figure 3, then with the similar argument as above, we obtain that $f^{-1}(x)$ is adjacent to v_1, and so (2.3) is not satisfied, again. Therefore after finite steps we should choose a vertex x adjacent to v_{j_1}, among the vertices that are adjacent to $G_2^{j_2}$, otherwise we conclude that the order of G_1 is infinite and this is a contradiction. By Figure 3 and above information, the vertex v_{j_1} is the only vertex that is adjacent to $G_2^{j_2}$ and is not among the adjacent vertices to v_{j_1}, in each step. Hence $v_{j_1} \sim v_{j_2}$, and the result follows. □

Lemma 2.4. *Let G_1 and G_2 be two connected graphs of order n_1 and n_2, respectively, and $n_1 > 1$. If f is an automorphism of $G_1 \star G_2$, then the restriction of f to G_1 is an automorphism of G_1.*

Proof. Since f is an automorphism, it is suffices to show that the restriction of f to G_1 is an automorphism of G_1. By contradiction, suppose that $f|_{G_1}$ is not an automorphism of G_1. Without loss of generality we assume that $f(v_1) = u_1^2$. Hence by Lemma 2.1, $d_{G_1}(v_2) > d_{G_1}(v_1)$. Since f preserves the degree of the vertices, $d_{G_1 \star G_2}(v_1) = d_{G_1 \star G_2}(u_1^2)$, and so by (2.1) and (2.2) we have $(1+n_2)d_{G_1}(v_1) = d_{G_2}(u_1) + d_{G_1}(v_2)$. Suppose that $N_{G_1}(v_1) = \{v_{1,1}, \ldots, v_{1,s_1}\}$, $N_{G_1}(v_2) = \{v_{2,1}, \ldots, v_{2,s_2}\}$, and $N_{G_2}(u_1) = \{u_{1,1}, \ldots, u_{1,t}\}$ where $(1+n_2)s_1 = t + s_2$ and $s_i = d_{G_1}(v_i)$, $i = 1, 2$, and also $t = d_{G_2}(u_1)$ (see Figure 4). Since f preserves the adjacency relation, $f(N_{G_1 \star G_2}(v_1)) =$

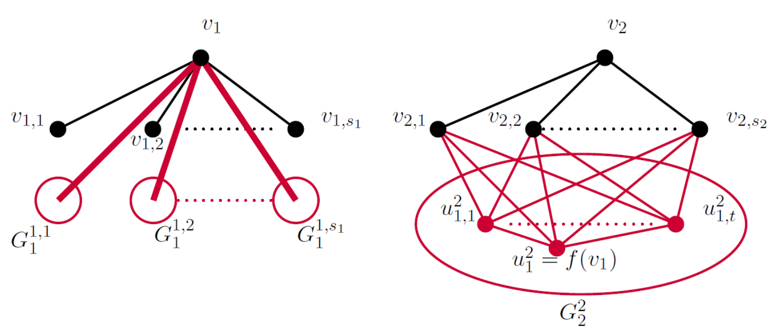

FIGURE 4. A piece of neighbourhood corona of G_1 and G_2 in the proof of Lemma 2.4.

$N_{G_1 \star G_2}(u_1^2)$, i.e.,

$$\begin{aligned} &\{f(v_{1,1}), \ldots, f(v_{1,s_1}), f(u_1^{1,1}), \ldots, f(u_{n_2}^{1,1}), \ldots, f(u_1^{1,s_1}), \ldots, f(u_{n_2}^{1,s_1})\} \\ &\qquad = \{u_{1,1}^2, \ldots, u_{1,t}^2, v_{2,1}, \ldots, v_{2,s_2}\}. \end{aligned} \tag{2.4}$$

Since $t < n_2$, there are vertices in the copies $G_2^{1,1}, \ldots, G_2^{1,s_1}$ such that they are mapped to the elements of the set $\{v_{2,1}, \ldots, v_{2,s_2}\}$, under the automorphism f. Without loss of generality we can assume that $f(u_{i_j}^{1,j}) = v_{2,j}$, where $1 \leqslant j \leqslant s_1$. We continue the proof by considering two cases for s_1 as follows:

Case 1: $s_1 > 1$.

Since v_2 is adjacent to the vertices $v_{2,1}, \ldots, v_{2,s_1}$, it follows that $f^{-1}(v_2)$ is adjacent to the vertices $u_{i_1}^{1,1}, \ldots, u_{i_{s_1}}^{1,s_1}$. Since $s_1 > 1$, thus $f^{-1}(v_2) \in G_1$ and $f^{-1}(v_2)$ is adjacent to the vertices $v_{1,1}, \ldots, v_{1,s_1}$. Hence v_2 is adjacent to the vertices $f(v_{1,1}), \ldots, f(v_{1,s_1})$, and by (2.4) we have

$$\{f(v_{1,1}), \ldots, f(v_{1,s_1})\} \subseteq \{v_{2,s_1+1}, \ldots, v_{2,s_2}\}. \tag{2.5}$$

Without loss of generality we assume that $f(v_{1,i}) = v_{2,s_1+i}$, where $1 \leqslant i \leqslant s_1$ (see Figure 5).

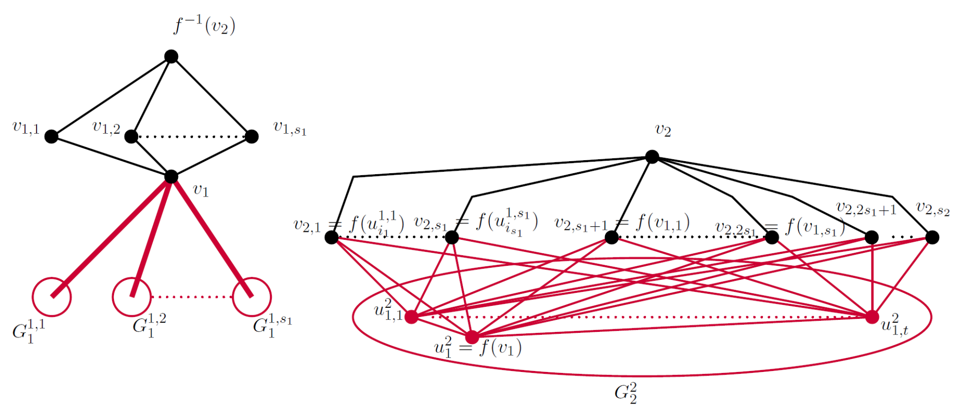

FIGURE 5. A piece of neighbourhood corona of G_1 and G_2 in the proof of Lemma 2.4.

Since $f^{-1}(v_2)$ is adjacent to the vertices $v_{1,1}, \ldots, v_{1,s_1}$, we can say that $f^{-1}(v_2)$ is adjacent to all vertices of $G_2^{1,1}, \ldots, G_2^{1,s_1}$, so v_2 is adjacent to all vertices of $f(G_2^{1,1}), \ldots, f(G_2^{1,s_1})$. Then by (2.4) we get

$$\{f(u_1^{1,1}), \ldots, f(u_{n_2}^{1,1}), \ldots, f(u_1^{1,s_1}), \ldots, f(u_{n_2}^{1,s_1})\} \subseteq \{v_{2,1}, \ldots, v_{2,s_2}\}, \tag{2.6}$$

and with respect to (2.4), (2.5), and (2.6) we have a contradiction.

Case 2: $s_1 = 1$.

Since f preserves the adjacency relation,

$$\{f(v_{1,1}), f(u_1^{1,1}), \ldots, f(u_{n_2}^{1,1})\} = \{u_{1,1}^2, \ldots, u_{1,t}^2, v_{2,1}, \ldots, v_{2,s_2}\}. \tag{2.7}$$

Since $t < n_2$, there exists a vertex in the copy $G_2^{1,1}$ such that it is mapped to an element of the set $\{v_{2,1}, \ldots, v_{2,s_2}\}$, under the automorphism f. Without loss of generality we can assume that $f(u_{i_1}^{1,1}) = v_{2,1}$. Since v_2 is adjacent to $v_{2,1}$, so $f^{-1}(v_2)$ is adjacent to $u_{i_1}^{1,1}$, and since $f^{-1}(v_2) \neq v_1$, so $f^{-1}(v_2) \in G_2^{1,1}$. Without loss of generality we can assume that

$f^{-1}(v_2) = u_{i_1 1}^{1,1}$ such that $u_{i_1 1}^{1,1}$ is adjacent to $u_{i_1}^{1,1}$ (see Figure 6). Since

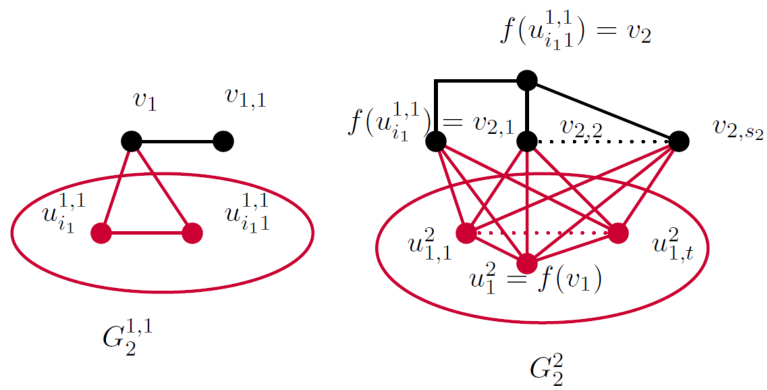

FIGURE 6. A piece of neighbourhood corona of G_1 and G_2 in the proof of Lemma 2.4.

$v_{1,1}$ is adjacent to the vertex v_1 and

$$\mathrm{dist}_{G_1 \star G_2}(v_{1,1}, u_{i_1}^{1,1}) = \mathrm{dist}_{G_1 \star G_2}(v_{1,1}, u_{i_1 1}^{1,1}) = 2,$$

thus $f(v_{1,1})$ is adjacent to the vertex u_1^2 and also

$$\mathrm{dist}_{G_1 \star G_2}(f(v_{1,1}), v_2) = dist_{G_1 \star G_2}(f(v_{1,1}), v_{2,1}) = 2. \tag{2.8}$$

Now by Equations (2.7) and (2.8) we have a contradiction. Therefore the restriction of each automorphism of $G_1 \star G_2$ to G_1 is an automorphism of G_1.

□

Corollary 2.5. *Let G_1 and G_2 be two connected graphs of order n_1 and n_2, respectively, such that $n_1 > 1$ and f an automorphism of $G_1 \star G_2$. Then the restriction of f to G_1 is an automorphism of G_1 and also there are the automorphism g of G_1 and the automorphisms $h_1, \ldots, h_{n_1}$ of G_2 such that $f(G_2^i) = (h_i(G_2))^k$, where $v_k = g(v_i)$ and $i, k = 1, \ldots, n_1$.*

Proof. By Lemmas 2.3 and 2.4, it is sufficient to prove that the copies of G_2 are mapped to each other under the automorphism f and it is true because f preserves the adjacency relation on each copy of G_2. □

The following corollary is an immediate consequence of Corollary 2.5 for graphs of the form $G \star K_1$.

Corollary 2.6. *Let G be a connected graph of order $n > 1$ and f be an arbitrary automorphism of $G \star K_1$. Then the restriction of f to G is an automorphism of G. Also $f(K_1^i) = K_1^{j_i}$ for some automorphism g of G such that $g(v_i) = v_{j_i}$ where $i, j_i = 1, 2, \ldots, n_1$.*

3. Study of $D(G_1 \star G_2)$ and $D'(G_1 \star G_2)$

In this section we use the results in Section 2 to study the distinguishing number and the distinguishing index of the neighbourhood corona of two graphs. First we consider the neighbourhood corona of an arbitrary graph with K_1. The following theorem gives an upper bound for $D(G \star K_1)$ and $D'(G \star K_1)$.

Theorem 3.1. *Let G be a connected graph of order $n > 1$. We have*
(i) $D(G \star K_1) \leqslant D(G)$,
(ii) $D'(G \star K_1) \leqslant D'(G)$.

Proof. (i) We shall define a distinguishing vertex labelling for $G \star K_1$ with $D(G)$ labels. First we label G in a distinguishing way with $D(G)$ labels. Next we assign the vertex $K_1^{v_i}$, the label of the vertex v_i where $1 \leqslant i \leqslant n$. This labelling is a distinguishing vertex labelling of $G \star K_1$, because if f is an automorphism of $G \star K_1$ preserving the labelling, then by Corollary 2.5, the restriction of f to G is an automorphism of G preserving the labelling. Since we labelled G in a distinguishing way at first, the restriction of f to G is the identity automorphism on G. On the other hand by Corollary 2.6 there exists an automorphism g of G such that $f(K_1^{v_i}) = K_1^{g(v_i)}$, $1 \leqslant i \leqslant n$. Regarding the labelling of copies of K_1, we can obtain that g is the identity automorphism on G, and so f is the identity automorphism on $G \star K_1$.

(ii) We define a distinguishing edge labelling for $G \star K_1$ with $D'(G)$ labels. First we label the edges of G in a distinguishing way with $D'(G)$ labels. By (2.1) and (2.2) we know that the degree of $K_1^{v_i}$ in $G \star K_1$ is equal to the degree of v_i in G where $1 \leqslant i \leqslant n$. Now we assign the edge between $K_1^{v_i}$ and $v_{i,j}$ where $v_{i,j} \in N_G(v_i)$, the label of the edges between v_i and $v_{i,j}$ where $j = 1, \dots, d_G(v_i)$. This labelling is a distinguishing edge labelling of $G \star K_1$, because if f is an automorphism of $G \star K_1$ preserving the labelling, then by Corollary 2.5, the restriction of f to G is an automorphism of G preserving the labelling. Since we labelled G in a distinguishing way at first, so the restriction of f to G is the identity automorphism on G. On the other hand, by Corollary 2.6 there exists an automorphism g of G such that $f(K_1^{v_i}) = K_1^{g(v_i)}$, $1 \leqslant i \leqslant n$. Regarding the labelling of the edges incident to each copies of K_1, we can obtain that g is the identity automorphism on G, and so f is the identity automorphism on $G \star K_1$. □

The bounds of $D(G \star K_1)$ and $D'(G \star K_1)$ in Theorem 3.1 are sharp. If we consider G as the star graph $K_{1,n}$, $n > 1$, then $K_{1,n} \star K_1$ is a graph as shown in Figure 7. Using the degree of the vertices of $K_{1,n} \star K_1$ we can get the automorphism group of $K_{1,n} \star K_1$ and then it can be concluded that $D(K_{1,n} \star K_1) = n = D(K_{1,n})$, and also $D'(K_{1,n} \star K_1) = n = D'(K_{1,n})$.

In Theorem 3.1, the sharp upper bounds for $D(G \star K_1)$ and $D'(G \star K_1)$ have been given, but we did not present lower bounds for these parameters. Actually, there are graphs whose distinguishing number can be arbitrarily larger than the distinguishing number of its neighbourhood corona with K_1.

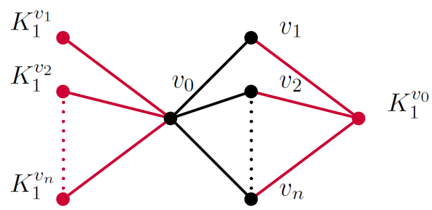

FIGURE 7. The neighbourhood corona of $K_{1,n}$ and K_1.

In other words, we can show that there exists a connected graph G of order $n > 1$ such that the value of $D(G \star K_1)/D(G)$ can be arbitrarily small. To do this we need the following two theorems. Recall that the friendship graph F_n ($n \geqslant 2$) can be constructed by joining n copies of K_2 with a common vertex.

Theorem 3.2 ([2]). *The distinguishing number of the friendship graph F_n ($n \geq 2$) is*

$$D(F_n) = \left\lceil \frac{1+\sqrt{8n+1}}{2} \right\rceil .$$

Now we obtain the exact value of the distinguishing number of neighborhood corona of F_n with K_1.

Theorem 3.3. *The distinguishing number of $F_n \star K_1$ ($n \geq 2$) is*

$$D(F_n \star K_1) = \left\lceil \sqrt{\frac{1+\sqrt{8n+1}}{2}} \right\rceil .$$

Proof. Let $V(F_n) = \{v_0, v_1, v_2, \ldots, v_{2n-1}, v_{2n}\}$ where the vertex v_0 is the central vertex and v_{2i-1} and v_{2i} are the vertices of the base of the triangles in F_n where $1 \leqslant i \leqslant n$. So $d_{F_n}(v_0) = 2n$ and $d_{F_n}(v_i) = 2$ where $1 \leqslant i \leqslant 2n$. By (2.1) and (2.2) we have $d_{F_n \star K_1}(v_0) = 4n$ and $d_{F_n \star K_1}(v_i) = 4$, also $d_{F_n \star K_1}(K_1^{v_0}) = 2n$ and $d_{F_n \star K_1}(K_1^{v_i}) = 2$ where $1 \leqslant i \leqslant 2n$ (see Figure 8).

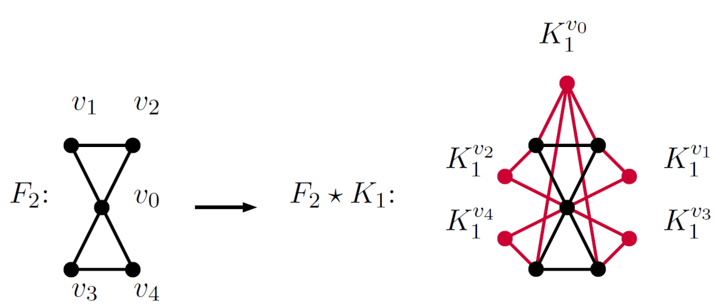

FIGURE 8. The graphs F_2 and $F_2 \star K_1$.

If f is an automorphism of $F_n \star K_1$, then f fixes the vertices v_0 and $K_1^{v_0}$ (if $n = 2$ we can get the same result by Corollary 2.5). So we assign the vertices v_0 and $K_1^{v_0}$ the label 1. Let (x_i, y_i, z_i, w_i) be the label of the vertices $(K_1^{v_{2i}}, v_{2i-1}, v_{2i}, K_1^{v_{2i-1}})$ where $1 \leqslant i \leqslant n$. Suppose that $L = \{(x_i, y_i, z_i, w_i) \mid 1 \leqslant i \leqslant n, x_i, y_i, z_i, w_i \in \mathbb{N}\}$, is a labelling of the vertices of $F_n \star K_1$ except for the vertices v_0 and $K_1^{v_0}$. If L is a distinguishing labelling of $F_n \star K_1$ then:

(i) For every $i = 1, \ldots, n$, we require $x_i \neq w_i$ or $y_i \neq z_i$. Otherwise, the automorphism f_i of $F_n \star K_1$ such that f_i maps $K_1^{v_{2i}}$ and $K_1^{v_{2i-1}}$ to each other, two vertices v_{2i-1} and v_{2i} to each other, and fixes the remaining vertices, preserves the labelling.

(ii) For every i and j in $\{1, \ldots, n\}$, with $i \neq j$, we require $(x_i, y_i, z_i, w_i) \neq (x_j, y_j, z_j, w_j)$ and $(x_i, y_i, z_i, w_i) \neq (w_j, z_j, y_j, x_j)$. Otherwise, the automorphism f_{ij} and g_{ij} of $F_n \star K_1$ by the following definitions preserve the labelling.
- The automorphism f_{ij} maps $K_1^{v_{2i}}$ and $K_1^{v_{2j}}$ to each other and also $K_1^{v_{2i-1}}$ and $K_1^{v_{2j-1}}$ to each other. The map f_{ij} maps v_{2i} and v_{2j} to each other, it also maps v_{2i-1} and v_{2j-1} to each other and fixes the remaining vertices of $F_n \star K_1$.
- The automorphism g_{ij} maps $K_1^{v_{2i}}$ and $K_1^{v_{2j-1}}$ to each other and also $K_1^{v_{2i-1}}$ and $K_1^{v_{2j}}$ to each other. The map g_{ij} maps v_{2i} and v_{2j-1} to each other, it also maps v_{2i-1} and v_{2j} to each other and fixes the remaining vertices of $F_n \star K_1$.

Using the label set $\{1, \ldots, s\}$ we can make at most $(s^4 - s^2)/2$ of the 4-ary's (x, y, z, w) satisfying (i) and (ii) because the number of 4-ary's (x, y, z, w) such that $x \neq w$ is $s(s-1)s^2$, and the number of 4-ary's (x, y, z, w) such that $y \neq z$ is $s(s-1)s^2$. On the other hand, the number of 4-ary's (x, y, z, w) such that $x \neq w$ and $y \neq z$ is $(s(s-1))^2$. So the maximum number of 4-ary's (x, y, z, w) satisfying (i) is

$$(s(s-1)s^2 + s(s-1)s^2) - (s(s-1))^2 = s^4 - s^2.$$

Among these 4-ary's we should choose the 4-ary's satisfying (ii) too. Therefore the number of 4-ary's (x, y, z, w) satisfying (i) and (ii) that can be made by the label set $\{1, \ldots, s\}$ is $(s^4 - s^2)/2$. Therefore $D(F_n \star K_1) \geqslant \min\{s : (s^4 - s^2)/2 \geqslant n\}$. By an easy computation, we see that

$$\min\left\{s : \frac{s^4 - s^2}{2} \geqslant n\right\} = \left\lceil \sqrt{\frac{1 + \sqrt{8n+1}}{2}} \right\rceil .$$

Now we present a distinguishing vertex labelling with this number of labels. We assign v_0 and $K_1^{v_0}$ the label 1. We should label the remaining vertices such that the identity automorphism preserves the labelling only. Denoting each pentagon with the vertices $K_1^{v_{2i}}, v_{2i-1}, v_{2i}, K_1^{v_{2i-1}}, v_0$ in $F_n \star K_1$ where $1 \leqslant i \leqslant n$, by a general pentagonshown in Figure 9 and we call it a blade and continue the labelling. At first, we want to know the maximum number

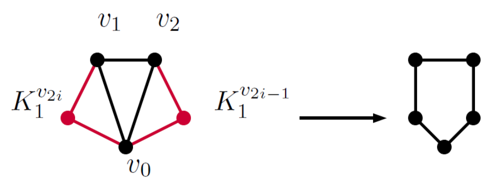

FIGURE 9. The considered pentagon (or a cycle of size 5) in the proof of Theorem 3.3.

of blades that can be labelled in a distinguishing way by 1 and 2. As we can see in Figure 10, the maximum number of blades that can be labelled in distinguishing way, by $1, 2$, is $\underline{6}$.

In order to preserve the labelling under the identity automorphism only, we should use another label to assign the next blade. As mentioned earlier, the maximum number of blades that can be labelled by each set $\{1, 3\}, \{2, 3\}$ is six. Now we want to know the maximum number of blades that can be labelled by the presence of $\{1, 2, 3\}$ at the same time in the blade. This number is 18.Because it is sufficient to label the blade with the labels 1; 2; 3 and repeat label 1. As shown in Figure 10, we can label six blades. Obviously, we can do the same by allowing repetitions of 2 and 3. Therefore the maximum number of blades that can be labelled by the presence of $\{1, 2, 3\}$ at the same time is $\underline{18}$. Until now, we labelled 36 blades.

$$\underbrace{6}_{\{1,2\}} + \underbrace{6}_{\{1,3\}} + \underbrace{6}_{\{2,3\}} + \underbrace{18}_{\{1,2,3\}} = 36$$

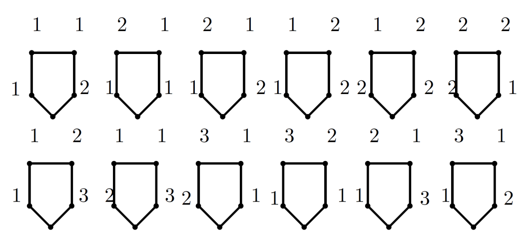

FIGURE 10. Distinguishing labelling of blades with the labels $\{1, 2\}$ and $\{1, 2, 3\}$, respectively.

If we want to label the next blade, we should add a new label, 4. The maximum number of blades that can be labelled by each set $\{1, 4\}, \{2, 4\}, \{3, 4\}$ is six. Also, the maximum number of blades that can be labelled by each set $\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$ is eighteen. We can see that the maximum number of blades that can be labelled by presence of $\{1, 2, 3, 4\}$ at the same time is $\underline{12}$ as shown in Figure 11.

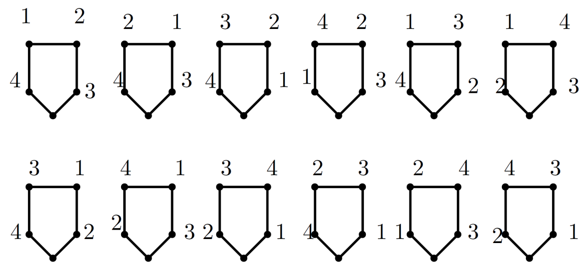

FIGURE 11. The distinguishing labelling of blades with the labels $\{1, 2, 3, 4\}$.

Thus we have labelled 120 blades now:

$$36 + \underbrace{6}_{\{1,4\}} + \underbrace{6}_{\{2,4\}} + \underbrace{6}_{\{2,4\}} + \underbrace{18}_{\{1,2,4\}} + \underbrace{18}_{\{1,3,4\}} + \underbrace{18}_{\{2,3,4\}} + \underbrace{12}_{\{1,2,3,4\}} = 120.$$

Therefore the relationship between the number of labels that has been used, $\mathbf{d}(F_n \star K_1)$, and n are described by the following sequence:

$$\{\mathbf{d}(F_n \star K_1)\} = \{0, \underbrace{2}_{\text{6-times}}, \underbrace{3}_{\text{30-times}}, \underbrace{4}_{\text{84-times}}, \dots, m, \dots, m, \dots\}.$$

where the number of the repetitions m in the above sequence is $(m-1)6 + \binom{m-1}{2}18 + \binom{m-1}{3}12$, with $m \geqslant 1$.

In fact, $\mathbf{d}(F_n \star K_1) = \min\{k : \sum_{i=1}^{k} \left(\binom{i-1}{1}6 + \binom{m-1}{2}18 + \binom{m-1}{3}12\right) \geqslant n\}$. By an easy computation, we see that

$$\begin{aligned}
&\min\left\{k : \sum_{i=1}^{k}\left(\binom{i-1}{1}6 + \binom{m-1}{2}18 + \binom{m-1}{3}12\right) \geqslant n\right\} \\
&= \min\{k : (k^4 - k^2)/2 \geqslant n\} \\
&= \left\lceil \sqrt{\frac{1+\sqrt{8n+1}}{2}} \right\rceil .
\end{aligned}$$

Therefore we have the result. □

Now we are ready to state and prove the following theorem:

Theorem 3.4. *There exists a connected graph G of order $n > 1$ such that the value of $D(G \star K_1)/D(G)$ can be arbitrarily small.*

Proof. By Theorems 3.2 and 3.3 it can be seen that

$$\lim_{n\to\infty} \frac{D(F_n \star K_1)}{D(F_n)} = \lim_{n\to\infty} \frac{\left\lceil \sqrt{\frac{1+\sqrt{8n+1}}{2}} \right\rceil}{\left\lceil \frac{1+\sqrt{8n+1}}{2} \right\rceil} = 0$$

Therefore we have the result. □

The following theorem is one of the main result of this paper and gives an upper bound for the distinguishing number of the neighbourhood corona of two arbitrary graphs:

Theorem 3.5. *Let G_1 and G_2 be two connected graphs of orders n_1 and n_2, respectively, such that $n_1 > 1$. Then $D(G_1 \star G_2) \leqslant \max\{D(G_1), D(G_2)+M\}$, where*

$$M = \min\left\{k : \sum_{m=0}^{k} y_m \geqslant D(G_1)\right\},$$
$$y_m = \begin{cases} 1 & m = 0, \\ D(G_2) & m = 1, \\ D(G_2) + \sum_{i=1}^{m-1} \binom{m-1}{i}\binom{D(G_2)}{i+1} & m \geqslant 2. \end{cases}$$

Proof. We define a distinguishing vertex labelling for $G_1 \star G_2$ with $\max\{D(G_1), D(G_2) + M\}$ labels. First we label G_1 with $D(G_1)$ labels in a distinguishing way. For the labelling of copies of G_2, we partition the vertices of G_1 by the distinguishing labelling of G_1, i.e., we partition the vertices of G_1 into $D(G_1)$ classes such that the $[i]$th class contains the vertices of G_1 having the label i in the distinguishing labelling of G_1, where $1 \leqslant i \leqslant D(G_1)$. Let $[i] = \{v_{i1}, \ldots, v_{is_i}\}$, where s_i is the size of $[i]$th class and $1 \leqslant i \leqslant D(G_1)$. By this partition we label the copies of G_2 as follows: First, we label the vertices of G_2 with $D(G_2)$ labels in a distinguishing way, next we do the following changes on the labelling of G_2. Before the labelling of the copies of G_2, we introduce the notation $G_2^{[i]}$ for the set $\{G_2^{i1}, \ldots, G_2^{is_i}\}$, i.e., $G_2^{[i]}$ is the set of copies of G_2 corresponding to the elements of the $[i]$th class, where $1 \leqslant i \leqslant D(G_1)$. In fact we partition the copies of G_2 into $D(G_1)$ classes, that $G_2^{[i]}$ is the notation of $[i]$th class. Now we present the labelling of copies of G_2 by the following steps:

Step 1) We label all of the copies of G_2 which are in $G_2^{[1]}$, in exactly the same way as the distinguishing labelling of G_2.

Step 2) For the labelling of the copies in $G_2^{[i]}$, where $2 \leqslant i \leqslant D(G_2) + 1$, we use of the new label $D(G_2) + 1$ in such a way that the label $i - 1$ in the all elements of $G_2^{[i]}$ is replaced by the new label $D(G_2) + 1$, where $2 \leqslant i \leqslant D(G_2) + 1$.

Step 3) For the labelling of the copies in $G_2^{[i]}$, where $D(G_2) + 2 \leqslant i \leqslant 2D(G_2) + 1$, we do the same action as Step 2, with the new label $D(G_2) + 2$, instead of the label $D(G_2) + 1$.

Step 4) By choosing two labels among the labels $\{1, \ldots, D(G_2)\}$, and replacing them by the two new labels $D(G_2) + 1$ and $D(G_2) + 2$, we can label the elements of $\binom{D(G_2)}{2}$ other classes of the classes $G_2^{[i]}$.

Step 5) We do the same work as Step 2 with the new label $D(G_2)+3$ instead of labels $D(G_2) + 1$. Next we label $2\binom{D(G_2)}{2}$ other classes $G_2^{[i]}$, with the two new labels $D(G_2)+1$ and $D(G_2)+3$, also with the labels $D(G_2)+2$ and $D(G_2)+3$, exactly the same as Step 4.

Step 6) Now we choose three labels among the labels $\{1,\ldots,D(G_2)\}$, and replace them by the three new labels $D(G_2)+1$, $D(G_2)+2$, and $D(G_2)+3$.

By continuing this method we conclude that the number of classes can be labelled with the labels $1,\ldots,D(G_2)+m$, $m \geqslant 1$, such that the label $D(G_2)+m$ is used in the labelling of each element of classes, is y_m where

$$y_m = \begin{cases} 1 & m = 0, \\ D(G_2) & m = 1, \\ D(G_2) + \sum_{i=1}^{m-1} \binom{m-1}{i}\binom{D(G_2)}{i+1} & m \geqslant 2. \end{cases}$$

Therefore the number of labels that have been used for the labelling of all copies of G_2, is $D(G_2)+M$ where $M = \min\{k : \sum_{m=0}^{k} y_m \geqslant D(G_1)\}$. This labelling is a distinguishing vertex labelling of $G_1 \star G_2$, because if f is an automorphism of $G_1 \star G_2$ preserving the labelling, then by Corollary 2.5, $f|_{G_1}$ is an automorphism of G_1 preserving the labelling. Since we labelled G_1 in a distinguishing way at first, f is the identity automorphism on G_1. Regarding the labelling of copies of G_2 and since f preserves the labelling of the copies of G_2, so f maps each copy of G_2 to itself. The map f is the identity automorphism on each copy of G_2, because each copy of G_2 was labelled in a distinguishing way. Therefore f is the identity automorphism on $G_1 \star G_2$. □

The following corollary is an immediate consequence of Theorem 3.5.

Corollary 3.6. *Let G_1 and G_2 be two connected graphs of orders n_1 and n_2, respectively, such that $n_1 > 1$. If $D(G_1) = 1$, then $D(G_1 \star G_2) \leqslant D(G_2)$.*

Proof. It is sufficient to note that if $D(G_1) = 1$, then the value of M in Theorem 3.5 is zero. □

We end the paper by presenting an upper bound for the distinguishing index of the neighbourhood corona of two graphs:

Theorem 3.7. *Let G_1 and G_2 be two connected graphs of orders n_1 and n_2, respectively, such that $n_1 > 1$. Then $D'(G_1 \star G_2) \leqslant \max\{D'(G_1), D'(G_2)\}$.*

Proof. We define an edge distinguishing labelling of $G_1 \star G_2$ with $\max\{D'(G_1), D'(G_2)\}$ labels. To obtain such a labelling we first label the edge set of G_1 and G_2 in a distinguishing way with $D'(G_1)$ and $D'(G_2)$ labels, respectively. For the labelling of the edges between each copy of G_2 and G_1 we use of the labelling of the edge set of G_1 as follows:

Let $N_{G_1}(v_k) = \{v_{k1},\ldots,v_{1|N_{G_1}(v_k)|}\}$, where $1 \leqslant k \leqslant n_1$. By the notation of the vertices of G_1 and the copies of G_2, we assign all edges $v_{kj_k}u_i^k$, $1 \leqslant i \leqslant n_2$, the label of the edge $v_{kj_k}v_k$ in the distinguishing labelling of the edge set of G_1, where $1 \leqslant k \leqslant n_1$ and $1 \leqslant j_k \leqslant |N_{G_1}(v_k)|$. This labelling is a distinguishing edge labelling of $G_1 \star G_2$, because if f is an automorphism of $G_1 \star G_2$ preserving the labelling, then by Corollary 2.5, the restriction of f to G_1 is an automorphism of G_1 preserving the labelling. Since we labelled G_1 in a distinguishing way at first, f is the identity automorphism on G_1. Regarding the labelling of the edges between the copies of G_2 and G_1 and by Corollary 2.5, we conclude that f maps each copy of G_2 to itself. Since we labelled each copy of G_2 in a distinguishing way, at first, so the map f is the identity automorphism on each copy of G_2, and so f is the identity automorphism on $G_1 \star G_2$. □

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Department of Mathematics, Yazd University, 89195-741, Yazd, Iran
E-mail address: alikhani@yazd.ac.ir

Department of Mathematics, Yazd University, Yazd, Iran
E-mail address: s.soltani1979@gmail.com