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SCHRÖDER PARTITIONS, SCHRÖDER TABLEAUX AND WEAK POSET PATTERNS

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ABSTRACT. We introduce the notions of *Schröder shapes* and *Schröder tableaux*, which provide an analog of the classical notions of Young shapes and Young tableaux. We investigate some properties of the partial order given by containment of Schröder shapes. Then we propose an algorithm that is the natural analog of the well-known RS correspondence for Young tableaux, and we characterize those permutations whose insertion tableaux have some special shapes. The last part of the article relates the notion of the Schröder tableau with those of interval order and weak containment (and strong avoidance) of posets. We end our paper with several suggestions for possible further work.

1. INTRODUCTION

Given a positive integer n, a *partition* of n is a finite sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_r$. When λ is a partition of n we also write $\lambda \vdash n$. A graphical way of representing partitions is given by Young shapes. The *Young shape* of the above partition $\lambda \vdash n$ consists of r left-justified rows having $\lambda_1, \ldots, \lambda_r$ boxes (also called cells) stacked in decreasing order of length. The set of all Young shapes can be endowed with a poset structure by containment (of top-left justified shapes). Such a poset turns out to be a lattice, called the *Young lattice*. A *standard Young tableau* with n cells is a Young shape whose cells are filled in with positive integers from 1 to n in such a way that entries in each row and each column are (strictly) increasing.

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Young tableaux are among the most investigated combinatorial objects. The widespread interest in Young tableaux is certainly due both to their intrinsic combinatorial beauty (which is witnessed by several surprising facts concerning, for instance, their enumeration, such as the hook length formula [14] and the RSK algorithm [22, 24, 19]) and to their usefulness in several algebraic contexts, typically in the representation theory of groups and related matters (such as Schur functions and the Littlewood–Richardson rule [20]).

Apart from their classical definition, there are several alternative ways to introduce Young tableaux. In the present paper, we are interested in the possibility of defining standard Young tableaux in terms of a certain lattice structure on Dyck paths. The main advantage of this point of view lies in the possibility of giving an analogous definition in a modified setting, in which Dyck paths are replaced by some other class of lattice paths. Here we will try to see what happens if we replace Dyck paths with Schröder paths, just scratching the surface of a theory that, in our opinion, deserves to be better studied.

Given a Cartesian coordinate system, a *Dyck path* is a lattice path starting from the origin, ending on the x-axis, never falling below the x-axis, and using only two kinds of steps, $u(p) = (1,1)$ and $d(own) = (1,-1)$. A Dyck path can be encoded by a word w on the alphabet $\{u,d\}$ such that in every prefix of w the number of u's is greater than or equal to the number of d's and the total number of u and d in w is the same (the resulting language is called a *Dyck language* and its words *Dyck words*). The *length* of a Dyck path is the length of the associated Dyck word (which is necessarily an even number).

Consider the set $\mathbf{D}_n$ of all Dyck paths of length $2n$; it can be endowed with a very natural poset structure, by declaring $P \leq Q$ whenever P lies weakly below Q in the usual two-dimensional drawing of Dyck paths (for any $P, Q \in \mathbf{D}_n$). This partial order actually induces a distributive lattice structure on $\mathbf{D}_n$, to be denoted $\mathcal{D}_n$ and called the *Dyck lattice of order n*. This can be shown both directly, using the combinatorics of lattice paths (see [12]), and as a consequence of the fact that $\mathcal{D}_n$ is order-isomorphic to (the dual of) the Young lattice of the staircase partition $(n-1, n-2, \dots, 2, 1)$ (that is the principal down-set generated by such a staircase partition in the Young lattice). Referring to the latter approach, any $P \in \mathbf{D}_n$ uniquely determines a Young shape, which can be obtained by taking the region included between P and the maximum path of $\mathcal{D}_n$, slicing it into square cells using diagonal lines of slope 1 and -1 passing through all points having integer coordinates, and finally rotating the sheet of paper by 45° anticlockwise (see Figure 1).

It is well-known that there is a bijection between standard Young tableaux of a given shape and saturated chains in the Young lattice starting from the empty shape and ending with that shape. Translating this fact on Dyck lattices, we can thus state that standard Young tableaux of a given shape are in bijection with saturated chains (inside a Dyck lattice of suitable order)

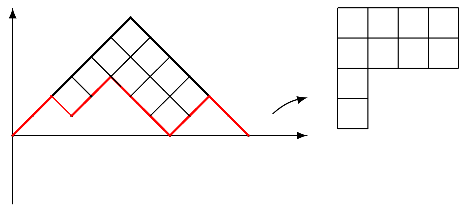

FIGURE 1. A Dyck path (red) and the associated Young shape.

starting from the Dyck path associated with that shape and ending with the maximum of the lattice. This fact has been extensively used in [10, 11] in order to enumerate saturated chains of small length in Dyck lattices. This suggests trying to find an analog of this approach in which Dyck paths are replaced by other types of paths. As already mentioned, the case treated in the present paper is that of Schröder paths.

In Section 2 we introduce the notion of Schröder shapes and study some properties of the poset of Schröder shapes (which are in some sense analogous to those of the Young lattice). In Section 3 we introduce the notion of the Schröder tableaux and we define an algorithm which, given a permutation, produces a pair of Schröder tableaux having the same Schröder shape; this is analogous to the classical RS algorithm. In particular, we will address the problem of determining which permutations are mapped into the same Schröder insertion tableau, and we solve it for a few special shapes. Section 4 offers an alternative description of the notion of Schröder tableaux in terms of two seemingly unrelated concepts: one is well-known (interval orders) whereas the other one (weak pattern poset and strong poset avoidance) is much less studied; we then give an overview of a possible combinatorial approach to the study of weak poset containment and strong poset avoidance, and provide a link between these notions and Schröder tableaux. Finally, we devote Section 5 to the presentation of some directions of further research.

An extended abstract of the present work has appeared in the proceedings of the conference IWOCA 2015 [8].

2. The poset of Schröder partitions

A *Schröder shape* is a set of triangular cells in the plane obtained from a Young shape by drawing the NE-SW diagonal of each of its (square) cells, and possibly adding below the first column and at the end of some rows one more triangular cell provided that, in a group of rows having equal length, only the first (topmost) one can have an added triangle. The number of cells of a Schröder shape is called the *order* of that shape. An example of a Schröder shape is illustrated in Figure 2.

A Schröder shape has triangular cells of two distinct types, which will be referred to as *lower triangular cells* and *upper triangular cells*. In particular, rows having an odd number of cells necessarily terminate with an upper triangular cell. A Schröder shape determines a unique integer partition,

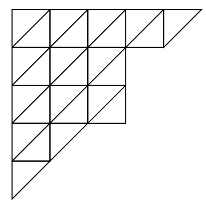

Figure 2. A Schröder shape of order 25.

whose parts are the number of cells in the rows of the shape. For instance, the partition associated with the shape in Figure 2 is $(9,6,6,3,1)$. As a consequence of the definition of a Schröder shape, it is clear that not every partition can be represented using a Schröder shape. More precisely, we have the following result, whose proof is completely trivial and so it is left to the reader.

Proposition 2.1. *An integer partition can be represented with a Schröder shape if and only if its odd parts are* simple *(i.e. have multiplicity 1).*

Those integer partitions which can be represented with a suitable Schröder shape will be called *Schröder partitions*. The set of all Schröder partitions will be denoted **Sch**, and the set of Schröder partitions of order n with $\mathbf{Sch}_n$. From now on we will frequently refer to Schröder shapes and to Schröder partitions interchangeably, when no confusion is likely to arise.

From the enumerative point of view, the number of Schröder partitions is known, and is recorded in [26] as sequence A006950. In particular, the generating function of Schröder partitions is given by

$$\prod_{k>0} \frac{1+x^{2k-1}}{1-x^{2k}}.$$

There are several combinatorial interpretations for the resulting sequence, however an appropriate reference for the present one (in terms of Schröder partitions) appears to be [7]. In that paper the author proves a far more general result concerning partitions such that the multiplicity of each odd part is in a prescribed set and the multiplicity of each even part is unrestricted.

It is interesting to notice that this sequence is also relevant from an algebraic point of view. Indeed, as remarked in the comments for sequence A006950 in [26], it coincides with the sequence of numbers of nilpotent conjugacy classes in the Lie algebras $o(n)$ of skew-symmetric $n \times n$ matrices. This suggests that Schröder partitions have a role in representation theory that certainly deserves to be better investigated.

Though the formalism of Schröder shapes seems not to add relevant information on the enumerative combinatorics of Schröder partitions, it suggests at least an interesting family of maps on integer partitions, which turns out to define a family of involutions if suitably restricted. Consider the family of maps $(c_n)_{n\in\mathbf{N}}$ defined on the set of all integer partitions as follows:

given a partition λ and a positive integer n, $c_n(\lambda)$ is the integer partition $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ (of the same size as λ) whose ith part μ_i is given by the sum of the n columns of (the Young shape of) λ from the $((i-1)n+1)$th one to the (in)th one. So, for instance, $c_3((7,6,6,6,4,3,3,1)) = (22,13,1)$. Since each of the above maps preserves the size of a partition, it is clearly an endofunction when restricted to the set of all integer partitions of size n. Notice that c_1 is the well-known conjugation map (which exchanges rows with columns in a Young shape). Even though c_1 is an involution (on the set of all partitions), it is easy to see that all the other c_n's are not involutions. However, it is possible to characterize the set of those partitions for which c_n^2 acts as the identity map. The next proposition, which was incorrectly formulated in [8], is now stated and proved correctly.

Proposition 2.2. *Given $n \in \mathbf{N}$ and an integer partition λ (whose ith part will be denoted λ_i, as usual), we have that $c_n^2(\lambda) = \lambda$ if and only if for all $k \geq 0$, there exists at most one index i such that $kn < \lambda_i < (k+1)n$.*

Proof. For any given λ, suppose that there exists at least one part of $c_n(\lambda)$ which is not multiple of n, and let μ' be one of them. More precisely, let k be the unique nonnegative integer such that $kn < \mu' < (k+1)n$. This means that λ has a set of n consecutive columns whose sum is equal to μ'. Since $\mu' \not\equiv 0 \pmod{n}$, this implies that such n columns are not all equal. In particular, the rightmost column must have less than $k+1$ cells. Now, since in a Young shape columns are in decreasing order of length, the sum of the successive n columns of λ is at most kn, thus, on the right of μ', there are no other parts of $c_n(\lambda)$ strictly greater than kn. Using a similar argument, we observe that, in the set of columns of λ that sum up to μ', the first (leftmost) column must have at least $k+1$ cells, and so the sum of the previous n columns of λ is at least $(k+1)n$; as a consequence, every part before μ' is at least $(k+1)n$. We have thus proved that the condition in the above statement holds for every partition in the image of c_n. This is enough to conclude that, if $c_n^2(\lambda) = \lambda$, then necessarily the same condition holds for λ (which lies in the image of c_n).

Conversely, split each row of λ into clusters containing n consecutive cells, except the last cluster which contains at most n cells. Denoting with μ_i the ith part of $c_n(\lambda)$, we have that μ_i is obtained by taking the ith cluster from each row, and the hypothesis implies that, among the rows whose contribution is nonzero, there is at most one row whose contribution is strictly less than n. The construction of $c_n(\lambda)$ from λ is illustrated in Figure 3 for the partition $\lambda = (9,7,6,6,6,4,3,3,2)$ and $n = 3$: cells with the same label have to be grouped together, and the resulting partition $c_3(\lambda) = (26,16,4)$ is depicted on the right of Figure 3.

Now, similar to the above, to construct $c_n^2(\lambda)$, we have to split each row of $c_n(\lambda)$ into clusters. Notice that, as a consequence of our hypothesis, if the ith row of $c_n(\lambda)$ has a (necessarily unique) cluster containing strictly less than n cells, then this is precisely the unique cluster with less than n cells

a	a	a	b	b	b	c	c	c
a	a	a	b	b	b	c		
a	a	a	b	b	b			
a	a	a	b	b	b			
a	a	a	b	b	b			
a	a	a	b					
a	a	a						
a	a	a						
a	a							

a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	b	b	b	b	b	b	b	b	b	b	b	b	b	b	b										
c	c	c	c																						

FIGURE 3. From λ to $c_n(\lambda)$.

among all the ith clusters of all rows of λ. Therefore, constructing $c_n^2(\lambda)$ from $c_n(\lambda)$ we recover exactly the starting partition λ, as desired. □

As already mentioned, as a special case of the above proposition we have that the set of all integer partitions is the set of fixed points of the map c_1^2 (where c_1 is the conjugation map) because the condition of the proposition becomes empty in this case. Another consequence is recorded in the following corollary, which shows the role of Schröder partitions in this context.

Corollary 2.3. *The set of Schröder partitions is the set of fixed points of the map* c_2^2.

Proof. Just observe that, setting $n = 2$ in the previous proposition, the requirement in order to have $c_2^2(\lambda) = \lambda$ is that there is at most one part of λ between two consecutive even numbers, which means precisely that odd parts have to be simple. □

Before moving on to their order structure, we remark that Schröder partitions can be investigated in the context of *overpartitions* [6]. An overpartition can be defined as an integer partition in which the last occurrence of each part can be overlined. There is a bijection that maps an overpartition λ into a Schröder partition as follows: for each part λ_i of λ, λ_i is mapped into $2\lambda_i$ if it is not overlined, whereas it is mapped into $2\lambda_i - 1$ if it is overlined. This suggests that techniques and results from overpartitions may be useful in understanding the combinatorics of Schröder partitions.

The set **Sch** of all Schröder shapes can be naturally endowed with a poset structure, by declaring $\lambda \leq \mu$ whenever the set of cells of the shape λ is a subset of the set of cells of the shape μ, provided that we draw the two shapes in such a way that their top-left cells coincide. This is equivalently (and perhaps more formally) expressed in terms of Schröder partitions: if $\lambda = (\lambda_1, \dots, \lambda_h)$ and $\mu = (\mu_1, \dots, \mu_k)$, then $\lambda \leq \mu$ when $h \leq k$ and, for all $i \leq h$, $\lambda_i \leq \mu_i$. Therefore the poset $\mathscr{Sch}$ of Schröder shapes is actually a subposet of the Young lattice. However, it seems not at all a trivial one; notice, in particular, that an interval of the Young lattice whose endpoints are Schröder partitions does not contain only Schröder partitions (apart from very simple cases). In general, it appears to be very hard (if not impossible) to infer nontrivial properties of the Schröder poset from properties of the Young lattice. The rest of this section is devoted to developing some elements of the theory of the Schröder poset along the lines suggested by the classical theory of its more noble relative, the Young lattice.

One of the most fundamental properties the Schröder poset shares with the Young lattice is the fact that it is a distributive lattice. We will obtain this result as a consequence of a more general one, which is of independent interest and can be seen as a slight generalization of Lemma 2.1 in [2][1].

Theorem 2.4. *Given a function $f : \mathbf{N} \to \mathbf{N} \cup \{\infty\}$, denote with $\mathbf{P}_f$ the set of integer partitions in which part i appears at most $f(i)$ times. Then $\mathbf{P}_f$ is a distributive sublattice of the Young lattice (with partwise join and meet).*

Proof. Since every sublattice of a distributive lattice is distributive, it will be enough to show that $\mathbf{P}_f$ is a sublattice of the Young lattice.

Given two partitions $\lambda, \mu \in \mathbf{P}_f$, their join in the Young lattice is the partition $\lambda \vee \mu$ whose ith part is the maximum between λ_i and μ_i, for all i. We will now show that $\lambda \vee \mu$ is in $\mathbf{P}_f$.

By contradiction, suppose that part i appears more than $f(i)$ ($\neq \infty$) times in $\lambda \vee \mu$, and denote with $(\lambda \vee \mu)_t = \cdots = (\lambda \vee \mu)_{t+j} = i$ all such parts in $\lambda \vee \mu$. Since $\lambda, \mu \in \mathbf{P}_f$, it cannot happen that $(\lambda \vee \mu)_{t+s} = \lambda_{t+s}$, for all $s = 0, \dots, j$, and the same holds with λ replaced by μ. In other words, there exist two indices k, r, with $0 < k \leq r < j$, such that (without loss of generality) $\lambda_t = \lambda_{t+1} = \cdots = \lambda_{t+r} = i > \lambda_{t+r+1}$ and $\mu_{t+k-1} > \mu_{t+k} = \mu_{t+k+1} = \cdots = \mu_{t+j} = i$. But this would imply, in particular, that $\lambda_{t+k-1} = i = (\lambda \vee \mu)_{t+k-1} \geq \mu_{t+k-1} > \mu_{t+k} = i$, which is impossible. We can thus conclude that $\lambda \vee \mu$ is in $\mathbf{P}_f$.

Using a similar argument one can also show that the meet of two partitions belonging to $\mathbf{P}_f$ in the Young lattice is again a partition of $\mathbf{P}_f$, thus completing the proof. $\square$

Corollary 2.5. *The Schröder poset $\mathscr{S}\!\mathscr{ch}$ is a distributive lattice.*

Proof. Apply the previous theorem with f defined by setting $f(n) = \infty$ when n is even and $f(n) = 1$ when n is odd. $\square$

Recall that, using Birkhoff's representation theorem, elements of a finite distributive lattice can be seen as down-sets[2] of the poset of its join-irreducibles[3]. Though the Schröder lattice is infinite, the above approach can be useful to study some of its down-sets. We refer to [9], where this point of view is applied also to other similar classes of lattices.

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[1]We also notice that the Schröder poset is a partition ideal of order 1, in the terminology introduced in [2].

[2]In a poset $\mathscr{P}$, a down-set I is a subset of $\mathscr{P}$ such that, if $x \in I$ and $y \leq x$, then $y \in I$.

[3]In a lattice $\mathscr{L}$, a join-irreducible x is an element of $\mathscr{L}$ such that, for any $y, z \in \mathscr{L}$, $x = y \vee z$ implies that $x = y$ or $x = z$.

The Young lattice is the prototypical example of a differential poset. Following [27], an *r-differential poset* (for some positive integer r), is a locally finite, ranked poset $\mathcal{P}$ having a minimum and such that:

- for any two distinct elements x, y of $\mathcal{P}$, if there are exactly k elements covered by both x and y, then there are exactly k elements which cover both x and y;
- if x covers exactly k elements, then x is covered by exactly $k + r$ elements.

As noted in [27], in the first of the above conditions, if $k \geq 2$ then $\mathcal{P}$ is not a lattice. The Young lattice is a 1-differential poset. More specifically, it is the unique 1-differential distributive lattice. Thus, it is clear that $\mathscr{Sch}$ is not a 1-differential poset, since we have proved that it is a distributive lattice (and it is clearly not isomorphic to the Young lattice). However, it belongs to a wider class of posets which we believe to be an interesting generalization of differential posets.

Let φ be a map sending a positive integer k to an interval $\varphi(k)$ of positive integers. We say that a poset $\mathcal{P}$ is a *φ-differential poset* when it is an infinite, locally finite, ranked poset with a minimum such that:

(1) for any two distinct elements x, y of $\mathcal{P}$, if there are exactly k elements covered by both x and y, then there are exactly k elements which cover both x and y;
(2) if x covers exactly k elements, then x is covered by l elements, for some $l \in \varphi(k)$.

When there exists a positive integer r such that $\varphi(k) = \{k + r\}$, for all k, a φ-differential poset is just an r-differential poset.

The next proposition shows that $\mathscr{Sch}$ is indeed a φ-differential distributive lattice, for a suitable φ (condition (1) in the above definition is trivially satisfied with $k = 1$).

Proposition 2.6. *Let λ be a Schröder partition covering k Schröder partitions in $\mathscr{Sch}$. Then λ is covered by l Schröder partitions with $\lceil (k+1)/2 \rceil \leq l \leq 2k$.*

Proof. Given λ in $\mathscr{Sch}$, we denote with $\uparrow \lambda$ the number of elements of $\mathscr{Sch}$ covering λ and with $\downarrow \lambda$ the number of elements of $\mathscr{Sch}$ which are covered by λ. From the hypothesis we have that $\downarrow\lambda = k$.

In the rest of the proof we slightly modify our notation for partitions. Namely, we will add to each partition a smallest part equal to 0. So, for instance, we will write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1})$, with $\lambda_{r+1} = 0$. A part λ_i of λ will be called *up-free* whenever either

- it is odd, or
- it is even and $\lambda_{i-1} \neq \lambda_i, \lambda_i + 1$.

Similarly, it will be called *down-free* whenever either

- it is odd, or

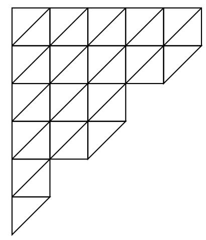


FIGURE 4. A Schröder shape which maximizes $\uparrow\lambda$.

- it is even and $\lambda_{i+1} \neq \lambda_i, \lambda_i - 1$.

In particular, each odd part of λ is both up-free and down-free (for this reason, we will sometimes refer to odd parts as *trivial* up-free (or down-free) parts). Observe that, concerning the special (even) part $\lambda_{r+1} = 0$, it is never down-free by convention, whereas it is assumed to be up-free when $\lambda_r \neq 1$.

In order to determine $\uparrow\lambda$, we observe that we have to find those parts of λ to which we can add 1 without losing the property of being a Schröder partition. These are precisely all up-free parts. Similarly, $\downarrow\lambda$ is given by the number of down-free parts.

If we want to maximize $\uparrow\lambda$, it is then clear that we have to choose a Schröder partition λ having many nontrivial up-free parts and few nontrivial down-free parts (odd parts are irrelevant). Observe moreover that we can restrict ourselves to the case of λ having all distinct parts since several repeated (even) parts is equivalent to having only one part of the same cardinality (all parts except for the top one cannot be modified). Concerning the greatest part of λ, λ_1, we notice that it has to be even, otherwise it would be down-free. Moreover, in order to have few partitions immediately below λ, we should try to make $\lambda_1 = 2n$ not down-free. To do this, just choose $\lambda_2 = \lambda_1 - 1 = 2n - 1$ (recall that we are assuming λ to have all distinct parts). Observe that, in this way, λ_2 is odd (and so trivially down-free), however it is not difficult to realize that any other choice of λ_2 would have produced a down-free part (an even part strictly larger than another even part is certainly down-free). Now, concerning λ_3, we want it to be up-free but not down-free. The first condition is fulfilled if and only if $\lambda_3 \neq \lambda_2 - 1 = 2n - 2$; for the second condition, we must choose λ_3 even and such that $\lambda_4 = \lambda_3 - 1$. Without loss of generality, we can set $\lambda_3 = 2n - 4$, so that $\lambda_4 = 2n - 5$. We can now argue in a completely analogous way for all the remaining parts of λ until we have $\downarrow\lambda = k$ (notice that n has to be large enough to reach this goal). In the end, we obtain that a partition λ which maximizes $\uparrow\lambda$ has odd-indexed parts $\lambda_{2i+1} = 2n - 4i$ and even-indexed parts $\lambda_{2i} = 2n - 4i - 1$ (see Figure 4 for an example).

A direct computation then shows that, in the best possible cases (which occur when the smallest part of λ is 1 or 2), we get $\uparrow\lambda = 2k$, as desired.

A similar approach allows also to determine a lower bound for $\uparrow\lambda$. The only difference with the previous arguments is that now we would like to have a partition λ having many down-free parts and few up-free parts. It turns out that the role of odd and even parts are swapped in the above arguments. Specifically, it can be shown that the largest part λ_1 of λ has to be odd and that $\lambda_2 = \lambda_1 - 1$. In the end, we obtain a partition having odd-indexed parts $\lambda_{2i+1} = 2n-4i-1$ and even-indexed parts $\lambda_{2i} = 2n-4i-2$. Similar to before, a direct computation shows that, when the smallest part of λ is 1 or 2, we get the desired lower bound. The task of providing all the details is then left to the reader. □

The bounds mentioned in the above proposition are clearly strict. As we have mentioned in the proof, Figure 4 shows a Schröder shape that maximizes $\uparrow\lambda$, whereas a Schröder shape that minimizes $\uparrow\lambda$ is the one associated with the partition $(9,8,5,4,1)$.

3. An RSK-like algorithm for Schröder tableaux

From the algorithmic point of view, the main application of Young tableaux is in the context of the RSK algorithm. This algorithm, named after Robinson, Schensted, and Knuth, takes as an input a word (on the alphabet of positive integers) of length n and produces two semistandard Young tableaux with n cells having the same shape. For what concerns us, we will deal with a special case of the RSK algorithm, often referred to as *Robinson–Schensted correspondence* (briefly, RS correspondence), in which the input is a permutation of length n and the output is given by a pair of standard Young tableaux. A brief description of such an algorithm is given below (Algorithm 1, where $\pi = \pi_1\pi_2\cdots\pi_n$ is a generic permutation of length n).

The RSK algorithm is extensively described in the literature. For instance, the interested reader can find a modern and elegant presentation of it in [3]. Among other things, one of the most beautiful properties of the RS correspondence is that it establishes a bijection between permutations of length n and pairs of standard Young tableaux with n cells having the same shape. This fact bears important enumerative consequences, as well as strictly algebraic ones. For a given permutation π, the tableaux of the pair (P,Q) returned by the RS algorithm are usually referred to as the *insertion tableau* (the tableau P) and the *recording tableau* (the tableau Q). As a consequence, we have the following nice result, which can again be found in [3].

Theorem 3.1. *Denote with f^λ the number of standard Young tableaux of shape λ. Then we have:*

$$n! = \sum_{\lambda\vdash n} (f^\lambda)^2.$$

A *standard Schröder tableau* (from now on, simply *Schröder tableau*) with n cells is a Schröder shape whose cells are filled in with positive integers from

Algorithm 1: RS(π)

```
P := [π_1];
Q := [1];
for k from 2 to n do
    α := π_k;
    for i ≥ 1 do
        if α is bigger than all elements in the ith row of P then
            append a cell with π_k inside at the end of the ith row of P;
            append the cell [k] at the end of the ith row of Q;
            break;
        else
            write α in the cell of the ith row containing the smallest
              element β bigger than α;
            α := β;
        end
    end
end
```

1 to n in such a way that entries in each row and each column are (strictly) increasing.

We propose a natural analog of the RS algorithm for Schröder tableaux. The main difference (which is due to the specific underlying shape of a Schröder tableaux) lies in the fact that there are two distinct ways of managing the insertion of a new element in the tableau, depending on whether the cell it should be inserted in is an upper triangle or a lower triangle. As a consequence, our algorithm does not establish a bijection between permutations and pairs of Schröder tableaux (for instance, the two permutations 12 and 21 generate the same pair of Schröder tableaux); nevertheless, due to the strict analogy with the RS correspondence, we believe that it is very likely to have interesting combinatorial properties. A description of our algorithm is given below (Algorithm 2, where π is as in Algorithm 1).

Example 3.2. *Consider the permutation* $\pi = 465193287$. *The pair* (P, Q) *of Schröder tableaux produced by applying the algorithm Sch to* π *is illustrated in Figure 5.*

In this section we aim at starting the investigation of the combinatorial properties of this RS-analog. More specifically, we will address the following problems: given a Schröder shape P, can we characterize those permutations having a Schröder tableau of shape P as their insertion tableau? How many of them are there? This problem seems to be quite difficult in its full generality; here we will deal with very few simple cases, for which we can provide complete answers.

Algorithm 2: Sch(π)

```
P := the 1-cell Schröder tableau with $\pi_1$ written in the cell;
Q := the 1-cell Schröder tableau with 1 written in the cell;
for $k$ from 2 to $n$ do
    $\alpha := \pi_k$;
    for $i \geq 1$ do
        if $\alpha$ is bigger than all elements in the $i$th row of $P$ then
            append a cell (either an upper or a lower triangle) with $\pi_k$
              inside at the end of the $i$th row of $P$;
            append a cell (either an upper or a lower triangle) with $k$
              inside at the end of the $i$th row of $Q$;
            break;
        else
            let $A$ be the cell of the $i$th row containing the smallest
              element bigger than $\alpha$;
            if $A$ is an upper triangle then
                if $A$ has a twinned lower triangle immediately below it,
                  set $\beta$ := content of such lower triangle;
                move the content of $A$ to the lower triangle
                  immediately below $A$, possibly by creating such a
                  triangle if it does not exist;
                write $\alpha$ in $A$;
                if $\beta$ exists, set $\alpha := \beta$, else break;
            else
                $\beta$ := content of $A$;
                write $\alpha$ in $A$;
                $\alpha := \beta$;
            end
        end
    end
end
```

Figure 5. How our RS-like algorithm works.

3.1. **Permutations with given Schröder insertion shape.** In this subsection we collect some starting results concerning permutations whose Schröder insertion tableaux have simple shapes.

The first case we investigate is that of a Schröder shape consisting of a single row (which can terminate either with an upper or a lower triangle). To state our result we first need to recall a classical definition.

Given a permutation $\pi = \pi_1 \cdots \pi_n$, we say that π_i is a *left-to-right maximum* (or, briefly, *LR maximum*) whenever $\pi_i = \max(\pi_1, \dots, \pi_i)$. The proof of the next proposition can be found in [8].

Proposition 3.3. *Let $\pi = \pi_1 \cdots \pi_n$ be a permutation of length n. The Schröder insertion tableau of π has a single row if and only if, for all $i \le n$:*

(1) *if i is odd, then π_i is a LR maximum of π;*
(2) *if i is even, then π_i is a LR maximum of the permutation obtained from π by removing π_{i-1} (and suitably renaming the remaining elements).*

The permutations π of length n whose Schröder insertion tableau have a single row can therefore be simply characterized as follows: for all i, $\{\pi_{2i+1}, \pi_{2i+2}\} = \{2i+1, 2i+2\}$. As a consequence of this fact, a formula for the number of such permutations follows immediately.

Proposition 3.4. *The set of permutations of length n whose Schröder insertion tableau consists of a single row has cardinality $2^{\lfloor \frac{n}{2} \rfloor}$.*

The second case we consider is the natural counterpart of the previous one, that is Schröder shapes having a single column. Despite the similarities with the previous case, it turns out that the set of permutations having Schröder insertion tableau of this form can be nicely described in terms of *pattern avoidance*.

Given two permutations σ and $\tau = \tau_1 \cdots \tau_n$ (of length k and n respectively, with $k \le n$), we say that there is an *occurrence* of σ in τ when there exists indices $i_1 < i_2 < \cdots < i_k$ such that $\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}$ is order isomorphic to σ. When there is an occurrence of σ in τ, we also say that τ *contains the pattern* σ. When τ does not contain σ, we say that τ *avoids the pattern* σ. The set of all permutations of length n avoiding a given pattern σ is denoted with $Av_n(\sigma)$. Some useful references for the combinatorics of patterns in permutations are [4] and [18], whereas similar notions of patterns in set partitions and in compositions and words are studied in [21] and [16], respectively.

Proposition 3.5. *Let $\pi = \pi_1 \cdots \pi_n$ be a permutation of length n. The Schröder insertion tableau of π has a single column if and only if $\pi \in Av_n(123, 213)$.*

Proof. The Schröder insertion tableau of π has a single column if and only if, for all $i \le n$, $\pi_i < \min(\{\pi_1, \dots, \pi_{i-1}\} \setminus \min\{\pi_1, \dots, \pi_{i-1}\})$ (i.e., π_i is smaller than the second minimum of the set of all previous elements). Thus π can be factored into subpermutations (made of consecutive elements of π), say $\pi = \tilde{\pi}_1 \cdots \tilde{\pi}_r$, in such a way that each factor $\tilde{\pi}_i$ is isomorphic to a permutation of the form $1t(t-1)\cdots 32$ (for some t) and each element of $\tilde{\pi}_i$ is greater than each element of $\tilde{\pi}_{i+1}$ (for all i). In the language of permutation patterns, this is usually expressed by saying that π is a *skew sum* of permutations of the form $1t(t-1)\cdots 32$. It is now a known fact (see, for instance, [1]) that such permutations are precisely those avoiding the two patterns 123 and 213. $\square$

Many classes of permutations avoiding a given set of patterns have been enumerated. The above one is among them, see [25].

Proposition 3.6. *The set of permutations of length n whose Schröder insertion tableau consists of a single column has cardinality 2^{n-1}.*

We close this section by illustrating one more case which is, in some sense, a generalization of both the cases described above. Namely, we consider the case of what can be called *Schröder hooks*, that is Schröder shapes having at most one row and one column with more than one upper triangular cell. Since this case is considerably more difficult than the previous ones, we need some preparation and our results will be less elegant. Nevertheless, the strategy employed reveals some features of our algorithms that are interesting in themselves.

Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ (that will be fixed until the end of this section), π_i is a *quasi-left-to-right minimum* (briefly, a QLTR minimum) of π when either $i = 1,2$ or $\pi_i < \min(\{\pi_1,\dots,\pi_{i-1}\}\setminus\{\min(\pi_1,\dots,\pi_{i-1})\})$. In other words, a QLTR minimum of π is an element of π which is smaller than the second smallest element of π preceding it. For instance, in the permutation $\rho = \underline{4}\,\underline{1}\,7\,\underline{2}\,9\,3\,6\,5\,8$, the QLTR minima are the underlined elements, i.e. $4,1,2$.

Given a natural number i, the *ith QLTR sequence of* π is recursively defined as follows:

- when $i = 1$, it is the sequence of the QLTR minima of π;
- when $i > 1$, it is the sequence of the QLTR minima of the permutation obtained from π by deleting the elements of its jth QLTR sequence, for all $j < i$.

The ith QLTR sequence of a permutation can be interpreted as a permutation as well, by simply replacing its kth smallest element with k. When no confusion is likely to arise, we will call "ith QLTR sequence" both the sequence and the associated permutation.

Every permutation π can be written as the shuffle of its QLTR sequences. Considering again the permutation ρ above, such a decomposition is the following: $\rho = \underline{4}\,\underline{1}\,\tilde{7}\,\underline{2}\,\tilde{9}\,\tilde{3}\,\tilde{6}\,\tilde{5}\,\hat{8}$.

Lemma 3.7. *The ith QLTR sequence of a permutation π avoids 123 and 213, for all i.*

Proof. Observe that, for each i, the ith sequence of π consists of those elements which enter the Schröder insertion tableau of π in column i. So, using an argument completely analogous to that of Proposition 3.5, we get the thesis. $\square$

We now label the elements of π by recording the column in which they enter the Schröder insertion tableau. We say that the *c-label* of π_i is c_j when it enters the Schröder insertion tableau of π in column j. The *c-word* of π is then the word obtained from π by replacing each π_i with its c-label. Moreover, the *bumping word* of π is obtained from the c-word by deleting the two occurrences of c_j corresponding to the two smallest elements whose c-label is c_j, for every j, and the *bumping sequence* of π is the sequence of elements of π corresponding to its bumping word. Therefore the c-word of our running example ρ is $c_1^2c_2c_1c_2^4c_3$, its bumping word is $c_1c_2^3$ and its bumping sequence is $4\,7\,9\,6$.

We come finally to our last definition. The *ordered bumping sequence* is obtained from the bumping sequence of π by rearranging in decreasing order the element having the same c-label (and keeping their relative positions). So the ordered bumping sequence of ρ is $4\,9\,7\,6$.

We are now ready to state our main result on Schröder hooks.

Proposition 3.8. *The Schröder insertion tableau of π is a Schröder hook if and only if the ordered bumping sequence of π avoids 123 and 213.*

Proof. Saying that the Schröder insertion tableau of π is a Schröder hook is equivalent to saying that every element of π which is bumped down from the first to the second row always goes to the first column. Now observe that, for every j, the first two elements of π having c-label j are inserted into the jth cell of the first row without causing any element to be bumped down. On the other hand, all the successive elements that are inserted into the same cell bump down an element having the same c-label. This means that the bumping word of π records the c-labels of the elements that are successively bumped down into the second row. However, the bumping sequence *is not* the sequence of the bumped elements. Instead, each element π_i of the bumping sequence having c-label j bumps down the second smallest element among those preceding π_i and having c-label j. In fact, when π_i is inserted, all the previous elements having c-label j have been bumped down, except for the two smallest ones, and of course π_i bumps down the largest of the two. As a consequence, we observe that the elements of the jth QLTR sequence of π are bumped down in decreasing order, with the two smallest ones which remain in the first row. Therefore, summing up the above considerations, we have that the kth element of π which is bumped down from the first row has a c-label equal to the kth letter of the bumping word; moreover, since the set of elements having the same c-label are bumped down in decreasing order, the ordered bumping sequence is precisely the sequence of the elements that are bumped down in the correct order. Now, it is clear that the Schröder insertion tableau of π is a Schröder hook if and only if all the elements of π that are bumped down from the first row are placed into the first column, and this happens if and only if the ordered bumping sequence represents a permutation whose insertion Schröder tableau has a single column. From Proposition 3.5, we know that this happens precisely when the ordered bumping sequence avoids both 123 and 213, as desired. □

4. An alternative view of Schröder tableaux

Following our treatment, Schröder tableaux can be interpreted as upper saturated chains in Schröder lattices (where upper means that the maximum of the chain is the maximum of the lattice). To be more specific, given any Schröder shape λ, the principal down-set [4] $\langle\lambda\rangle$ generated by λ in $\mathscr{Sch}$ is a lattice (we use the terminology *Schröder lattices* to refer to such lattices). In $\langle\lambda\rangle$, each saturated chain starting from $\mu \in \langle\lambda\rangle$ and ending with λ can be clearly encoded using a Schröder tableau of shape λ (removing triangles following the order of their integer labels recovers the chain from λ to μ).

Now we propose a different description of Schröder tableaux, relying on at least two main ingredients: interval orders (which are a well-known class of posets) and a notion of *weak pattern* for posets, which is not entirely new in its own right, but appears to have never been considered from a strictly order-theoretic point of view.

4.1. **Interval orders.** A poset $\mathcal{P}$ is called an *interval order* when it is isomorphic to a collection of intervals of the real line, with partial order relation given as follows: for any two intervals I, J, it is declared that $I < J$ whenever all elements of I are less than all elements of J. In other words, the interval I lies completely on the left of J. For the purpose of the present article, all intervals will be closed, and the minimum and the maximum will be natural numbers. Notice that, under these hypotheses, the set of all maxima and minima of the intervals of a given interval order can be chosen to be an initial segment of the natural numbers.

The notion of interval order is now very classical and was introduced by Fishburn [13]. Though the main motivation for the introduction of such a concept came from social choice theory, it soon revealed its intrinsic interest, especially from a combinatorial point of view. To support this statement (and without giving any detail), we only recall here the characterization of interval orders as partially ordered sets avoiding the (induced) subposet $\mathbf{2}+\mathbf{2}$, and the more recent enumeration of finite interval orders [5].

An immediate link between interval orders and Schröder tableaux is given by the fact that every Schröder tableau can be associated with a set of intervals. Given a Schröder shape λ, two cells A and B of λ are called *twin* when they are adjacent and their union is a square. Equivalently, two adjacent cells A and B are twin cells when their common edge is a diagonal edge. Moreover, a (necessarily upper triangular) cell of λ is called *lonely* when it is the last cell of an odd row. Notice that the set of cells of λ can be partitioned into twin cells and lonely cells. Now, given a Schröder

[4] For a given element x of $\mathscr{P}$, the principal down-set generated by x is the smallest down-set of $\mathscr{P}$ containing x, that is the set of all elements of $\mathscr{P}$ smaller than or equal to x.

tableau S having n cells, consider the set of intervals $\mathcal{I}_S$ defined as follows: $I = [a,b] \in \mathcal{I}_S$ when both a and b are fillings of a pair of twin cells of S or a is the filling of a lonely cell and $b = n+1$. For instance, for the Schröder tableau S on the right in Figure 5, which has 9 cells, we have $\mathcal{I}_S = \{[1,2],[5,8],[3,4],[9,10],[6,7]\}$. The benefit of endowing the set of intervals associated with a Schröder tableau with its interval order will be discussed in the next subsections.

4.2. **Weak patterns in posets and strong pattern avoidance.** The study of classes of posets that contain or avoid certain subposets is a major trend in order theory. Classically, a poset $\mathscr{Q}$ *contains* another poset $\mathscr{P}$ whenever $\mathscr{Q}$ has a subposet isomorphic to $\mathscr{P}$. Borrowing the terminology from permutations, we could also say that $\mathscr{Q}$ contains the *pattern* $\mathscr{P}$. On the other hand, we say that $\mathscr{Q}$ *avoids* $\mathscr{P}$ whenever $\mathscr{Q}$ does not contain $\mathscr{P}$. The notion of pattern containment defines a partial order on the set $\mathfrak{X}$ of all (finite) posets, and we will write $\mathscr{P} \sqsubseteq \mathscr{Q}$ to mean that $\mathscr{P}$ is contained in $\mathscr{Q}$. Instead, the class of all finite posets avoiding a given poset $\mathscr{P}$ will be denoted $Av(\mathscr{P})$.

Here the use of the word "subposet" might be controversial. Technically speaking, what we have called "subposet" is sometimes called "induced subposet". Formally, we say that $\mathscr{P}$ is an *induced subposet* of $\mathscr{Q}$ when there is an injective function $f : \mathscr{P} \to \mathscr{Q}$ which is both order-preserving and order-reflecting: for all x, y, $x \leq y$ in $\mathscr{P}$ if and only if $f(x) \leq f(y)$ in $\mathscr{Q}$. Loosely speaking, this means that $\mathscr{Q}$ contains an isomorphic copy of $\mathscr{P}$. In what follows we will fully adhere to such terminology: we write $\mathscr{P} \sqsubseteq \mathscr{Q}$ to mean $\mathscr{Q}$ has an induced subposet isomorphic to $\mathscr{P}$.

What is useful for us is however a weaker version of the above notion of pattern. We say that $\mathscr{P}$ is *weakly contained* in $\mathscr{Q}$ (or that $\mathscr{P}$ is a *weak pattern* of $\mathscr{Q}$) when there exists an injective order-preserving function $f : \mathscr{P} \to \mathscr{Q}$. This can be also expressed by saying that $\mathscr{P}$ is a (not necessarily induced) subposet of $\mathscr{Q}$. It is clear that a pattern is also a weak pattern. On the other hand, we say that $\mathscr{Q}$ *strongly avoids* $\mathscr{P}$ whenever $\mathscr{Q}$ does not weakly contain $\mathscr{P}$. This is also expressed by writing $\mathscr{Q} \in SAv(\mathscr{P})$.

The partial order relation (on the set $\mathfrak{X}$ of all finite posets) defined by weak containment will be denoted $\leq$. This notion of poset containment is not entirely new. Typically, it has been considered in the context of families of sets, rather than generic posets, and many investigations in this field concern the study of finite families of sets that strongly avoid one or more finite posets, often with a special focus on extremal properties (see, for instance, [15, 17] to cite just a few). Here, however, we consider this order relation from a purely order-theoretic point of view, to initiate the investigation of the poset $(\mathfrak{X}, \leq)$. Some rather easy facts are the following:

- $(\mathfrak{X}, \leq)$ has a minimum, which is the empty poset, and does not have maximum.

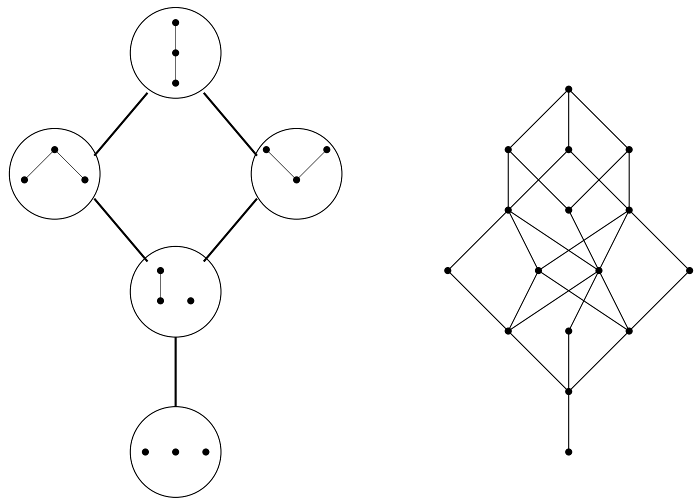


FIGURE 6. Hasse diagrams of $\mathfrak{X}_3$ (with explicit representation of each element) and $\mathfrak{X}_4$.

- Given $\mathscr{P}, \mathscr{Q} \in \mathfrak{X}$, if $\mathscr{P} \le \mathscr{Q}$ then the ground set of $\mathscr{P}$ has at most as many elements as $\mathscr{Q}$.
- $(\mathfrak{X}, \le)$ is a ranked poset, and the rank function is the sum of the number of elements of the ground set and the number of order relations between them.
- Denoting $\mathfrak{X}_n$ the set of all posets of size[5] n, the restriction of $\le$ to $\mathfrak{X}_n$ gives a poset with a minimum (the discrete poset on n elements) and a maximum (the chain having n elements). Also, $\mathfrak{X}_n$ has exactly one atom, which is the poset of size n having a single covering relation. Notice that, if we replace $\le$ with $\sqsubseteq$, the resulting poset structure on $\mathfrak{X}_n$ would be trivial (more precisely, discrete). We claim that the study of the posets $(\mathfrak{X}_n, \le)$ might be a potentially very interesting field of research. In Figure 6 we illustrate the posets $(\mathfrak{X}_n, \le)$ for a couple of small values of n.

The main aim of the present section is to initiate the study of the notion of strong pattern avoidance introduced above. Recall that $SAv(\mathscr{P})$ denotes the class of all posets strongly avoiding $\mathscr{P}$. Moreover, $SAv_n(\mathscr{P})$ is the subset of $SAv(\mathscr{P})$ consisting of the posets of size n. Finally, the above expression can be easily adapted to the case of strong avoidance of several posets (just by listing all the posets which are required to be avoided). Observe that, for given posets $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_s$, $SAv(\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_s)$ is a *down-set* of $(\mathfrak{X}, \le)$, which means that, if $\mathscr{Q} \in SAv(\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_s)$ and $\mathscr{P} \le \mathscr{Q}$, then $\mathscr{P} \in SAv(\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_s)$. In what follows we will always deal with the strong avoidance of a single poset. The generalization of a

[5]The size is the number of elements of the ground set.

given statement to several posets is easy (when meaningful) and so it is left to the reader.

Proposition 4.1. *For every poset $\mathscr{P} \in \mathfrak{X}_n$, $SAv(\mathscr{P}) = Av(\langle \mathscr{P} \rangle_n)$, where $\langle \mathscr{P} \rangle_n$ is the up-set*[6] *generated by $\mathscr{P}$ in $\mathfrak{X}_n$.*

Proof. The fact that $\mathscr{Q} \in SAv(\mathscr{P})$ is equivalent to the following: there cannot be an induced subposet $\mathscr{R}$ of size n of $\mathscr{Q}$ which contains $\mathscr{P}$ as a subposet. In other words, this means that $\mathscr{Q}$ cannot contain any induced subposet $\mathscr{R}$ of size n which is a refinement of $\mathscr{P}$, that is $\mathscr{Q} \in Av(\langle \mathscr{P} \rangle_n)$, as desired. □

The above proposition, rather than saying that strong avoidance can just be expressed in terms of classical avoidance, suggests us that the formalism of strong poset avoidance allows us to express certain problems concerning classical avoidance in a much-simplified way: avoiding several posets is sometimes equivalent to strongly avoiding just one of them.

We next characterize some classes of posets strongly avoiding certain patterns. Before starting, we give some definitions that will be useful.

Let Q, R be subsets of the ground set of a poset $\mathscr{P}$. We say that Q is *weakly below* R, and write $Q \ngtr R$, whenever for all $x \in Q$ and $y \in R$, we have $x \ngtr y$ in $\mathscr{P}$. Given subsets $P_1, P_2, \dots, P_r$ of $\mathscr{P}$, we say that $(P_1, P_2, \dots, P_r)$ is a *weakly ordered partition* of $\mathscr{P}$ when it is a set partition of the ground set of $\mathscr{P}$ such that $P_1 \ngtr P_2 \ngtr \cdots \ngtr P_r$. Replacing every symbol $\ngtr$ with $\leq$ in the two above definitions, we obtain the definitions of "Q is *below* R" and "$(P_1, P_2, \dots, P_r)$ is an *ordered partition* of $\mathscr{P}$".

Suppose that $\mathscr{P}$ and $\mathscr{Q}$ are two posets. The *disjoint union* $\mathscr{P} \cup \mathscr{Q}$ is the poset whose ground set is the disjoint union of the ground sets of $\mathscr{P}$ and $\mathscr{Q}$ and such that $x \leq y$ in $\mathscr{P} \cup \mathscr{Q}$ whenever either $x \leq y$ in $\mathscr{P}$ or $x \leq y$ in $\mathscr{Q}$. The *linear sum* $\mathscr{P} \oplus \mathscr{Q}$ is the poset whose ground set is the disjoint union of the ground sets of $\mathscr{P}$ and $\mathscr{Q}$ and such that $x \leq y$ in $\mathscr{P} \oplus \mathscr{Q}$ whenever either $x \leq y$ in $\mathscr{P}$ or $x \leq y$ in $\mathscr{Q}$ or else $x \in \mathscr{P}$ and $y \in \mathscr{Q}$.

Finally, recall that the *height* of a poset is the maximum cardinality of a chain.

Proposition 4.2.

(1) *If $\mathscr{P}$ is the discrete poset of size n, then $SAv(\mathscr{P})$ is the class of all posets of size $\leq n-1$.*
(2) *If $\mathscr{P}$ is the poset of size n containing a single covering relation, then $SAv(\mathscr{P})$ contains all posets of size $\leq n-1$ and all (finite) discrete posets.*
(3) *If $\mathscr{P}$ is the chain of size n, then $SAv(\mathscr{P})$ is the class of all finite posets of height $\leq n-1$.*

[6]In a poset $\mathscr{P}$, an up-set F is a subset of $\mathscr{P}$ such that, if $x \in F$ and $y \geq x$, then $y \in F$. For a given subset A of $\mathscr{P}$, the up-set generated by A is the smallest up-set of $\mathscr{P}$ containing A.

Proof.

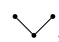
(1) Clearly, any poset $\mathscr{Q}$ of size at most $n-1$ strongly avoids $\mathscr{P}$, since there cannot exist any injective function from $\mathscr{P}$ to $\mathscr{Q}$. Moreover, if $\mathscr{Q}$ has size at least n, then any injective map from $\mathscr{P}$ to $\mathscr{Q}$ trivially preserves the order ($\mathscr{P}$ does not have any order relation among its elements, except of course the trivial ones coming from reflexivity), and so $\mathscr{Q} \notin SAv(\mathscr{P})$.

(2) This is essentially a consequence of the previous proposition. Indeed, every poset $\mathscr{Q}$ of size less than n trivially strongly avoids $\mathscr{P}$; moreover, if $\mathscr{Q}$ has size at least n and strongly avoids $\mathscr{P}$, then it cannot contain any covering relation, i.e. it is a discrete poset.

(3) If there is an injective function from $\mathscr{P}$ to a certain poset $\mathscr{Q}$ which preserves the order, then every pair of elements in the image of $\mathscr{P}$ must be comparable, i.e., $f(\mathscr{P})$ has to be a chain. This immediately yields the thesis. □

Notice that, in the last case of the above proposition, that is when $\mathscr{P}$ is a chain, clearly $SAv(\mathscr{P}) = Av(\mathscr{P})$, since the up-set generated by $\mathscr{P}$ in $\mathfrak{X}_n$ consists of $\mathscr{P}$ alone.

A finite poset is called a *flat* when it consists of a (possibly empty) antichain with an added maximum.

Proposition 4.3. *If* $\mathscr{P}$ = , *then* $SAv(\mathscr{P})$ *is the class of all disjoint unions of flats. As a consequence,* $|SAv_n(\mathscr{P})| = p_n$*, the number of integer partitions of* n*.*

Proof. Thanks to Proposition 4.1 and Proposition 4.2, we observe that $SAv(\mathscr{P}) = Av($  ,  $)$, and so in particular, if $\mathscr{Q} \in SAv(\mathscr{P})$, then $\mathscr{Q}$ has height at most 1. Moreover, any subset of cardinality 3 of $\mathscr{Q}$ cannot have minimum; thus, any three elements in the same connected component are either an antichain or one of them is greater than the remaining two. This means that each connected component of $\mathscr{Q}$ is a flat.

Concerning enumeration, the class of posets of size n whose connected components are flats is in bijection with the class of integer partitions of n: just map each such poset into the integer partition of the cardinality of its ground set whose parts are the cardinalities of the connected components (and observe that the order structure of a flat is completely determined by its cardinality). From this observation, the thesis follows. □

We close this section with some general results which allow us to understand the class $SAv(\mathscr{P})$ when $\mathscr{P}$ is built from simpler posets using classical operations.

Proposition 4.4. *Let* $\mathscr{P}, \mathscr{Q}$ *be two posets.*

(1) $\mathscr{R} \in SAv(\mathscr{P} \uplus \mathscr{Q})$ *if and only if, for every partition* (R_1, R_2) *into two blocks of the ground set of* $\mathscr{R}$*, denoting with* $\mathscr{R}_1, \mathscr{R}_2$ *the associated induced subposets,* $\mathscr{R}_1 \in SAv(\mathscr{P})$ *or* $\mathscr{R}_2 \in SAv(\mathscr{Q})$*.*

(2) *If $\mathscr{R} \in SAv(\mathscr{P} \oplus \mathscr{Q})$, then for every ordered partition (R_1, R_2) into two blocks of $\mathscr{R}$, $\mathscr{R}_1 \in SAv(\mathscr{P})$ or $\mathscr{R}_2 \in SAv(\mathscr{Q})$. If $\mathscr{R} \notin SAv(\mathscr{P} \oplus \mathscr{Q})$, then there exists a weakly ordered partition (R_1, R_2) into two blocks of $\mathscr{R}$ such that $\mathscr{R}_1 \notin SAv(\mathscr{P})$ and $\mathscr{R}_2 \notin SAv(\mathscr{Q})$.*

Proof.

(1) An occurrence of $\mathscr{P} \cup \mathscr{Q}$ in $\mathscr{R}$ consists of an occurrence of $\mathscr{P}$ and an occurrence of $\mathscr{Q}$ whose ground sets are disjoint and with no requirements about the order relations among pairs of elements (x, y) such that $x \in \mathscr{P}$ and $y \in \mathscr{Q}$. Therefore, if $\mathscr{R} \in SAv(\mathscr{P} \cup \mathscr{Q})$ and (R_1, R_2) is a partition of the ground set of $\mathscr{R}$, then it is clear that if $\mathscr{R}_1$ weakly contains $\mathscr{P}$, then necessarily $\mathscr{R}_2$ strongly avoids $\mathscr{Q}$. Vice versa, if $\mathscr{R}$ weakly contains $\mathscr{P} \cup \mathscr{Q}$, then clearly there exists an occurrence of $\mathscr{P}$ whose complement weakly contains $\mathscr{Q}$.

(2) An occurrence of $\mathscr{P} \oplus \mathscr{Q}$ in $\mathscr{R}$ consists of an occurrence of $\mathscr{P}$ and an occurrence of $\mathscr{Q}$ whose ground sets are disjoint and such that every element of $\mathscr{P}$ is less than every element of $\mathscr{Q}$. Thus, if $\mathscr{R} \in SAv(\mathscr{P} \oplus \mathscr{Q})$ and (R_1, R_2) is an ordered partition of $\mathscr{R}$ such that R_1 weakly contains $\mathscr{P}$, then necessarily R_2 strongly avoids $\mathscr{Q}$, since the ground set of $\mathscr{P}$ lies below R_2. On the other hand, if $\mathscr{R}$ weakly contains $\mathscr{P} \oplus \mathscr{Q}$, then the partition (R_1, R_2) of $\mathscr{R}$ in which R_1 is the down-set generated by an occurrence of $\mathscr{P}$ (and, of course, R_2 is the complement of R_1, and so an up-set) is a weakly ordered partition having the required properties. □

Another simple, general result involving the disjoint union of posets is the following.

Proposition 4.5. *If $\mathscr{Q} \in SAv(\mathscr{P})$ and $\mathscr{P}$ is connected, then $\mathscr{Q}$ is the disjoint union of a family of posets strongly avoiding $\mathscr{P}$.*

Proof. Indeed, take $\mathscr{Q} \in SAv(\mathscr{P})$ and suppose that $\mathscr{Q}$ is not connected (otherwise the thesis is trivial). Since $\mathscr{P}$ is connected, any occurrence of $\mathscr{P}$ in $\mathscr{Q}$ would be connected too (since such an occurrence is $\mathscr{P}$ with possibly some added order relations), so, for $\mathscr{Q}$ to strongly avoid $\mathscr{P}$, each connected component of $\mathscr{Q}$ has to strongly avoid $\mathscr{P}$, as desired. □

4.3. **How Schröder tableaux come into play.** Our introduction of weak poset patterns is motivated by the role they have in the description of Schröder tableaux. Let S be a Schröder tableau, and let $\mathcal{I}_S$ be the associated set of intervals, as defined in Subsection 4.1. Consider the interval order associated with $\mathcal{I}_S$, to be denoted $\mathcal{I}_S$ as well. The map $S \mapsto \mathcal{I}_S$ from Schröder tableaux to interval orders is clearly neither injective nor surjective: for instance, the Schröder tableau on the right in Figure 5 is mapped to the same interval order as the tableau obtained by adding into the second row a lower triangular cell containing 10, whereas the interval order $\{[1,3],[2,4]\}$ cannot be the image of any Schröder tableau. Therefore two natural questions concerning such a map arise.

(1) Given an interval order $\mathcal{I}$, does there exist a Schröder tableau S such that $\mathcal{I} = \mathcal{I}_S$?
(2) In the case of a positive answer to the previous question, how many Schröder tableaux associated with a given interval order are there?

Below we answer the first question. Recall that $\mathbf{N}$ denotes the set of natural numbers (without 0), which will be endowed with its usual total order structure.

Theorem 4.6. *Let $\mathcal{I}$ be an interval order of size n. There exists a Schröder tableau S such that $\mathcal{I} = \mathcal{I}_S$ if and only if $\mathcal{I}$ weakly contains a down-set of size n of $\mathbf{N} \times \mathbf{N}$.*

Proof. Suppose first that $\mathcal{I} = \mathcal{I}_S$, for some Schröder tableau S. This means that we can use the map $S \mapsto \mathcal{I}_S$ to label the set C of all pairs of twin cells and of all lonely cells of S with the elements of $\mathcal{I}$, so that C can be identified with $\mathcal{I}$. Consider the function $f : \mathcal{I} \to \mathbf{N} \times \mathbf{N}$ mapping I to the pair (n_I, m_I) such that n_I (resp. m_I) is the row (resp. column) of the pair of twin cells or of the lonely cell associated with I (as usual, rows and columns are enumerated from top to bottom and from left to right, respectively). We show that $f(\mathcal{I})$ is a down-set of $\mathbf{N} \times \mathbf{N}$: indeed, if $I \in \mathcal{I}$ and $(n, m) \in \mathbf{N} \times \mathbf{N}$ are such that $(n, m) \leq f(I) = (n_I, m_I)$, then the tableau S has at least n_I rows and m_I columns, so in particular there exists a pair of twin cells or a lonely cell at the crossing of row n and column m (since $n \leq n_I$ and $m \leq m_I$); denoting with J the associated interval of $\mathcal{I}$, we then have that $(n, m) = f(J) \in f(\mathcal{I})$, as desired. By construction f is injective, since two distinct intervals of $\mathcal{I}$ correspond to two distinct cells of S, which of course cannot lie both in the same row and in the same column of S. We can thus consider the inverse $g : f(\mathcal{I}) \to \mathcal{I}$ of f on $f(\mathcal{I})$. We now show that g is order-preserving: indeed, consider $I, J \in \mathcal{I}$ and suppose that $(n_I, m_I) = f(I) \leq f(J) = (n_J, m_J)$; the tableau S has a pair of twin cells or a lonely cell at the crossing of row n_J and column m_I (since $m_I \leq m_J$), and the associated interval, call it K, is such that $K \leq J$ in $\mathcal{I}$; moreover, since $n_I \leq n_J$, it is not difficult to realize that $I \leq K$, hence we conclude that $I \leq J$. Therefore we have proved that $f(\mathcal{I})$ is a down-set (of size n) of $\mathbf{N} \times \mathbf{N}$ and that $g : f(\mathcal{I}) \to \mathcal{I}$ is an injective order-preserving map, i.e., $f(\mathcal{I})$ is a weak pattern of $\mathcal{I}$. This is exactly the thesis.

In the other direction, suppose that $\mathcal{I}$ weakly contains a down-set D of $\mathbf{N} \times \mathbf{N}$ of size n. In other words, D is a coarsening of $\mathcal{I}$ (i.e., it is obtained from $\mathcal{I}$ by possibly only removing some order relations, leaving untouched its ground set). This means that there is an order-preserving injective map $g : D \to \mathcal{I}$. As we have already recalled, it is not restrictive to suppose that the endpoints of the elements of $\mathcal{I}$ are all distinct and constitute an initial segment of $\mathbf{N}$. Consider the tableau S having a pair of twin cells in row n and column m if and only if $(n, m) \in D$ and, in such case, the two cells are filled in with the endpoints of the interval $g(n, m) \in \mathcal{I}$. It is clear that S is a Schröder tableau: since D is a down-set, rows are left-justified; moreover, rows and columns are increasing because g is order-preserving. Now it is not difficult to realize that $\mathcal{I} = \mathcal{I}_S$, which is precisely what we wanted to show. $\square$

Remark: Notice that, in the second part of the above proof, we construct a Schröder tableau S *having no lonely cells* such that $\mathcal{I} = \mathcal{I}_S$. This again shows the fact that there can be several Schröder tableaux associated with the same interval order.

5. Further work

The algebraic and combinatorial properties of the distributive lattice $\mathscr{Sch}$ of Schröder shapes need to be further investigated. In particular, the analogies with differential posets should be much deepened, for instance trying to understand the role of the lowering and raising operators (a fundamental tool for computations in differential posets) in the Schröder lattice, or even in the more general setting of φ-differential posets.

We have just started the characterization and enumeration of permutations having a given Schröder insertion tableau. Many more shapes should be investigated. Moreover, we still have to understand the role of the recording tableau.

Can we find a nice closed formula for the number of Schröder tableaux of a given shape? In the case of Young tableaux, there is a famous *hook formula*, which however seems to be unlikely in our case, since we have numerical evidence that, for certain shapes, this number has large prime factors.

The alternative presentation of Schröder tableaux in terms of interval orders and weak poset patterns might have more secrets to reveal. For instance, the enumeration of Schröder tableaux associated with a given interval order remains entirely to be done. In a different direction, the topic of strong pattern avoidance for posets seems to be an interesting line of research on its own, independent from its relationship with Schröder tableaux.

The analogies between Young tableaux and Schröder tableaux should be investigated more, especially from a purely algebraic point of view. Combinatorial objects related to Young tableaux, such as Schur functions and the plactic monoid, as well as algorithmic and algebraic constructions, such as Schützenberger's *jeu de taquin* [23], the Littlewood–Richardson rule and the Schubert calculus on Grassmannians and flag varieties could have some interesting counterparts in the context of Schröder tableaux.

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