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ANCHORED HYPERSPACES AND MULTIGRAPHS

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ABSTRACT. Consider a multigraph X as a metric space and $p \in X$. The *anchored hyperspace at* p is the set

$$C_p(X) = \{A \subseteq X : p \in A,\ A \text{ connected and compact}\}.$$

In this paper we prove that $C_p(X)$ is a polytope, considering the Hausdorff metric H. And reciprocally, if X is a locally connected and compact metric space such that $C_p(X)$ is a polytope, for each $p \in X$, then X must be a multigraph.

1. Introduction

We say that S is a *face* of a n-cell $M = [0,1]^n$ if $S = S_1 \times S_2 \times \cdots \times S_n$, where either $S_i = [0,1]$, or $S_i = \{0\}$, or $S_i = \{1\}$ and for at least one i, $S_i \neq [0,1]$. For example $[0,1]^2$ has eight faces: $\{0\} \times \{0\}$, $\{1\} \times \{1\}$, $\{0\} \times \{1\}$, $\{1\} \times \{0\}$, $\{0\} \times [0,1]$, $[0,1] \times \{0\}$, $[0,1] \times \{1\}$, $\{1\} \times [0,1]$. In general $[0,1]^n$ has $3^n - 1$ faces. A *polytope* is a metric space which can be written as a finite union of finite-dimensional cells such that the intersection of any two of them is either empty or a union of faces. A *multigraph* is a polytope whose cells are 1-cells and 0-cells.

Remark 1.1. Is a well-known fact that the faces of a n-cell is itself a polytope.

If X is a metric space and $p \in X$, the *anchored hyperspace* $C_p(X)$ is the subspace $2^X = \{A \subseteq X : A \neq \emptyset,\ A \text{ closed}\}$, whose elements are all compact and connected subsets of X containing p. We say that a metric space X is *Cpp* if $C_p(X)$ is a polytope for each $p \in X$. Next, some results that will be useful for our purposes (see [7]).

A family $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ of sets in a topological space X is called *neighborhood finite*, (*nbd-finite* for short) if each point of X has a neighborhood V such that $V \cap A_\alpha \neq \emptyset$ for at most finite indices α.

Let X be a topological space and $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ a covering of X such that either:

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(1) The sets A_α are all open.
(2) The sets A_α are all closed and form a nbd-finite family.

For each $\alpha \in \mathcal{A}$ let $f_\alpha : A_\alpha \to Y$ be continuous and assume that $f_\alpha|_{A_\alpha \cap A_\beta} = f_\beta|_{A_\alpha \cap A_\beta}$ for each α, $\beta \in \mathcal{A}$, then there exists a unique continuous map $f : X \to Y$ which is an extension of each f_α.

Let $\{B_\alpha\}$ be an open, or a closed nbd-finite covering of Y. Let $f : X \to Y$ be continuous and assume that for each $\alpha \in \mathcal{A}$, $f|_{f^{-1}(B_\alpha)}$ is a homeomorphism of $f^{-1}(B_\alpha)$ and B_α. Then X is homeomorphic to Y.

The *free* union $X + Y$ of disjoint topological spaces X, Y is the set $X \cup Y$ with topology: $U \subset X + Y$ is open if and only if $U \cap X$ is open in X and $U \cap Y$ is open in Y. Since $X \cap Y = \emptyset$, clearly $B \subset X + Y$ is closed if and only if $B \cap X$ is closed in X and $B \cap Y$ is closed in Y.

Lemma 1.2 (Transgression Theorem)**.** *Let $p : X \to Y$ be an identification map and $h : X \to Z$ continuous. Assume that hp^{-1} is single-valued (that is, h is constant on each fiber $p^{-1}(y)$). Then $hp^{-1} : Y \to Z$ is continuous, and in addition, the diagram*

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "h"] & Y \arrow[dl, "hp^{-1}"] \\ Z & \end{tikzcd}$$

is commutative.

Let X and Y be two disjoint spaces, $A \subset X$ a closed subset, and $f : A \to Y$ continuous. In $X + Y$, generate an equivalence relation R by $a \sim f(a)$ for each $a \in A$. The quotient space $(X + Y)/R$ is said to be "X attached to Y" and is written $X \cup_f Y$; f is called the *attaching* map.

We also need the following results (see [7])

Lemma 1.3. *Let $p : X + Y \to X \cup_f Y$ be the projection map, and let $C \subset X + Y$ be such that $C \cap X$ is closed in X. Then $p(C)$ is closed in $X \cup_f Y$ if and only if $(C \cap Y) \cup f(C \cap A)$ is closed in Y.*

Lemma 1.4. *Let $Q : X + Y \to X \cup_f Y$ be the projection map. Then Y is embedded as a closed set in $X \cup_f Y$ and $Q|_Y$ is an embedding.*

2. Polytopes

In modern times, polytopes and their related concepts have important applications in computer graphics, optimization and many other fields ([3], [2], [10], [13]).

Proposition 2.1. *Let $Q : X + Y \to X \cup_f Y$ be the projection map, with X compact and Y a Hausdorff space. If f embeds A in a closed set $f(A)$ of Y, then X is embedded as a closed subset of $X \cup_f Y$ and $Q|_X$ is an embedding.*

Proof. The map $Q|_X$ is obviously continuous and one-to-one. Now, let $C \subset X$ be closed, then $(C \cap Y) \cup f(C \cap A) = f(C \cap A)$ is closed in $f(A)$, as this set is closed in Y we have that $(C \cap Y) \cup f(C \cap A)$ is closed in Y. It follows by Lemma 1.3 that $Q(C)$ is closed in $X \cup_f Y$ and therefore $Q|_X$ is an embedding. □

Proposition 2.2. *Let $X = C_1 \cup C_2 \cup \cdots \cup C_n$ and $Y = D_1 \cup D_2 \cup \cdots \cup D_n$ be disjoint spaces and $f : X \to Y$ a homeomorphism such that:*

(1) For each i, the space C_i is a finite-dimensional cell,
(2) for each pair i, j we have $C_i \cap C_j$ is empty or is a union of faces,
(3) for each i, we have that $f(C_i) = D_i$,
(4) suppose we have chosen for each cell C_i a face $\mathcal{F}C_i$ (and therefore for each cell D_i we have chosen a face $\mathcal{F}D_i = f(\mathcal{F}C_i)$).

If g is the restriction of f to $\mathcal{F}(C_1) \cup \cdots \cup \mathcal{F}(C_n)$, then the space $X \cup_g Y$ is a finite union of cells where each two have empty intersection or is a finite union of faces.

Proof. Let $\mathfrak{X} = \mathfrak{C}_1 \cup \mathfrak{C}_2 \cup \ldots \mathfrak{C}_\mathfrak{n}$ and $\mathfrak{Y} = \mathfrak{D}_1 \cup \mathfrak{D}_2 \cup \ldots \mathfrak{D}_\mathfrak{n}$ be homeomorphic copies of X and Y contained in $X \cup_f Y$ whose existence is ensured by Lemma 1.4 and Proposition 2.1, where $Q(C_i) = \mathfrak{C}_i$, $Q(D_i) = \mathfrak{D}_i$, $Q(\mathcal{F}C_i) = \mathcal{F}\mathfrak{C}_i$, and $Q(\mathcal{D}_i) = \mathcal{F}\mathfrak{D}_i$. Evidently for each pair i, j the set $\mathfrak{C}_i \cap \mathfrak{C}_i$ is empty or is union of faces and the same occurs with the sets $\mathfrak{D}_i \cap \mathfrak{D}_j$. Now, if $x \in \mathfrak{C}_i \cap \mathfrak{D}_j$, then there exists $a \in \mathcal{F}C_i$ such that $x = [a] = [f(a)]$ and $f(a) \in D_j \cap \mathcal{F}D_i$, therefore $[f(a)] = x \in \mathfrak{D}_j \cap \mathcal{F}\mathfrak{D}_i$, this shows that $\mathfrak{C}_i \cap \mathfrak{D}_j \subset \mathfrak{D}_j \cap \mathcal{F}\mathfrak{D}_i$. On the other hand, if $x \in \mathfrak{D}_j \cap \mathcal{F}\mathfrak{D}_i$, then $x \in \mathcal{F}\mathfrak{D}_i$. It follows that there is a point $a \in \mathcal{F}C_i$ such that $[a] = [f(a)] = x$ therefore $x \in \mathcal{F}\mathfrak{C}_i$, which shows that $x \in \mathfrak{C}_i \cap \mathfrak{D}_j$ and therefore $\mathfrak{C}_i \cap \mathfrak{D}_j = \mathfrak{D}_j \cap \mathcal{F}\mathfrak{D}_i$.

As well, we have $D_j \cap \mathcal{F}D_i \subset D_j \cap D_i = F_1 \cup F_2 \cup \ldots F_m$ where each F_i is a face by (2). On the other hand $D_j \cap \mathcal{F}D_i = D_j \cap (D_i \cap \mathcal{F}D_i) = (D_i \cap D_j) \cap \mathcal{F}D_i = (F_1 \cup F_2 \cup \ldots F_m) \cap \mathcal{F}D_i$, by Remark 1.1 this last expression is a union of faces and therefore $\mathfrak{D}_j \cap \mathcal{F}\mathfrak{D}_i$ is union of faces, that is, $\mathfrak{C}_i \cap \mathfrak{D}_j$ is union of faces. Finally, since the projection map Q is surjective, we have $Q(X+Y) = Q(X) \cup Q(Y) = \mathfrak{X} \cup \mathfrak{Y} = \mathfrak{C}_1 \cup \mathfrak{C}_2 \cup \cdots \mathfrak{C}_\mathfrak{n} \cup \mathfrak{D}_1 \cup \mathfrak{D}_2 \cup \cdots \mathfrak{D}_\mathfrak{n}$. □

The following properties concerning polytopes are useful in general.

Proposition 2.3.

(1) If the components of a space are polyhedra, then the components of a union of its faces are polyhedra.
(2) If a subset of the cells forming a polytope are taken, then the components of their union are polyhedra.

Proof.
(1) Let F_1 and F_2 be two faces of cells C_1 and C_2 in a same component of X. If $F_1 \cap F_2$ is nonempty, either F_1 and F_2 is a common face of C_1 and C_2, or one is contained in other or $F_1 \cap F_2$ is a union of (in a manner of speaking) "lower dimensional subfaces".
(2) Each two such cells (with no empty intersection) intersects in the whole space in a union of their faces, so they intersect in each component of such a union.

□

Remark 2.4. If two polytopes are joined by some of their faces then this new space is a polytope.

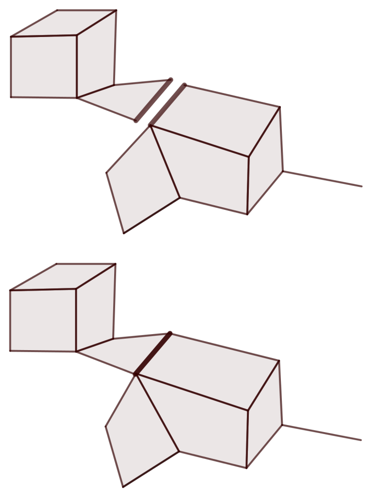

FIGURE 1

Multigraphs have been an object of study for several years, and recently there are many works on the subject (see, for example, [1], [5], [8]). Particularly, there are works in multigraphs viewed as metric spaces (see, [4], [14]).

In [6], R. Duda studied the characterization of the hyperspaces $C(G)$ when G is a connected multigraph. As previously mentioned, a multigraph is a polytope whose cells are arcs or points, these arcs are called *edges*, also a closed curve J is called an *edge* provided this closed curve contains a unique ramification point, these edges are called *loops*. We identify the end points of an edge with 0_J and 1_J respectively, if this edge is a loop we assume $0_J = 1_J$ (that is to say, in a loop end points coincide) is the unique ramification point in J. Given any multigraph G, a *subgraph* of G is a subspace formed for some edges and this subspace is itself a multigraph. A simple vertex is considered to be a subgraph of G. A *tree* is a subgraph not containing simple closed curves, an *internal tree* is a tree not containing terminal points of G, and therefore its terminal points are ramification points in G. Let $AI(G)$ denote the set of all internal trees of G. We establish that $E(G)$, $V(G)$, $O(G)$, $R(G)$, and $T(G)$, denote the sets of edges, vertices, ordinary points, ramification points, and terminal points, respectively. Given an internal tree T of G, we regard those edges J such that $J \cap T \neq \emptyset$ and J is not subset of T. Let us divide these edges in two types: $J_1, \ldots, J_n$ and $L_1, \ldots, L_m$, where each edge J_i has just one extreme (say 0_J) in T and it is not a loop. The edges

L_j are those whose two extreme points are in T, including the loops whose ramification point is in T. We define

$$D(1,T) = T \cup (\bigcup_{i=1}^{n} J_i) \cup (\bigcup_{j=1}^{m} L_j),$$

and we say that this is the *canonical representation* of $D(1,T)$. Given an internal tree $T \subset G$, let $\mathfrak{M}(T)$ be the family of all connected compact subsets of G of the form

$$((c_i)_{i=1}^n, (a_j, b_j)_{j=1}^m)_T = T \cup (\bigcup_{i=1}^{n} [0_{j_i}, c_i]) \cup (\bigcup_{j=1}^{m} [0_{L_j}, a_j] \cup [b_j, 1_j]),$$

where $c_i \in J_i$ and $a_j, b_j \in L_j$, $[0_{j_i}, c_i]$ is the subarc of J_i joining 0_{J_i} with c_i, $[0_{L_j}, a_j]$ is the subarc of J_i joining 0_{L_j} with a_j and $[b_j, 1_{L_j}]$ is the subarc of L_j joining b_j with 1_{L_j}.

We have the following lemma (for more details and proofs see [6]).

Lemma 2.5. *Let G be a connected multigraph, then*

(i) For each internal tree $T \subset G$, the family $\mathfrak{M}(T)$ is a $(n+2m)$-cell.

(ii) The hyperspace of connected compact subsets of G is

$$C(G) = [\bigcup_{T \in AI(G)} \mathfrak{M}(T)] \cup [\bigcup_{I \in E(G)} C(I)].$$

It is a well-known fact that $C(S^1) \cong C([0,1]) \cong [0,1]^2$ (see [6]) and therefore $C(I) \cong [0,1]^2$ for each edge $I \in E(G)$.

The following lemma is proved in [6].

Lemma 2.6. *Let G be a connected multigraph, then.*

(i) Two sets of the form $\mathfrak{M}(T)$ intersect at some of their faces.

(ii) Two sets of the form $C(I)$ (where $I \in E(G)$) intersect at some of their faces.

(iii) A set of the form $\mathfrak{M}(T)$ and one of the form $C(I)$ intersect at some of their faces.

Since the set $AI(G)$ and the set $E(G)$ are finite, using the previous lemma, the following theorem follows.

Theorem 2.7 ([6])**.** *A connected compact metric space G is a multigraph, if and only if $C(G)$ is a polytope.*

3. MULTIGRAPHS AND THEIR ANCHORED HYPERSPACES

Notice that if G is a multigraph and C the component of G containing p, then $C_p(G) = C_p(C)$. Therefore throughout this work we will assume all our multigraphs are connected. If G is a multigraph and $p \in G$, there are three possibilities (see Figure 2).

(1) The point p is in a loop L of G and $p \neq 0_L$,

(2) there exists a terminal edge J which is not a loop such that $p \in J$ and $p \neq 0_J$,
(3) if neither p is in a loop nor in a terminal edge, then there exists an internal tree T such that $p \in D(1, T)$.

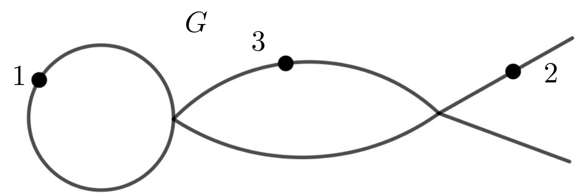

FIGURE 2

In the first case, let $\mathcal{L}_1$ be one of the two arcs in L connecting 0_L with p and let $\mathcal{L}_2$ be the other one. Define $\Gamma_1 = \{T \in AI(G) : 0_L \in T\}$. For $T \in \Gamma_1$, if $D(1,T) = T \cup (\bigcup_{i=1}^{n} J_i) \cup (\bigcup_{j=1}^{m} L_j)$ is the canonical representation of $D(1,T)$ we will always assume, without loss of generality, that $L = L_1$. Let $\mathfrak{M}_p^1(T)$ be the family of all those connected compact subsets of the form:

$$(T \cup \mathcal{L}_1) \cup ([p, a_1] \cup [b_1, 1_L]) \cup (\bigcup_{i=1}^{n} [0_{J_i}, c_i]) \cup (\bigcup_{j=2}^{m} [0_{L_j}, a_j] \cup [b_j, 1_{L_j}]),$$

where $a_1,\, b_1 \in \mathcal{L}_2,\, c_i \in J_i,\, a_j,\, b_j \in L_j$.

Lemma 3.1 ([12]). *Let $f : X \to Y$ be a continuous map into a compact connected metric space Y, then $\hat{f} : C(X) \to C(Y)$ defined by*

$$\hat{f}(A) = f(A) \text{ for each } A \in C(X),$$

is continuous.

Proposition 3.2. *For each $T \in \Gamma_1$, the family $\mathfrak{M}_p^1(T)$ is homeomorphic to the $(n + 2m)$-cell, $\mathfrak{M}(T) = T \cup (\bigcup_{i=1}^{n} [0_{j_i}, c_i]) \cup (\bigcup_{j=1}^{m} [0_{L_j}, a_j] \cup [b_j, 1_j])$ in Lemma 2.5 (i).*

Proof. Indeed, let $G_{\mathcal{L}_1} = G/\mathcal{L}_1$ be the space obtained from G by identifying $\mathcal{L}_1$ to 0_L. Then $G_{\mathcal{L}_1}$ is homeomorphic to G as shown below: Let $Q : G \to G_{\mathcal{L}_1}$ be the projection map and $f : G \to G$ defined as follows:

$$f(x) = \begin{cases} x & x \in \overline{G - L}, \\ 0_L & x \in \mathcal{L}_1, \\ g(x) & x \in \mathcal{L}_2, \end{cases}$$

where $g : \mathcal{L}_2 \to L$ is a continuous map such that $g(p) = 0_L = g(0_L)$ and restricted to $\mathcal{L}_2 - \{p, 0_L\}$ is one-to-one; hence f is continuous. Now, f is constant in each $Q^{-1}([x])$ and conversely Q is constant in each $f^{-1}(x)$, it follows from Lemma 1.2 that $f \circ Q^{-1}$ and $Q \circ f^{-1}$ are continuous and being mutual inverses, it follows that G and $G_{\mathcal{L}_1}$ are homeomorphic.

If $T \in \Gamma_1$, then $Q(T)$ is an internal tree containing $[0_L]$. Since G is connected and compact, $G_{\mathcal{L}_1}$ is also connected and compact. It follows from Lemma 3.1 that $\hat{Q} : C(G) \to C(G_{\mathcal{L}_1})$ defined by $\hat{Q}(A) = Q(A)$ is continuous. For each $T \in \Gamma_1$ the map $\hat{Q}$ is one-to-one on the set $\mathfrak{M}_p^1(T)$, which shows that $\hat{Q}_T = \hat{Q}|_{\mathfrak{M}_p^1(T)}$ is a homemorphism between $\mathfrak{M}_p^1(T)$ and $\mathfrak{M}(Q(T))$. Since $\mathfrak{M}(Q(T))$ is homeomorphic to $\mathfrak{M}(T)$ the proposition follows. $\square$

Similarly for each $T \in \Gamma_1$, we can define $\mathfrak{M}_p^2(T)$ as the family of all connected compact subsets of the form:

$$(T \cup \mathcal{L}_2) \cup ([p, a_1] \cup [b_1, 1_L]) \cup (\bigcup_{i=1}^{n} [0_{J_i}, c_i]) \cup (\bigcup_{j=2}^{m} [0_{L_j}, a_j] \cup [b_j, 1_j]),$$

where $a_1, b_1 \in \mathcal{L}_1$, and therefore the next lemma follows.

Lemma 3.3. *For each $T \in \Gamma_1$ the family $\mathfrak{M}_p^2(T)$ is a finite-dimensional cell.*

If p is a point satisfying Theorem 2.7, (2), let 0_J and 1_J be the endpoints of the edge $J = J_1$, where 1_J is the terminal vertex of G and let Γ_2 be the set of all internal trees T of G such that $0_J \in T$. In this case, for each $T \in \Gamma_2$, let $\mathfrak{N}_p^3(T)$ be the family of all connected compact subsets of G of the form

$$T \cup (\bigcup_{i=1}^{n} [0_{J_i}, c_i]) \cup (\bigcup_{j=1}^{m} [0_{L_j}, a_j] \cup [b_j, 1_{L_j}])),$$

where $c_i \in J_i$, for $i = 2, \ldots n$, $a_j, b_j \in L_j$ for $j = 1, \ldots, m$ and $c_1 \in [p, 1_J]$. As in the above case, a similar analysis shows that $\mathfrak{N}_p^3(T)$ is a finite-dimensional cell, and therefore we have the following lemma:

Lemma 3.4. *If $T \in \Gamma_2$, then $\mathfrak{N}_p^3(T)$ is a finite-dimensional cell.*

If p is a point satisfying Theorem 2.7, (3) consider the set Γ_3, whose members are all the internal trees of G which $p \in T$. For each $T \in \Gamma_3$, consider the cell $\mathfrak{M}(T)$ (Lemma 2.5 (i)). Evidently, each of these cells is contained in $C_p(G)$, whereby the following lemma is trivially true.

Lemma 3.5. *If G is a multigraph and $T \in \Gamma_3$, then the cell $\mathfrak{M}(T)$ is a subspace of $C_p(G)$.*

Let J be the edge containing p, and u and v the endpoints of J. Denote $\mathcal{L}_1$ the subarc of J containing u and p as endpoints and denote $\mathcal{L}_2$ the subarc of J containing p and v as endpoints. Let $G/\mathcal{L}_1$ and $G/\mathcal{L}_2$ be the quotient spaces obtained from G identifying the arcs $\mathcal{L}_1$ and $\mathcal{L}_2$ at one point, respectively.

Proposition 3.6. *The hyperspaces $C(G)$, $C(G/\mathcal{L}_1)$ and $C(G/\mathcal{L}_2)$ are homeomorphic.*

Proof. As in Proposition 3.2 if $Q : G \to G/\mathcal{L}_1$ is the projection map and $f : \mathcal{L}_2 \to J$ is a homeomorphism such that $f(p) = u$ and $f(v) = v$, then define $h_0 : G \to G$ by

$$h_0(x) = \begin{cases} u & x \in \mathcal{L}_1, \\ f(x) & x \in \mathcal{L}_2, \\ x & x \in \overline{G - J}. \end{cases}$$

We have by Lemma 1.2 that maps $h_0 \circ Q^{-1}$ and $Q \circ h_0^{-1}$ are continuous, and being mutual inverses, we have that G and $G/\mathcal{L}_1$ are homeomorphic graphs. By Lemma 3.1 we have that $\psi_1 = \widehat{h_0 \circ Q^{-1}}$ is a homeomorphism between $C(G/\mathcal{L}_1)$ and $C(G)$.

Now, consider the space $G/\mathcal{L}_2$ obtained from G identifying $\mathcal{L}_2$ at one point. Let $R : G \to G/\mathcal{L}_2$ be the projection map, and $g : \mathcal{L}_1 \to J$ a homeomorphism such that $g(p) = v$ and $g(u) = u$, then, for the map $h_1 : G \to G$ defined by:

$$h_1(x) = \begin{cases} v & x \in \mathcal{L}_2, \\ g(x) & x \in \mathcal{L}_1, \\ x & x \in \overline{G - J}, \end{cases}$$

again, by Lemma 1.2, the maps $h_1 \circ R^{-1}$ and $R \circ h_1^{-1}$ are continuous, and being mutual inverses, we have that G and $G/\mathcal{L}_2$ are homeomorphic. By Lemma 3.1, $\psi_2 = \widehat{h_1 \circ R^{-1}}$ is a homeomorphism between $C(G/\mathcal{L}_2)$ and $C(G)$. $\square$

Now, let Γ_4 be the set of all those internal trees T for which $p \in D(1,T) - (T \cup \bigcup_{i=1}^{n} J_i)$, where $T \cup (\bigcup_{i=1}^{n} J_i) \cup (\bigcup_{j=1}^{m} L_j)$ is the canonical representation of $D(1,T)$ and, without loss of generality, we will always assume that $p \in L_1$.

If $H \in \Gamma_4$, denote $\mathfrak{K}_p^1(H)$ the subspace of $C_p(G)$ whose members are all connected compact subsets of G of the form:

$$(H \cup \mathcal{L}_1) \cup ([p, a_1] \cup [b_1, 1_{L_1}]) \cup (\bigcup_{i=1}^{n} [0_{J_i}, c_i]) \cup (\bigcup_{j=2}^{m} [0_{L_j}, a_j] \cup [b_j, 1_{L_j}])),$$

where, $a_1, b_1 \in \mathcal{L}_2$, for $j = 2, \ldots, m$ we have $a_j, b_j \in L_j$ and, for $i = 1, \ldots n$ we have $c_i \in J_i$; Figure 3 shows a typical element of $\mathfrak{K}_p^1(H)$. Similarly, given $H \in \Gamma_4$, let $\mathfrak{K}_p^2(H)$ denote the subspace of $C_p(G)$ whose members are all those connected compact subsets in the form:

$$(H \cup \mathcal{L}_2) \cup ([0_{L_1}, a_1] \cup [b_1, p]) \cup (\bigcup_{i=1}^{n} [0_{J_i}, c_i]) \cup (\bigcup_{j=2}^{m} [0_{L_j}, a_j] \cup [b_j, 1_{L_j}])).$$

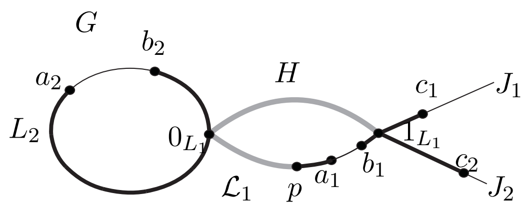

FIGURE 3

Proposition 3.7. *For each internal tree $H \in \Gamma_4$, the spaces $\mathfrak{K}_p^1(H)$ and $\mathfrak{K}_p^2(H)$ of $C_p(G)$ are finite-dimensional cells.*

Proof. For each $H \in \Gamma_4$, if $\hat{Q}$ and $\hat{R}$ are the induced maps of Q and R in the proof of Proposition 3.6, then the maps $\sigma_H : \mathfrak{K}_p^1(H) \to \mathfrak{M}(\hat{Q}(H))$ and $\rho_H : \mathfrak{K}_p^2(H) \to \mathfrak{M}(\hat{R}(H))$ given by $\sigma_H(A) = \hat{Q}(A)$ and $\rho_H(A) = \hat{R}(A)$ are homeomorphisms. $\square$

In order to continue building the cells that will endow a polytope structure to $C_p(G)$, suppose that p lies in a nonterminal edge J of G with endpoints, say u and v. Now, let Γ_5 be the set of all internal trees M of G for which, if $M \cup (\bigcup_{i=1}^n J_i) \cup (\bigcup_{j=1}^m L_j)$ is the canonical representation of $D(1, M)$, then for some index i (which we always assume without loss of generality that $i = 1$) we have $J_1 = J$ and if 0_{j_1} is the only point of $M \cap J_1$, then $0_{J_1} = u$.

Given $M \in \Gamma_5$, let $\mathfrak{M}_p(M)$ be the family of all connected compact subsets of $C_p(G)$ whose members have the form:

$$(M \cup \mathcal{L}_1 \cup [p, c_1]) \cup (\bigcup_{i=2}^{n} [0_{J_i}, c_i]) \cup (\bigcup_{j=1}^{m} ([0_{L_j}, a_j] \cup [b_j, 1_{L_j}]),$$

where $c_1 \in \mathcal{L}_2$, for $i \in \{2, \ldots, n\}$ $c_i \in J_i$ and for $j \in \{1, \ldots, m\}$ $a_j, b_j \in L_j$. Figure 4 shows a typical element of $\mathfrak{M}_p(M)$, where $M = \{u\}$.

We now define Γ_6 as the set of all those internal trees $N \subset G$ such that, if $N \cup (\bigcup_{i=1}^n J_i) \cup (\bigcup_{j=1}^m L_j)$ is the canonical representation of $D(1, N)$, then, for some index i (which we always assume without loss of generality that $i = 1$) we have $J = J_1$ and if 0_{J_1} is the only point in $N \cap J_1$, then $0_{J_1} = v$.

Given $N \in \Gamma_6$, let $\mathfrak{N}_p(N)$ be the subspace of $C_p(G)$ whose members are those connected compact subsets in the form:

$$(M \cup \mathcal{L}_2 \cup [p, c_1]) \cup (\bigcup_{i=2}^{n} [0_{J_i}, c_i]) \cup (\bigcup_{j=1}^{m} ([0_{L_j}, a_j] \cup [b_j, 1_{L_j}]),$$

where $c_1 \in \mathcal{L}_1$, for $i \in \{2, \ldots, n\}$ $c_i \in J_i$, and for $j \in \{1, \ldots, m\}$ $a_j, b_j \in L_j$.

Lemma 3.8. *For each $M \in \Gamma_5$ the space $\mathfrak{M}_p(M)$ is a finite-dimensional cell. Also, if $N \in \Gamma_6$, then, the space $\mathfrak{N}_p(N)$ is a finite-dimensional cell.*

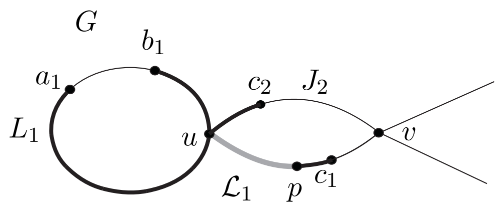

FIGURE 4

Proof. For each $M \in \Gamma_5$, the map $\beta_M : \mathfrak{M}_p(M) \to \mathfrak{M}(\hat{Q}(M))$ defined by $\beta_M(A) = \hat{Q}(A)$, where $\hat{Q}$ is the induced map of Proposition 3.6 is a homeomorphism. Since $\mathfrak{M}(\hat{Q}(M))$ is a finite-dimensional cell, $\mathfrak{M}_p(M)$ is also a finite-dimensional cell.

Similarly, the map $\lambda_N : \mathfrak{N}_p(N) \to \mathfrak{M}(\hat{R}(N))$ defined by $\lambda_N(A) = \hat{R}(A)$, where $\hat{R}$ is as in Proposition 3.6 is a homeomorphism. □

4. MULTIGRAPHS ARE CPP CONNECTED COMPACT SPACES

Recall that a space is Cpp if all its anchored hyperspaces are polytopes. In this section, we prove that multigraphs, are Cpp connected compact spaces. In the next theorem we assume G is not an arc or a closed simple curve.

Theorem 4.1. *Multigraphs are Cpp connected compact spaces.*

Proof. We have three cases from Section 3, in each case, $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$, and Γ_6 will be the sets of internal trees of G as they were previously defined in Section 3.

Case 1: The point p lies in a loop L of G and $p \neq 0_L$.

For each $T \in \Gamma_1$, let $\hat{Q}_T$ be the homeomorphism between $\mathfrak{M}^1_p(T)$ and $\mathfrak{M}(Q(T))$ described in the proof of Proposition 3.2. If $S, T \in \Gamma_1$, then $\hat{Q}_T = \hat{Q}_S$ for all points in $\mathfrak{M}^1_p(T) \cap \mathfrak{M}^1_p(S)$. Recall that there exists a unique continuous map $\hat{Q}_0$ which is an extension of every $\hat{Q}_T$, since each one of these is a homeomorphism it follows that $\hat{Q}_0$ is a homeomorphism between $\bigcup_{T\in\Gamma_1} \mathfrak{M}^1_p(T)$ and $\bigcup_{T\in\Gamma_1} \mathfrak{M}(Q(T))$. Similarly we have that $\bigcup_{T\in\Gamma_1} \mathfrak{M}^2_p(T)$ is homeomorphic to $\bigcup_{T\in\Gamma_1} \mathfrak{M}(Q(T))$.

Now, for all $T \in \Gamma_1$, consider $\mathcal{C}_T = \{A \in \mathfrak{M}(Q(T)) : Q(L) \subseteq A\}$. That is, $\mathcal{C}_T$ is the face of $\mathfrak{M}(Q(T))$ obtained making $b_1 = p$. Let $\bigcup_{T\in\Gamma_1} X_T$

and $\bigcup_{T\in\Gamma_1} Y_T$ be disjoint topological copies of $\bigcup_{T\in\Gamma_1}\mathfrak{M}(Q(T))$ and

$$f_X : \bigcup_{T\in\Gamma_1} X_T \to \bigcup_{T\in\Gamma_1} \mathfrak{M}(Q(T)), \quad f_Y : \bigcup_{T\in\Gamma_1} \mathfrak{M}(Q(T)) \to \bigcup_{T\in\Gamma_1} Y_T$$

be homeomorphisms such that, for all $T \in \Gamma_1$, $\mathcal{C}X_T = f_X^{-1}(\mathcal{C}_T)$ and $\mathcal{C}Y_T = f_Y(\mathcal{C}_T)$ are the faces of the cells X_T and Y_T homeomorphic to the face $\mathcal{C}_T$ of $\mathfrak{M}(Q(T))$.

If f is the restriction of $f_Y \circ f_X$ to $\bigcup \mathcal{C}X_T$, by Proposition 2.2, it follows that $\bigcup_{T\in\Gamma_5} X_T \cup_f \bigcup_{T\in\Gamma_5} Y_T$ is a finite union of cells, where each two are either disjoint or intersects in a union of their faces.

Now, the map,

$$R : (\bigcup \mathfrak{M}_p^1(T)) \cup (\bigcup \mathfrak{M}_p^2(T)) \to (\bigcup X_T) \cup_f (\bigcup Y_T),$$

defined by,

$$R(x) = \begin{cases} f_X^{-1}(Q(x)) & x \in \bigcup \mathfrak{M}_p^1(T), \\ f_Y(Q(x)) & x \in \bigcup \mathfrak{M}_p^2(T), \end{cases}$$

is a homeomorphism. By Proposition 2.2 the space,

$$(\bigcup \mathfrak{M}_p^1(T)) \cup (\bigcup \mathfrak{M}_p^2(T)),$$

is a finite union of cells where each two are either disjoint or intersect in a finite union of their faces.

Finally, note that for each internal tree $T \in \Gamma_1$ different from $\{0_L\}$, we have $C_p(L) \cap \mathfrak{M}_p^1(T) = \emptyset$ and $C_p(L) \cap \mathfrak{M}_p^2(T) = \emptyset$. If

$$\mathfrak{D} = \overline{C_p(L) - (\mathfrak{M}_p^1(\{0_L\}) \cup \mathfrak{M}_p^2(\{0_L\}))},$$

then once again the space,

$$(\bigcup \mathfrak{M}_p^1(T)) \cup (\bigcup \mathfrak{M}_p^2(T)) \cup \mathfrak{D},$$

is a finite union of cells, where each two are either disjoint or intersect in some union of their faces, since this union is the set $C_p(G)$, we have $C_p(G)$ that is a polytope.

Case 2: The point p is in a terminal edge J of G and $p \neq 0_J$.

In this case, $C_p(G) = (\bigcup_{T\in\Gamma_2} \mathfrak{N}_p^3(T)) \cup C_p(J)$ and we have two subcases. The first one is attained when p is the terminal point of J, and the second is attained when p is a point in the interior of J.

In the first subcase, the intersection of any two cells of the form $\mathfrak{N}_p^3(T)$ is empty or a union of their faces. This is because $\bigcup_{T\in\Gamma_2} \mathfrak{N}_p^3(T)$ is homeomorphic to $\bigcup \mathfrak{M}(T)$, where T runs over the internal trees of $G - p$ containing 0_J. On the other hand, $\mathfrak{N}_p^3(T) \cap C_p(J) = \{J\}$ is a face of both cells. The other subcase is treated similarly.

Case 3: If neither p is in a loop, nor a terminal edge, then there is an internal tree T such that $p \in D(1, T)$.

In this case we have that the space $C_p(G)$ coincides with:

$$C_p(J) \cup \bigcup_{T\in\Gamma_3} \mathfrak{M}(T) \cup \bigcup_{H\in\Gamma_4} \mathfrak{K}_p^1(H) \cup \bigcup_{H\in\Gamma_4} \mathfrak{K}_p^2(H) \cup \bigcup_{M\in\Gamma_5} \mathfrak{M}_p(M) \cup \bigcup_{N\in\Gamma_6} \mathfrak{N}_p(N).$$

It remains to be seen that the intersection of any two cells in the above union is a union of their faces. This involves many cases to consider. We will analyze a few of them, the others can be treated in a similar way.

Step 1. For $T, S \in \Gamma_3$ it follows immediately from Lemma 2.6 (i) that if the cells $\mathfrak{M}(T)$ and $\mathfrak{M}(S)$ have nonempty intersection, then the intersection is a union of their faces.

Step 2. Now, choose $H, K \in \Gamma_4$ and check that the intersection, $\mathfrak{K}_p^1(H) \cap \mathfrak{K}_p^2(K)$, is not empty, then it is a union of their faces.

Recall that the faces of a cell $[0,1]^n$ are those subsets obtained by restricting some of its coordinates, so that they can only take the values 0 or 1, and the remaining coordinates are free to take any value between 0 and 1.

What we show is that $\mathfrak{K}_p^1(H) \cap \mathfrak{K}_p^2(K)$ is a union of such faces, both, cell $\mathfrak{K}_p^1(H)$ and cell $\mathfrak{K}_p^2(K)$. We will prove this fact for $\mathfrak{K}_p^1(H)$; the proof for $\mathfrak{K}_p^2(K)$ is practically the same.

Let $D(1,H) = H \cup (\bigcup_{i=1}^n J_i) \cup (\bigcup_{j=1}^n L_j)$ be the canonical representation of $D(1,H)$. (Recall we are supposing that $L_1 = J$). Then, the members of $\mathfrak{K}_p^1(H)$ are connected compact subsets of the form,

$$(H \cup \mathcal{L}_1) \cup ([p, a_1] \cup [b_1, v]) \cup (\bigcup_{i=1}^n [0_{J_i}, c_i]) \cup (\bigcup_{j=2}^m [0_{L_j}, a_j] \cup [b_j, 1_{L_j}])).$$

According this, the faces of $\mathfrak{K}_p^1(H)$ are given when we fix $c_i = 0_{J_i}$ or $c_i = 1_{J_i}$ for some index $i \in \{1, \dots, n\}$ and when we fix $a_1 = p$, $b_1 = v$, or $a_1 = b_1$, or we fix $a_j = 0_{L_j}$, $b_j = 1_{L_j}$, or $a_j = b_j$ for some index $j \in \{2, \dots, m\}$.

Since $\mathfrak{K}_p^1(H) \cap \mathfrak{K}_p^2(K)$ is not empty, we can fix a point x in the intersection. Since $x \in \mathfrak{K}_p^2(K)$, we have that $\mathcal{L}_2 \subset x$. Thus, x when viewed as an element of $\mathfrak{K}_p^1(H)$, lies on the faces of $\mathfrak{K}_p^1(H)$ when $a_1 = b_1$. On the other hand, $K \subset x \subset H \cup (\bigcup_{i=1}^n J_i) \cup (\bigcup_{j=1}^m L_j)$. In this way, the internal tree K is formed by some edges in H, some edges J_i, and some edges L_j, where $j \neq 1$. In order to see what values are fixed, we will investigate each one of the edges J_i and L_j.

Given an edge J_i we have some cases to analyze:

Case 1: $J_i \subset K$.

In this case, any member $y \in \mathfrak{K}_p^1(H) \cap \mathfrak{K}_p^2(K)$ satisfies $J_i \subset K \subset y$. Thus, y as an element of $\mathfrak{K}_p^1(H)$, we must fix $c_i = 1_{J_i}$.

Case 2: J_i is not a subset of K and 0_{J_i}, $1_{L_j} \in K$.

Since K is formed by edges, since K is a subgraph of G, the intersection $K \cap J_i$ is precisely the vertices 0_{J_i} and 1_{J_i}, in this case c_i can take all values from 0_{J_i} to 1_{J_i}.

Case 3: J_i is not a subset of K and $0_{J_i} \in K$, $1_{J_i} \notin K$.

The growth of the members of $\mathfrak{K}_p^1(H)$ is the same as the members of $\mathfrak{K}_p^2(K)$ through the edge J_i. Thus, in this case c_i takes all values in the edge J_i.

Case 4: J_i is not a subset of K and $1_{J_i} \in K$, $0_{J_i} \notin K$.

In this case, c_i cannot be an element of $J_i - \{0_{J_i}, 1_{J_i}\}$ because corresponding element cannot belong to $\mathfrak{K}_p^2(K)$. Therefore in this case, $c_i = 0_{J_i}$ or $c_i = 1_{J_i}$.

Case 5: J_i is not a subset of K and $0_{J_i}, 1_{J_i} \notin K$.

No point in $J_i - \{0_{J_i}, 1_{J_i}\}$ belongs to $D(1, K)$ and therefore it does not belong to any element of $\mathfrak{K}_p^2(K)$. Thus, $c_i = 0_{J_i}$.

We have concluded the possible cases for J_i. Now let us see what happens with L_j, where $j \in \{2, \ldots, m\}$.

Case 6: $L_j \subset K$.

Any element $y \in \mathfrak{K}_p^1(H) \cap \mathfrak{K}_p^2(K)$ satisfies $L_j \subset K \subset y$. So, when viewed as elements of $\mathfrak{K}_p^1(H)$ we have $a_j = b_j$.

Case 7: L_j is not a subset of K and 0_{L_j}, $1_{L_j} \in K$.

Both the members of $\mathfrak{K}_p^1(H)$ and members of $\mathfrak{K}_p^2(K)$ can grow from the ends of L_j, so a_j and b_j can take all values in L_j.

Case 8: L_j is not a subset of K and $0_{L_j} \in K$, $1_{L_j} \notin K$.

No element of $\mathfrak{K}_p^2(K)$ can grow from the end 1_{L_j}, therefore we must have $b_j = 1_{L_j}$.

Case 9: L_j is not a subset of K, $0_{L_j} \notin K$ and $1_{L_j} \in K$.

No member of $\mathfrak{K}_p^2(K)$ can grow from the vertex 0_{L_j}, therefore we must have $a_j = 0_{L_j}$.

Case 10: L_j is not a subset of K, $0_{L_j}, 1_{L_j} \notin K$.

No element of $L_j - \{0_{L_j}, 1_{L_j}\}$ belongs to $D(1, K)$, therefore no element belongs to $\mathfrak{K}_p^2(K)$, therefore we must have $a_j = 0_{L_j}$ and $b_j = 1_{L_j}$.

Step 3. Now, let us see that for $H, K \in \Gamma_4$, if the cells $\mathfrak{K}_p^1(H)$, $\mathfrak{K}_p^1(K)$ have a nonempty intersection, then the intersection is a union of faces.

Indeed, the homeomorphisms $\sigma_H : \mathfrak{K}_p^1(H) \to \mathfrak{M}(\hat{Q}(H))$ and $\sigma_K : \mathfrak{K}_p^1(K) \to \mathfrak{M}(\hat{Q}(K))$ in the proof of Proposition 3.7 can be used to construct a unique continuous map $\sigma : \mathfrak{K}_p^1(H) \cup \mathfrak{K}_p^1(K) \to \mathfrak{M}(\hat{Q}(H)) \cup \mathfrak{M}(\hat{Q}(K))$ which extends σ_H and σ_K.

Furthermore, σ is a homeomorphism. Since $\mathfrak{M}(\hat{Q}(H))$ and $\mathfrak{M}(\hat{K}(H))$ are a pair of the cells that endow a polytope structure to $C(\hat{Q}(G))$, it follows that these cells intersect in a union of their faces. Therefore the cells $\mathfrak{K}_p^1(H)$ and $\mathfrak{K}_p^1(K)$ intersect in a union of their faces.

Step 4. Now we see that $T \in \Gamma_3$, $H \in \Gamma_4$, and the cells $\mathfrak{M}(T)$, and $\mathfrak{K}_p^1(H)$ either intersect in a union of their faces or do not intersect.

Let $Id : \mathfrak{M}(T) \to \mathfrak{M}(T)$ be the identity map and $\sigma_H : \mathfrak{K}_p^1(H) \to \mathfrak{M}(\hat{Q}(H))$ the map in Proposition 3.7. Consider $A \in \mathfrak{M}(T) \cap \mathfrak{K}_p^1(H)$. Recall that Γ_3 is the set of internal trees of G containing p. According this, we have that A contains the edge J (recall that J is the edge containing p in its interior), because of this, if A is represented as an element of $\mathfrak{K}_p^1(H)$, we must set $a_1 = b_1$. Thereby,

$$\begin{aligned}\sigma_H(A) = \hat{Q}(A) = \hat{Q}((H \cup \mathcal{L}_1 \cup [p, a_1] \cup [b_1, v]) \cup (\bigcup_{i=1}^{n} [0_{J_i}, c_i])\\ \cup (\bigcup_{j=2}^{m} [0_{L_j}, a_j] \cup [b_j, 1_{L_j}])))\\ = \hat{Q}(H) \cup \hat{Q}(\mathcal{L}_2) \cup \hat{Q}((\bigcup_{i=1}^{n} [0_{J_i}, c_i]) \cup (\bigcup_{j=2}^{m} [0_{L_j}, a_j] \cup [b_j, 1_{L_j}]))).\end{aligned}$$

So that

$$\begin{aligned}\psi_1 \circ \sigma_H(A) &= H \cup f(\mathcal{L}_1) \cup (\bigcup_{i=1}^{n} [0_{J_i}, c_i]) \cup (\bigcup_{j=2}^{m} [0_{L_j}, a_j] \cup [b_j, 1_{L_j}])\\ &= H \cup f(\mathcal{L}_2) \cup (\bigcup_{i=1}^{n} [0_{J_i}, c_i]) \cup (\bigcup_{j=2}^{m} [0_{L_j}, a_j] \cup [b_j, 1_{L_j}])\\ &= H \cup J \cup (\bigcup_{i=1}^{n} [0_{J_i}, c_i]) \cup (\bigcup_{j=2}^{m} [0_{L_j}, a_j] \cup [b_j, 1_{L_j}]) = A,\end{aligned}$$

where ψ_1 is the same as in proof of Proposition 3.6. Thus, maps Id and $\psi_1 \circ \sigma_H$ provide a unique continuous map $\alpha : \mathfrak{M}(T) \cup \mathfrak{K}_p^1(H) \to \mathfrak{M}(T) \cup \mathfrak{M}(H)$ which extends Id and $\psi_1 \circ \sigma_H$. Since α is a homeomorphism, because $\mathfrak{M}(T) \cup \mathfrak{M}(H)$ is a union of finite-dimensional cells whose intersections are either disjoint or are a union of faces (Lemma 2.6 (i)), then so is $\mathfrak{M}(T) \cup \mathfrak{K}_p^1(H)$ a union of finite-dimensional cells whose intersections are either disjoint or are a union of faces.

Step 5. Now, let us see that any two cells of type $\mathfrak{M}_p(M)$ either intersect in a union of their faces or have empty intersection. If $M_1, M_2 \in \Gamma_5$, the maps β_{M_1} and β_{M_2} in the proof of the Lemma 3.8 agree in the intersection of their domains, therefore there exists a unique continuous map $\beta : \mathfrak{M}_p(M_1) \cup \mathfrak{M}_p(M_2) \to \mathfrak{M}(\hat{Q}(M_1)) \cup \mathfrak{M}(\hat{Q}(M_2))$ which is an extension of β_{M_1} and β_{M_2}. The map β is a homeomorphism. It follows that cells $\mathfrak{M}_p(M_1)$ and $\mathfrak{M}_p(M_2)$ intersect one another like the cells $\mathfrak{M}(\hat{Q}(M_1))$ and $\mathfrak{M}(\hat{Q}(M_1))$ do. Since these intersect in a union of their faces, the cells $\mathfrak{M}_p(M_1)$ and $\mathfrak{M}_p(M_2)$ intersect in a union of their faces in the same manner.

The remaining cases are treated in a similar way. This concludes the proof of Theorem 4.1. □

The converse of Theorem 4.1, is also true in the class of all locally connected, connected and compact spaces and its proof is much simpler.

Lemma 4.2 ([11]). *Let X be a connected, locally connected, compact metric space. Then X is not a multigraph if and only if $C(X)$ contains a Hilbert cube.*

Since it is desirable for us that the Hilbert cube in the Lemma above be contained in a hyperspace $C_p(X)$, we need more results. Lemmas 4.3 and 4.4 can be found in [12].

Lemma 4.3. *Let A^0 and $A^1 \in 2^X$ be such that $A_0 \neq A_1$. Then, the following two statements are equivalent:*

(1) There exists an order arc in 2^X from A_0 to A_1;
(2) $A_0 \subseteq A_1$ and each component of A_1 intersects A_0.

Lemma 4.4. *If α is an order arc in 2^X beginning with $A_0 \in C(X)$, then $\alpha \subseteq C(X)$.*

Recall that, an ∞-*odd* is a connected and compact space B, containing a compact connected subspace A, such that $B - A$ contains infinitely many components.

Lemma 4.5 ([9]). *Let X be any connected compact metric space. Then $C(X)$ contains Hilbert cubes if and only if X contains ∞-odds.*

The converse of Lemma 4.5 is important to us, since from it, it will be clear that $C_p(X)$ contains a Hilbert cube for some p. We include here the proof for completeness.

Proof. Let B be an ∞-odd of X and $A \subseteq B$ a connected compact subspace such that $B - A$ contains infinitely components. Choose a numerable collection K_1, K_2, ... of such components. Each $A \cup K_n$ is a compact and connected subspace of X containing A. According to Lemmas 4.3 and 4.4, there exists an order arc $\alpha_n : [0, 1] \to C(X)$ from A to $A \cup K_n$.

Since each α_n is a continuous map, there exist $r_n \in (0, 1]$ such that $H(\alpha_n(r_n), A) < 1/n$ (where H is the Hausdorff metric on $C(X)$). Hence $\alpha_n(r_n) \subseteq N(1/n, A) = \{x \in X : \text{there exists } a \in A \text{ such that } d(x, a) < 1/n\}$. Define $\varphi : [0, 1]^{\mathbb{N}} \to C(X)$ by,

$$\varphi(t_1, t_2, \ldots) = \alpha_1(r_1 t_1) \cup \alpha_2(r_2 t_2) \cup \cdots .$$

Since each $\alpha_n(r_n t_n)$ is a connected compact subspace containing A, we have that $\varphi(t_1, t_2, \ldots)$ is a connected subset of X.

Furthermore, it can be proven that $\varphi(t_1, t_2, \ldots)$ is closed (hence compact), continuous and injective. Since $\operatorname{Dom}\varphi$ is compact and $\operatorname{Im}\varphi$ is contained in a Hausdorff space, we have that the map $\varphi : [0,1]^{\mathbb{N}} \to \varphi([0,1]^{\mathbb{N}})$ is a homeomorphism. So, $\varphi([0,1]^{\mathbb{N}})$ is a subspace homeomorphic to $[0,1]^{\mathbb{N}}$ which is contained in $C(X)$. This concludes the proof. □

Notice that, in the proof of the above Lemma, since $A \subseteq \varphi(z)$ for all $z \in [0,1]^{\mathbb{N}}$, if $p \in A$, then $\varphi([0,1]^{\mathbb{N}})$ is a Hilbert cube contained in $C_p(X)$. Therefore next corollary follows.

Corollary 4.6. *Let X be any connected, compact, metric space. If X contains an ∞-odd, then there exists $p \in X$, such that $C_p(X)$ contains a Hilbert cube.*

Theorem 4.7. *If X is locally connected, connected, compact, and Cpp, then X is a multigraph.*

Proof. If X is not a multigraph, it follows by Lemma 4.2, that $C(X)$ contains a Hilbert cube. By Lemma 4.5, X contains an ∞-odd and by Corollary 4.6, $C_p(X)$ contains a Hilbert cube for some $p \in X$, this is a contradiction since X is Cpp. This concludes the proof. □

Corollary 4.8. *A connected, locally connected, compact metric space X is a multigraph if and only if X is Cpp.*

Question. *Does there exists any nonlocally connected (and therefore not a multigraph), connected and compact metric space X, which is Cpp?*

Question. *Can the phrase* locally connected *in Theorem 4.7 be replaced by* arc-connected*?*

References

1. J. Barát and A. Gyárfás, *Rainbow matchings in bipartite multigraphs*, Periodica Mathematica Hungarica **74** (2017), 108–111.
2. A. Barvinok, S. J. Lee, and I. Novik, *Explicit constructions of centrally symmetric k-neighborly polytopes and large strictly antipodal sets*, Discrete Comput. Geom. **49** (2013), no. 3, 429–443.
3. A. Barvinok and I. Novik, *A centrally symmetric version of the cyclic polytope*, Discrete Comput. Geom. **39** (2008), no. 1–3, 76–99.
4. S. Bermudo, J. M. Rodríguez, and J. M. Sigarreta, *Gromov hyperbolic graphs*, Discrete Mathematics **313** (2013), 1575–1585.
5. D. Bryant and B. R. Smith, *Decompositions of complete multigraphs into cycles of varying length*, Journal of Combinatorial Theory **129** (2018), 79–106.
6. R. Duda, *On the hyperspace of subcontinua of a finite graph I*, Fund. Math. **62** (1968), no. 3, 265–286.
7. J. Dugundji, *Topology*, Allyn and Bacon Series in Advanced Mathematics, Allyn and Bacon, Boston, 1976.
8. M. Gentner and D. Rautenbach, *Feedback vertex sets in cubic multigraphs*, Discrete Mathematics **338** (2015), no. 12, 2179–2185.
9. A. Illanes, *Cells and cubes in hyperspaces*, Fund. Math. **130** (1988), no. 1, 57–65.

10. S. Klee, E. Nevo, I. Novik, and H. Zheng, *A lower bound theorem for centrally symmetric simplicial polytopes*, Discrete Comput. Geom. (2018), to appear. DOI 10.1007/s00454-018-9978-z.
11. S. B. Nadler, *Locating cones and Hilbert cubes in hyperspaces*, Fun. Math. **79** (1973), 233–250.
12. ______, *Hyperspaces of sets. a text with research questions*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 49, Marcel Dekker, Inc., 1978.
13. I. Novik, *A tale of centrally symmetric polytopes and spheres*, available online at `https://sites.math.washington.edu/~novik/publications/cs-survey.pdf`, May April.
14. J. M. Sigarreta, *Hyperbolicity in median graphs*, Proceedings Mathematical Sciences **123** (2013), no. 4, 455–467.

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