Contributions to Discrete Mathematics

Volume 14, Number 1, Pages 117–129
ISSN 1715-0868

NEW PARITY RESULTS OF SUMS OF PARTITIONS AND SQUARES IN ARITHMETIC PROGRESSIONS

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ABSTRACT. Recently, Ballantine and Merca proved that if $(a,b) \in \{(6,8),(8,12),(12,24),(15,40),(16,48),(20,120),(21,168)\}$, then $\sum_{ak+1 \text{ square}} p(n-k) \equiv 1 \pmod 2$ if and only if $bn+1$ is a square. In this paper, we investigate septuple $(a_1,a_2,a_3,a_4,a_5,a_6,a_7) \in \mathbb{N}^5 \times \mathbb{Q}^2$ for which $\sum_{a_1k+a_2 \text{ square}} p(a_3a_4^{\alpha}n + a_6a_4^{\alpha} + a_7 - k) \equiv 1 \pmod 2$ if and only if a_5n+1 is a square. We prove some new parity results of sums of partitions and squares in arithmetic progressions which are analogous to the results due to Ballantine and Merca.

1. Introduction

A partition of a nonnegative integer n is a nonincreasing sequence of positive integers that sum to n. Let $p(n)$ denote the number of partitions of n. As usual, set $p(0) = 1$. It is well known, by the work of Euler, that the generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1.1}$$

where throughout this paper, we always employ the standard notation

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

The parity of $p(n)$ has been investigated by a number of authors, including Hirschhorn [5, 6], Kolberg [8], Newman [9], Nicolas, Ruzsa, and Sárközy [10], Ono [11, 12], Subbarao [15], and Radu [14]. In fact, the parity of $p(n)$ seems to be quite random, and it is widely believed that $p(n)$ is even approximately

Received by the editors March 19, 2018, and in revised form July 13, 2018.
2010 *Mathematics Subject Classification.* 05A17; 11P83.
Key words and phrases. partitions, parity, arithmetic progressions.
This work was supported by the National Science Foundation of China (11971203, 11571143), the Natural Science Foundation of Jiangsu Province of China (BK20180044), Chinese Postdoctoral Science Foundation (2018T110444) and Jiangsu Overseas Research & Training Program for University Prominent Young & Middle-aged Teachers.

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half of the time. More precisely, Parkin and Shanks [13] conjectured that

$$\lim_{n\to+\infty} \frac{\#\{k \le n : p(k) \text{ is odd}\}}{n} = \frac{1}{2}. \tag{1.2}$$

Although there have been many works on the parity of $p(n)$, we are very far from proving this conjecture.

Questions regarding the parity of the partition function in arithmetic progressions have been investigated for many years. In 1966, Subbarao [15] conjectured that every arithmetic progression contains infinitely many integers M for which $p(M)$ is odd, as well as infinitely many integers N for which $p(N)$ is even. Moreover, he proved that for the progression $1 \pmod 2$ the conjecture is true. In the even case, the conjecture was settled by Ono [11]. Radu [14] completed the proof of the odd part of Subbarao's conjecture.

Recently, Ballantine and Merca [2] considered the parity of sums of partition numbers for square values in given arithmetic progressions. Ballantine and Merca's work differs from the articles [11, 14, 15] in that they considered the parity of single values of $p(n)$ in an arithmetic progression. Ballantine and Merca [2] proved that if $(a, b) \in \{(6, 8),\ (8, 12),\ (12, 24),\ (15, 40),\ (16, 48), (20, 120), (21, 168)\}$, then

$$\sum_{ak+1 \text{ square}} p(n-k) \equiv 1 \pmod 2$$

if and only if $bn + 1$ is a square.

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ denote the set of nonnegative integers, the set of integers and the set of rational numbers, respectively. In this paper, we investigate the septuple $(a_1, a_2, a_3, a_4, a_5, a_6, a_7) \in \mathbb{N}^5 \times \mathbb{Q}^2$ for which

$$\sum_{a_1k+a_2 \text{ square}} p(a_3 a_4^{\alpha} n + a_6 a_4^{\alpha} + a_7 - k) \equiv 1 \pmod 2$$

if and only if $a_5 n + 1$ is a square.

The main results of this paper can be stated as follows.

Theorem 1.1. *Let n, α be nonnegative integers. Then*

$$\sum_{24k+9 \text{ is a square}} p\left(2^{2\alpha+3} n + \frac{2^{2\alpha} - 1}{3} - k\right) \equiv 1 \pmod 2 \tag{1.3}$$

if and only if $24n + 1$ is a square.

Theorem 1.2. *Let n be a nonnegative integer. Then*

$$\sum_{120k+25 \text{ is a square}} p(4n - k) \equiv 1 \pmod 2 \tag{1.4}$$

if and only if $24n + 1$ is a square.

Theorem 1.3. *Let n, α be nonnegative integers. Then*

$$\sum_{88k+121 \text{ is a square}} p\left(2^{2\alpha+5} n + \frac{2(2^{2\alpha} - 1)}{3} - k\right) \equiv 1 \pmod 2 \tag{1.5}$$

if and only if $24n+1$ *is a square.*

Theorem 1.4. *Let* $n,\ \alpha$ *be nonnegative integers. Then*

$$\sum_{88k+121 \text{ is a square}} p\left(11\times 2^{2\alpha+4}n+\frac{5\times 2^{2\alpha+2}-2}{3}-k\right)\equiv 1 \pmod 2 \tag{1.6}$$

if and only if $24n+1$ *is a square.*

Theorem 1.5. *Let* n,α *be nonnegative integers. Then*

$$\sum_{312k+169 \text{ is a square}} p\left(4\times 13^{\alpha}n+\frac{13^{\alpha}-1}{2}-k\right)\equiv 1 \pmod 2 \tag{1.7}$$

if and only if $8n+1$ *is a square.*

Theorem 1.6. *Let* n,α *be nonnegative integers and let* $p\geq 3$ *be a prime. Then*

$$\sum_{312k+169 \text{ is a square}} p\left(4p^{2\alpha}n+\frac{p^{2\alpha}-1}{2}-k\right)\equiv 1 \pmod 2 \tag{1.8}$$

if and only if $8n+1$ *is a square.*

2. Proof of Theorem 1.1

It is well-known that

$$\sum_{n=0}^{\infty} q^{\frac{n^2+n}{2}}=\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{2.1}$$

Combining (1.1) and (2.1) yields

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{t+\frac{3s(s+1)}{2}=n,\ (t,s)\in\mathbb{N}^2} p(t)q^n &= \sum_{t=0}^{\infty}p(t)q^t\sum_{s=0}^{\infty}q^{\frac{3(s^2+s)}{2}}\\ &=\frac{(q^6;q^6)_\infty^2}{(q;q)_\infty(q^3;q^3)_\infty}.\end{aligned} \tag{2.2}$$

It follows from (2.6) in [16] that

$$\begin{aligned}(q;q)_\infty(q^3;q^3)_\infty =&\frac{(q^2;q^2)_\infty(q^8;q^8)^2(q^{12};q^{12})_\infty^4}{(q^4;q^4)_\infty^2(q^6;q^6)_\infty(q^{24};q^{24})_\infty^2}\\ &-q\frac{(q^4;q^4)_\infty^4(q^6;q^6)_\infty(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty(q^8;q^8)^2(q^{12};q^{12})_\infty^2}.\end{aligned} \tag{2.3}$$

By the binomial theorem,

$$(q;q)_\infty^2\equiv(q^2;q^2)_\infty \pmod 2\,. \tag{2.4}$$

Thanks to (2.2)–(2.4),

(2.5)
$$\begin{aligned}\frac{(q^6;q^6)_\infty^2}{(q;q)_\infty(q^3;q^3)_\infty} &= \frac{(q;q)_\infty(q^3;q^3)_\infty(q^6;q^6)_\infty^2}{(q;q)_\infty^2(q^3;q^3)_\infty^2}\\ &\equiv \frac{(q;q)_\infty(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q^2;q^2)}\\ &= \frac{(q^8;q^8)^2(q^{12};q^{12})_\infty^4}{(q^4;q^4)_\infty^2(q^{24};q^{24})_\infty^2} - q\frac{(q^4;q^4)_\infty^4(q^6;q^6)_\infty^2(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^2(q^8;q^8)^2(q^{12};q^{12})_\infty^2}\\ &\equiv (q^8;q^8)_\infty + q\frac{(q^{24};q^{24})_\infty^2}{(q^4;q^4)_\infty(q^{12};q^{12})_\infty} \pmod 2.\end{aligned}$$

Substituting (2.5) into (2.2) and extracting the terms involving q^{8n}, then replacing q^8 by q, we deduce that

$$\sum_{n=0}^{\infty} \sum_{t+\frac{3(s^2+s)}{2}=8n,\ (s,t)\in\mathbb{N}^2} p(t)q^n \equiv (q;q)_\infty \pmod 2. \tag{2.6}$$

One of the most famous identities in the theory of partitions is Euler's pentagonal number theorem

$$\sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{m(3m-1)}{2}} = (q;q)_\infty. \tag{2.7}$$

Combining (2.6) and (2.7) yields

$$\sum_{t+\frac{3(s^2+s)}{2}=8n,\ (s,t)\in\mathbb{N}^2} p(t) \equiv 1 \pmod 2 \tag{2.8}$$

if and only if $n = m(3m-1)/2$ for some integer m. It is easy to see that $24k+9$ is a square if and only if $k = 3(s^2+s)/2$ for some nonnegative integer s. Thus,

$$\begin{aligned}\sum_{t+\frac{3(s^2+s)}{2}=8n,\ (s,t)\in\mathbb{N}^2} p(t) &= \sum_{s\in\mathbb{N}} p\left(8n - \frac{3s(s+1)}{2}\right)\\ &= \sum_{24k+9 \text{ is a square}} p(8n-k).\end{aligned}$$

Moreover, $24n+1$ is a square if and only if $n = m(3m-1)/2$. Therefore, we can rewrite (2.8) as

$$\sum_{24k+9 \text{ is a square}} p(8n-k) \equiv 1 \pmod 2 \tag{2.9}$$

if and only if $24n+1$ is a square. Substituting (2.5) into (2.2) and extracting those terms in which the power of q is congruent to 1 modulo 4, then dividing by q and replacing q^4 by q, we have

$$\sum_{n=0}^{\infty} \sum_{t+\frac{3(s^2+s)}{2}=4n+1,\ (s,t)\in\mathbb{N}^2} p(t)q^n \equiv \frac{(q^6;q^6)_\infty^2}{(q;q)_\infty (q^3;q^3)_\infty} \pmod{2}. \tag{2.10}$$

It follows from (2.2) and (2.10) that for $n\in\mathbb{N}$,

$$\sum_{t+\frac{3(s^2+s)}{2}=4n+1,\ (s,t)\in\mathbb{N}^2} p(t) \equiv \sum_{t+\frac{3(s^2+s)}{2}=n,\ (s,t)\in\mathbb{N}^2} p(t) \pmod{2}. \tag{2.11}$$

By (2.11) and mathematical induction, for $n,\alpha\in\mathbb{N}$,

$$\sum_{t+\frac{3(s^2+s)}{2}=4^\alpha n+\frac{4^\alpha-1}{3},\ (s,t)\in\mathbb{N}^2} p(t) \equiv \sum_{t+\frac{3(s^2+s)}{2}=n,\ (s,t)\in\mathbb{N}^2} p(t) \pmod{2}.$$

Furthermore, the above identity can be rewritten as

$$\sum_{24k+9 \text{ is a square}} p\left(4^\alpha n+\frac{4^\alpha-1}{3}-k\right) \equiv \sum_{24k+9 \text{ is a square}} p(n-k) \pmod{2}. \tag{2.12}$$

Replacing n by $8n$ in (2.12) and using (2.9), we arrive at (1.3). This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

It follows from (1.1) and (2.7) that

$$\begin{aligned}\sum_{n=0}^{\infty} \sum_{t+\frac{5s(3s-1)}{2}=n,\ (s,t)\in\mathbb{Z}\times\mathbb{N}} p(t)q^n &= \sum_{t=0}^{\infty} p(t)q^t \sum_{s=-\infty}^{\infty} q^{\frac{5s(3s-1)}{2}} \\ &\equiv \sum_{t=0}^{\infty} p(t)q^t \sum_{s=-\infty}^{\infty} (-1)^s q^{\frac{5s(3s-1)}{2}} \pmod{2} \\ &= \frac{(q^5;q^5)_\infty}{(q;q)_\infty}.\end{aligned} \tag{3.1}$$

Xia and Yao [17] proved that

$$\frac{(q^5;q^5)_\infty}{(q;q)_\infty} = \frac{(q^8;q^8)_\infty (q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^2 (q^{40};q^{40})_\infty} + q\frac{(q^4;q^4)_\infty^3 (q^{10};q^{10})_\infty (q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^3 (q^8;q^8)_\infty (q^{20};q^{20})_\infty}; \tag{3.2}$$

see also [7]. By (2.4) and (3.2),

$$\frac{(q^5;q^5)_\infty}{(q;q)_\infty} \equiv (q^4;q^4)_\infty + q\frac{(q^{10};q^{10})_\infty^3}{(q^2;q^2)_\infty} \pmod{2}. \tag{3.3}$$

Substituting (3.3) into (3.1) and extracting the terms involving q^{4n}, then replacing q^4 by q, we arrive at

$$\sum_{n=0}^{\infty} \sum_{t+\frac{5s(3s-1)}{2}=4n} p(t)q^n \equiv (q;q)_\infty \pmod 2. \tag{3.4}$$

In view of (2.7) and (3.4),

$$\sum_{n=0}^{\infty} \sum_{s=-\infty}^{\infty} p\left(4n - \frac{5s(3s-1)}{2}\right) q^n \equiv \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{m(3m-1)}{2}} \pmod 2. \tag{3.5}$$

Therefore,

$$\sum_{s=-\infty}^{\infty} p\left(4n - \frac{5s(3s-1)}{2}\right) \equiv 1 \pmod 2 \tag{3.6}$$

if and only if $n = m(3m-1)/2$. It is easy to see that $k = 5s(3s-1)/2$ for some integer s if and only if $120k+25$ is a square. Thus, we can rewrite (3.6) as

$$\sum_{120k+25 \text{ is a square}} p(4n-k) \equiv 1 \pmod 2 \tag{3.7}$$

if and only if $n = m(3m-1)/2$ for some integer m. Theorem 1.2 follows from (3.7) and the fact that $n = m(3m-1)/2$ for some integer m if and only if $24n+1$ is a square. This completes the proof.

4. Proofs of Theorems 1.3 and 1.4

In order to prove Theorems 1.3 and 1.4, we first prove the following lemma.

Lemma 4.1. *We have*

$$\begin{aligned} \frac{1}{(q;q)_\infty (q^{11};q^{11})_\infty} &\equiv \frac{(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty (q^{44};q^{44})_\infty} + q\frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty^3} \\ &+ q^6 \frac{(q^{66};q^{66})_\infty^3}{(q^{22};q^{22})_\infty (q^{44};q^{44})_\infty} + q^{15} \frac{(q^{132};q^{132})_\infty^3}{(q^4;q^4)_\infty (q^{44};q^{44})_\infty} \pmod 2. \end{aligned} \tag{4.1}$$

Proof. From (36.8) in Berndt's book [3, p. 69], we deduce that if μ is even, then

$$\begin{aligned} \psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) &= \varphi(q^{\mu(\mu^2-\nu^2)})\psi(q^{2\mu}) \\ &+ \sum_{m=1}^{\mu/2-1} q^{\mu m^2 - \nu m} f(q^{(\mu+2m)(\mu^2-\nu^2)}, q^{(\mu-2m)(\mu^2-\nu^2)}) f(q^{2\nu m}, q^{2\mu-2\nu m}) \\ &+ q^{\mu^3/4 - \mu\nu/2} \psi(q^{2\mu(\mu^2-\nu^2)}) f(q^{\mu\nu}, q^{2\mu-\mu\nu}), \end{aligned} \tag{4.2}$$

where

$$\varphi(q)=\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2},\qquad \psi(q)=\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty},$$

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Setting $\mu=6$ and $\nu=5$ in (4.2), we get

$$\begin{aligned}\psi(q)\psi(q^{11})=&\varphi(q^{66})\psi(q^{12})+qf(q^{88},q^{44})f(q^{10},q^2)\\&+q^{14}f(q^{110},q^{22})f(q^{20},q^{-8})+q^{39}\psi(q^{132})f(q^{30},q^{-18}).\end{aligned}\tag{4.3}$$

It is easy to verify that

$$f(q^{88},q^{44})=\frac{f_{88}f_{132}^2}{f_{44}f_{264}},\quad f(q^{10},q^2)=\frac{f_4^2f_6f_{24}}{f_2f_8f_{12}},$$

$$f(q^{20},q^{-8})=q^{-8}\frac{f_8f_{12}^2}{f_4f_{24}},\quad f(q^{30},q^{-18})=q^{-24}\frac{f_{12}^5}{f_6^2f_{24}^2}.$$

Substituting the above identities into (4.3), we have

$$\begin{aligned}\frac{(q^2;q^2)_\infty^2(q^{22};q^{22})_\infty^2}{(q;q)_\infty(q^{11};q^{11})_\infty}=&\frac{(q^{24};q^{24})_\infty^2(q^{132};q^{132})_\infty^5}{(q^{12};q^{12})_\infty(q^{66};q^{66})_\infty^2(q^{264};q^{264})_\infty^2}\\&+q\frac{(q^4;q^4)_\infty^2(q^6;q^6)_\infty(q^{24};q^{24})_\infty(q^{88};q^{88})_\infty(q^{132};q^{132})_\infty^2}{(q^2;q^2)_\infty(q^8;q^8)_\infty(q^{12};q^{12})_\infty(q^{44};q^{44})_\infty(q^{264};q^{264})_\infty}\\&+q^6\frac{(q^8;q^8)_\infty(q^{12};q^{12})_\infty^2(q^{44};q^{44})_\infty^2(q^{66};q^{66})_\infty(q^{264};q^{264})_\infty}{(q^4;q^4)_\infty(q^{22};q^{22})_\infty(q^{24};q^{24})_\infty(q^{88};q^{88})_\infty(q^{132};q^{132})_\infty}\\&+q^{15}\frac{(q^{12};q^{12})_\infty^5(q^{264};q^{264})_\infty^2}{(q^6;q^6)_\infty^2(q^{24};q^{24})_\infty^2(q^{132};q^{132})_\infty}.\end{aligned}\tag{4.4}$$

Lemma 4.1 follows from (2.4) and (4.4). This completes the proof. □

We are now in a position to prove Theorems 1.3 and 1.4.

Based on (1.1) and (2.1),

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{t+\frac{11s(s+1)}{2}=n,\ (s,t)\in\mathbb{N}^2}p(t)q^n&=\sum_{t=0}^{\infty}p(t)q^t\sum_{s=0}^{\infty}q^{\frac{11s(s+1)}{2}}\\&=\frac{(q^{22};q^{22})_\infty^2}{(q;q)_\infty(q^{11};q^{11})_\infty}\equiv\frac{(q^{44};q^{44})_\infty}{(q;q)_\infty(q^{11};q^{11})_\infty}\pmod 2.\end{aligned}\tag{4.5}$$

Thanks to (4.1) and (4.5),

$$\sum_{n=0}^{\infty}\sum_{t+\frac{11s(s+1)}{2}=n,\ (s,t)\in\mathbb{N}^2}p(t)q^n\equiv\frac{(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty}+q\frac{(q^6;q^6)_\infty^3(q^{44};q^{44})_\infty}{(q^2;q^2)_\infty^3}$$

$$+ q^6 \frac{(q^{66};q^{66})_\infty^3}{(q^{22};q^{22})_\infty} + q^{15} \frac{(q^{132};q^{132})_\infty^3}{(q^4;q^4)_\infty} \pmod 2,$$

which yields

$$\sum_{n=0}^{\infty} \sum_{t+\frac{11s(s+1)}{2}=2n,\ (s,t)\in\mathbb{N}^2} p(t)q^n \equiv \frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty} + q^3 \frac{(q^{33};q^{33})_\infty^3}{(q^{11};q^{11})_\infty} \pmod 2. \tag{4.6}$$

In view of (2.4), (2.5), and (4.6),

$$\begin{aligned}\sum_{n=0}^{\infty} \sum_{t+\frac{11s(s+1)}{2}=2n,\ (s,t)\in\mathbb{N}^2} p(t)q^n &\equiv \frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty} + q^3 \frac{(q^{66};q^{66})_\infty^2}{(q^{11};q^{11})_\infty (q^{33};q^{33})_\infty} \\ &\equiv \frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty} + q^3 (q^{88};q^{88})_\infty + q^{14} \frac{(q^{132};q^{132})_\infty^3}{(q^{44};q^{44})_\infty} \pmod 2.\end{aligned} \tag{4.7}$$

It follows from (4.7) that

$$\sum_{n=0}^{\infty} \sum_{t+\frac{11s(s+1)}{2}=4n,\ (s,t)\in\mathbb{N}^2} p(t)q^n \equiv \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} + q^7 \frac{(q^{66};q^{66})_\infty^3}{(q^{22};q^{22})_\infty} \pmod 2 \tag{4.8}$$

and

$$\sum_{n=0}^{\infty} \sum_{t+\frac{11s(s+1)}{2}=176n+6,\ (s,t)\in\mathbb{N}^2} p(t)q^n \equiv (q;q)_\infty \pmod 2. \tag{4.9}$$

Based on (2.4), (2.5), and (4.8),

$$\begin{aligned}\sum_{n=0}^{\infty} \sum_{t+\frac{11s(s+1)}{2}=4n,\ (s,t)\in\mathbb{N}^2} p(t)q^n &\equiv \frac{(q^6;q^6)_\infty^2}{(q;q)_\infty (q^3;q^3)_\infty} + q^7 \frac{(q^{66};q^{66})_\infty^3}{(q^{22};q^{22})_\infty} \\ &\equiv (q^8;q^8)_\infty + q \frac{(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty} + q^7 \frac{(q^{66};q^{66})_\infty^3}{(q^{22};q^{22})_\infty} \pmod 2,\end{aligned} \tag{4.10}$$

which implies

$$\sum_{n=0}^{\infty} \sum_{t+\frac{11s(s+1)}{2}=32n,\ (s,t)\in\mathbb{N}^2} p(t)q^n \equiv (q;q)_\infty \pmod 2 \tag{4.11}$$

and

$$\sum_{n=0}^{\infty} \sum_{t+\frac{11s(s+1)}{2}=8n+4\ (s,t)\in\mathbb{N}^2} p(t)q^n \equiv \frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty} + q^3 \frac{(q^{33};q^{33})_\infty^3}{(q^{11};q^{11})_\infty} \pmod 2. \tag{4.12}$$

By (2.7), (4.9), and (4.11),

(4.13)
$$\sum_{t+\frac{11s(s+1)}{2}=176n+6,\ (s,t)\in\mathbb{N}^2} p(t) \equiv \sum_{t+\frac{11s(s+1)}{2}=32n,\ (s,t)\in\mathbb{N}^2} p(t) \equiv 1 \pmod 2$$

if and only if $n = m(3m-1)/2$ for some integer m. Note that $n = m(3m-1)/2$ for some integer m if and only if $24n+1$ is a square. Hence, we can rewrite (4.13) as

$$\sum_{88k+121 \text{ is a square}} p(176n+6-k) \equiv \sum_{88k+121 \text{ is a square}} p(32n-k) \equiv 1 \pmod 2 \tag{4.14}$$

if and only if $24n+1$ is a square. Based on (4.6) and (4.12),

$$\sum_{t+\frac{11s(s+1)}{2}=8n+4\ (s,t)\in\mathbb{N}^2} p(t) \equiv \sum_{t+\frac{11s(s+1)}{2}=2n\ (s,t)\in\mathbb{N}^2} p(t) \pmod 2. \tag{4.15}$$

By (4.15) and mathematical induction, we see that for $n,\ \alpha\in\mathbb{N}$,

(4.16)
$$\sum_{t+\frac{11s(s+1)}{2}=2^{2\alpha+1}n+\frac{2^{2\alpha+2}-4}{3},\ (s,t)\in\mathbb{N}^2} p(t) \equiv \sum_{t+\frac{11s(s+1)}{2}=2n,\ (s,t)\in\mathbb{N}^2} p(t) \pmod 2.$$

Since $k = 11s(s+1)/2$ for some nonnegative integer s if and only if $88k+121$ is a square, then we can rewrite (4.16) as

$$\sum_{88k+121 \text{ is a square}} p\left(2^{2\alpha+1}n+\frac{2^{2\alpha+2}-4}{3}-k\right) \equiv \sum_{88k+121 \text{ is a square}} p(2n-k) \pmod 2. \tag{4.17}$$

Replacing n by $88n+3$ and $16n$ in (4.17) and using (4.14), we arrive at (1.5) and (1.6). The proofs of Theorems 1.3 and 1.4 are complete.

5. Proofs of Theorems 1.5 and 1.6

By (1.1) and (2.7),

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{t+\frac{13s(3s-1)}{2}=n,\ (s,t)\in\mathbb{Z}\times\mathbb{N}} p(t)q^n &= \sum_{t=0}^{\infty} p(t)q^n \sum_{s=-\infty}^{\infty} q^{\frac{13s(3s-1)}{2}} \\ &\equiv \sum_{t=0}^{\infty} p(t)q^n \sum_{s=-\infty}^{\infty} (-1)^s q^{\frac{13s(3s-1)}{2}} \pmod 2 \\ &= \frac{(q^{13};q^{13})_\infty}{(q;q)_\infty}. \end{aligned} \tag{5.1}$$

Calkin et al. [4] proved that

$$\frac{(q^{13};q^{13})_\infty}{(q;q)_\infty} \equiv (q^4;q^4)_\infty^3 + q(q^2;q^2)_\infty^5 (q^{26};q^{26})_\infty + q^6 (q^{52};q^{52})_\infty^3 + q^7 \frac{(q^{26};q^{26})_\infty^7}{(q^2;q^2)_\infty} \pmod 2. \tag{5.2}$$

In view of (5.1) and (5.2),

$$\sum_{n=0}^{\infty} \sum_{t+\frac{13s(3s-1)}{2}=n,\ (s,t)\in\mathbb{Z}\times\mathbb{N}} p(t)q^n \equiv (q^4;q^4)_\infty^3 + q(q^2;q^2)_\infty^5 (q^{26};q^{26})_\infty + q^6 (q^{52};q^{52})_\infty^3 + q^7 \frac{(q^{26};q^{26})_\infty^7}{(q^2;q^2)_\infty} \pmod 2,$$

which yields

$$\sum_{n=0}^{\infty} \sum_{t+\frac{13s(3s-1)}{2}=2n,\ (s,t)\in\mathbb{Z}\times\mathbb{N}} p(t)q^n \equiv (q^2;q^2)_\infty^3 + q^3 (q^{26};q^{26})_\infty^3 \pmod 2. \tag{5.3}$$

By (2.4) and (5.3),

$$\sum_{n=0}^{\infty} \sum_{t+\frac{13s(3s-1)}{2}=4n,\ (s,t)\in\mathbb{Z}\times\mathbb{N}} p(t)q^n \equiv (q;q)_\infty^3 \equiv \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \pmod 2. \tag{5.4}$$

By (2.1) and (5.4), we see that

$$\sum_{t+\frac{13s(3s-1)}{2}=4n,\ (s,t)\in\mathbb{Z}\times\mathbb{N}} p(t) \equiv 1 \pmod 2 \tag{5.5}$$

if and only if $n = m(m+1)/2$ for some nonnegative integer m. Moreover, since $n = m(m+1)/2$ for some nonnegative integer m if and only if $8n+1$ is a square, then we can rewrite (5.5) as

$$\sum_{312k+169 \text{ is a square}} p(4n-k) \equiv 1 \pmod 2 \tag{5.6}$$

if and only if $8n+1$ is a square. Ahmed and Baruah [1] proved that if $p \geq 3$ is a prime, then

$$(q;q)_\infty^3 = \sum_{\substack{j=0,\\ j\neq \frac{p-1}{2}}}^{p-1} (-1)^j \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{\frac{pn(pn+2k+1)}{2}} + (-1)^{\frac{p-1}{2}} p q^{\frac{p^2-1}{8}} (q^{p^2};q^{p^2})_\infty^3. \tag{5.7}$$

Setting $p = 13$ in (5.7) and replacing q by q^2 in (5.7), then substituting the result identity into (5.3), extracting the terms in which the power of q is congruent to 3 modulo 13, dividing by q^3, and replacing q^{13} by q, we deduce that

(5.8)
$$\sum_{n=0}^{\infty} \sum_{t+\frac{13s(3s-1)}{2}=26n+6,\ (s,t)\in\mathbb{Z}\times\mathbb{N}} p(t)q^n \equiv (q^2;q^2)_\infty^3 + q^3(q^{26};q^{26})_\infty^3 \pmod 2.$$

Combining (5.3) and (5.8) yields

(5.9)
$$\sum_{t+\frac{13s(3s-1)}{2}=26n+6,\ (s,t)\in\mathbb{Z}\times\mathbb{N}} p(t) \equiv \sum_{t+\frac{13s(3s-1)}{2}=2n,\ (s,t)\in\mathbb{Z}\times\mathbb{N}} p(t) \pmod 2.$$

Applying (5.7) and picking out the terms in which the power of q is congruent to $(p^2-1)/4$ modulo p^2 from (5.3), then dividing by $q^{(p^2-1)/4}$ and replacing q^{p^2} by q, we have

$$\sum_{n=0}^{\infty} \sum_{t+\frac{13s(3s-1)}{2}=2p^2n+\frac{p^2-1}{2},\ (s,t)\in\mathbb{Z}\times\mathbb{N}} p(t)q^n \equiv (q^2;q^2)_\infty^3 + q^3(q^{26};q^{26})_\infty^3 \pmod 2. \tag{5.10}$$

In view of (5.3) and (5.10),

(5.11)
$$\sum_{t+\frac{13s(3s-1)}{2}=2p^2n+\frac{p^2-1}{2},\ (s,t)\in\mathbb{Z}\times\mathbb{N}} p(t) = \sum_{t+\frac{13s(3s-1)}{2}=2n,\ (s,t)\in\mathbb{Z}\times\mathbb{N}} p(t) \pmod 2.$$

Since $k = 13s(3s-1)/2$ for some integer s if and only if $312k+169$, then we can rewrite (5.9) and (5.11) as

(5.12)
$$\sum_{312k+169 \text{ is a square}} p(26n+6-k) \equiv \sum_{312k+169 \text{ is a square}} p(2n-k) \pmod 2$$

and

(5.13)
$$\sum_{312k+169 \text{ is a square}} p\left(2p^2n+\frac{p^2-1}{2}-k\right) = \sum_{312k+169 \text{ is a square}} p(2n-k).$$

By (5.12), (5.13) and mathematical induction, we see that for $n,\ \alpha\in\mathbb{N}$,

$$\sum_{312k+169 \text{ is a square}} p\left(2\times 13^\alpha n+\frac{13^\alpha-1}{2}-k\right) \equiv \sum_{312k+169 \text{ is a square}} p(2n-k) \pmod 2 \tag{5.14}$$

and

(5.15)
$$\sum_{312k+169 \text{ is a square}} p\left(2p^{2\alpha}n+\frac{p^{2\alpha}-1}{2}-k\right)=\sum_{312k+169 \text{ is a square}} p(2n-k).$$

Replacing n by $2n$ in (5.14) and (5.15), then utilizing (5.6), we get (1.7) and (1.8). This completes the proofs of Theorems 1.5 and 1.6.

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