Contributions to Discrete Mathematics

Volume 16, Number 1, Pages 128–137
ISSN 1715-0868

EQUIVALENT CLASSES OF DEGREE SEQUENCES FOR TRIANGULATED POLYHEDRA AND THEIR CONVEX REALIZATION

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Abstract. We define an equivalence on the set of all degree sequences of a triangulated polyhedron with a fixed number of vertices and compute them and their cardinal via an algorithm. We also prove that each class is realizable as a convex polyhedron.

1. Introduction

We consider the triangulated compact polyhedra of genus 0 that is homeomorphic to the unit sphere. In a previous study ([3]), we determined all distinct triangulations of a fixed set of points of S^2 as well as their degree sequences. Since several such sequences may actually represent the same triangulation (depending on the way you look at the sphere) we propose grouping the degree sequences into equivalent classes corresponding to the same triangulation. This yields a number of classes that are far smaller than the number of triangulations that represent the different possible polyhedra and proves that each of these classes is realizable as a convex polyhedron.

We first recall some basic definitions (see [1], [2]).

Definition 1.1. *A* triangulation *of a planar set of points V is a subdivision of the plane determined by a maximal set of noncrossing edges whose vertex set is V. Two triangulations T and T' of V are called* positively equivalent *if there exists a one-to-one map $\varphi : V \to V$ that sends the triangles of T to the triangles of T' and preserves their orientations.*

In the above definition, the word *maximal* indicates that any edge that does not belong to triangulation must intersect the interior of at least one of the edges of the triangulation. The compact triangulated polyhedra of genus 0 are homeomorphic to the unit sphere S^2 and thus induce triangulations of it. Each such triangulation has an associated degree sequence, which depends on the way the vertices are arranged.

Received by the editors June 15, 2018, and in revised form November 29, 2020.
2000 *Mathematics Subject Classification.* 51F, 51N, 52B, 52C.
Key words and phrases. Polyhedra, convex polyhedra, degree sequence, triangulation.

Definition 1.2. *Let* $V_n = \{v_0, \dots, v_n\}$ *be a set of* $n+1$ *points of* S^2*, and let* $T = \{t_1, \dots, t_{2n-2}\}$ *be a triangulation of* S^2 *where all* t_i *are triangles whose vertices are in* V_n*. An* order σ *is a permutation of* $\{0, \dots, n\}$ *and the* degree sequence *of the triangulation* T *associated with order* σ *is the* $(n+1)$*-tuple* $C_\sigma = (\deg(v_{\sigma(0)}), \dots, \deg(v_{\sigma(n)}))$.

Note that there is $2n-2$ triangles in a triangulation ([3]). The main advantage to the degree sequence is that it is simpler than the associated triangulation yet characterizes it uniquely, as will see in the next section. Let t be a triangle of a triangulation, through which one looks in order to obtain the associated *Schlegel* diagram (see Figure 1). In order to count positively equivalent degree sequences, our first task is to find a canonical order σ_t which depends only on the way you look at the sphere. This reduces the original problem to that of triangulating the triangle t with $n-2$ points in it.

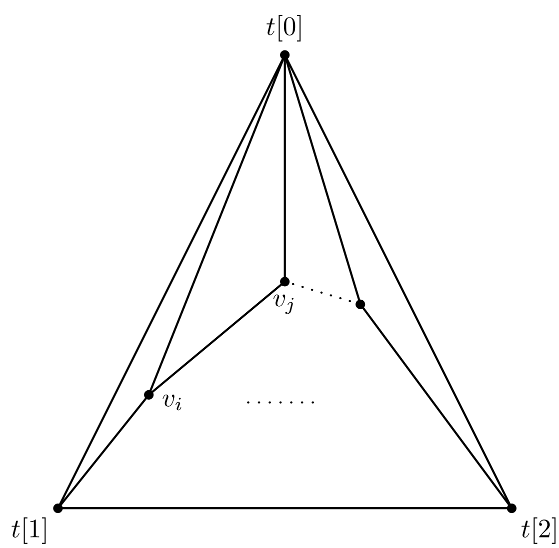

FIGURE 1. A part of the Schlegel diagram of a triangulation seen from the triangle $t = (t[0], t[1], t[2])$.

This is done by means of *edge flips* (see [4],[7]) as dictated by Wagner's theorem. Recall that if a triangulation contains two triangles (a, b, c) and (a, c, d) which form a convex quadrilateral (a, b, c, d), then the *edge flip* consists in replacing the diagonal edge $[a, c]$ with the diagonal $[b, d]$, thereby yielding a new triangulation. In particular, Wagner's theorem asserts any two triangulations can be obtained from one another through a sequence of edge flips.

Since a valid degree sequence of a triangulation T is obtained when T is seen from any one of its oriented triangles (see Section 2), then clearly T has several degree sequences that should thus be, in effect, considered equivalent (see Section 3). We will thus introduce equivalence classes to that end.

Finally, we will show that each such class can be realized as a convex polyhedron. Taken together, these two steps therefore give the list of distinct convex triangulated polyhedra with a fixed number of vertices.

2. Canonical numbering

We fix a triangulation T all of whose triangles have vertices in V_n. Looking through one of the positively oriented triangles $t = (a, b, c)$ of T, we associate it with the order $(0, 1, 2)$. As stated earlier, the triangulation T is then just a triangulation of t with $n-2$ points in it. A natural (canonical) method to number these remaining points consists in looking down at the sphere from above the triangle t and turning in the trigonometric direction around vertex a. If a has degree D, we obtain the ordered list $(0, 1, 2, \ldots, D)$ denoting the labels of points a, b, c and all vertices adjacent to a in T.

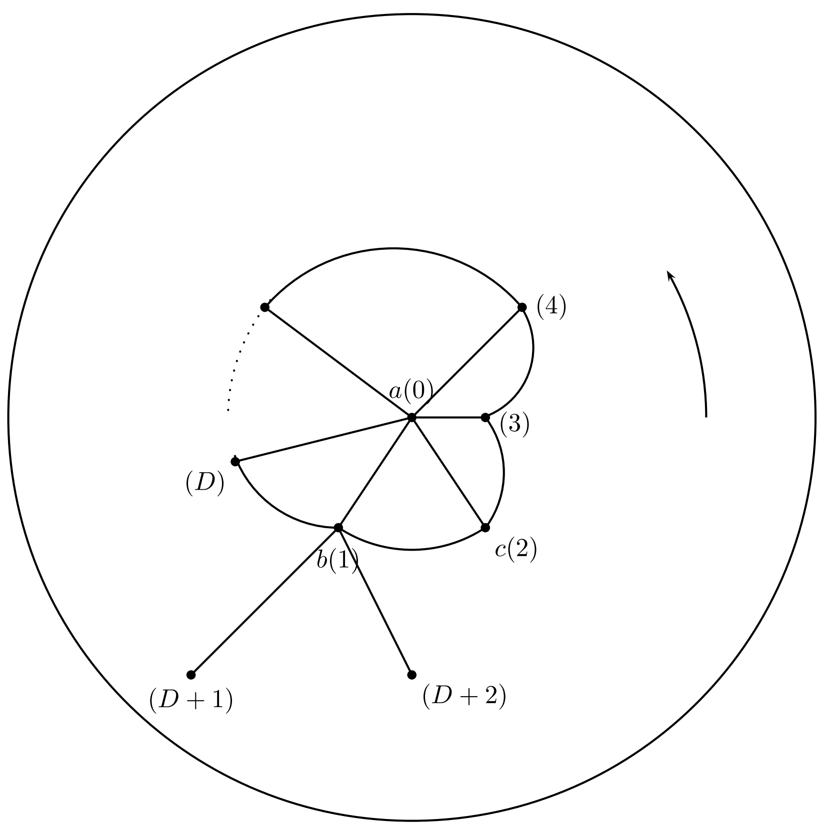

Figure 2. Canonical numbering of the points of the triangulation seen from triangle (a, b, c).

We may now perform this process again, this time centering on vertices b and then c but without considering those vertices that have already been labeled. Having labeled all vertices this way yields what we call the *canonical numbering related to the triangle* t of all vertices. We obtain the corresponding canonical degree sequence C as the list of degrees of the vertices ordered by their canonical numbering (see Figure 2). Figure 3 illustrates the Schlegel diagram for this numbering.

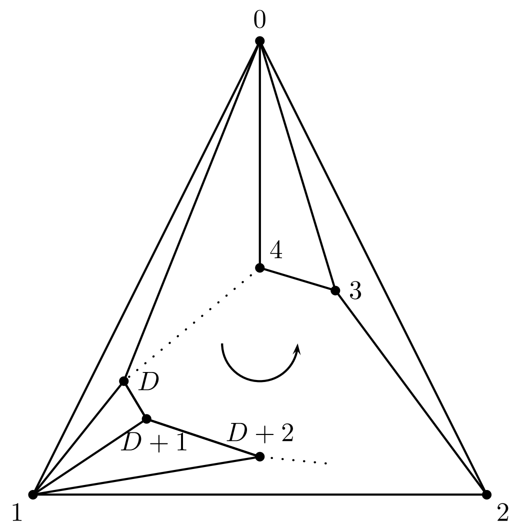

FIGURE 3. Schlegel diagram for the canonical numbering.

Conversely, starting from the Schlegel diagram of a triangulation T, we can number the vertices as follows. Denote by $(0, 1, 2)$ the starting oriented triangle. Then denote by 3 the unique vertex v not in $(0, 1, 2)$ such that $(0, 2, v)$ is a triangle of T, and continue this process until all vertices are exhausted.

Before we turn to computing canonical numberings of triangulations, we show their primary role of the former in distinguishing the latter with the following result:

Proposition 2.1. *Let T and T' be two triangulations whose vertices are all in V_n. Let C (resp. C') be the canonical degree sequence of T (resp. T') corresponding to the canonical numbering related to triangle $t \in T$ (resp. $t' \in T'$). Then $C = C'$ if and only if T and T' are positively equivalent.*

Proof. We prove the proposition by induction on n, the number of vertices in the triangulations T and T'. The result is trivial for $n = 3$ since the triangulation of a triangle is clearly unique. Now let $n \geq 4$ and suppose that the proposition is holds up to rank $n - 1$ included. For the sake of simplicity, we denote by $(0, \dots, n)$ (resp. $(0', \dots, n')$) the vertices of V_n

ordered by the canonical numbering defined by $t \in T$ (resp. $t' \in T'$) . If $\deg(0) = 3$, then the triangulation $\tilde{T}$ of $V_{n-1} = V_n \setminus \{0\}$ obtained by removing all triangles containing 0 in T has the same degree sequence as the triangulation $\tilde{T}'$ of V_{n-1} obtained on removing all triangles containing $0'$ in T'. By the induction hypothesis, this implies that T and T' are positively equivalent triangulations. If $D := \deg(0) \geq 4$, then there exists at least one vertex i (resp. i') in the set $(3, \dots, D-2)$ of vertices adjacent to 0 (resp. $(3', \dots, D'-2)$ adjacent to $0'$) such that $\deg(i) \geq 4$ (resp. $\deg(i') \geq 4$). Let i_0 (resp. j_0) be the smallest vertex label satisfying this condition, then $i_0 = j_0$. Performing an edge flip on the diagonal $[0, i_0]$ (resp. $[0', i'_0]$) of the quadrilateral $(0, i_0+1, i_0, i_0-1)$ (resp. $(0', i'_0+1, i'_0, i'_0-1)$) decreases the degree of vertex 0 (resp. $0'$). Repeating this process until $\deg(0) = 3$, the proposition follows by induction as above. □

Let $T = [[T[0][0], T[0][1], T[0][2]], \dots, [T[2n-3][0], T[2n-3][1], T[2n-3][2]]$ be a triangulation with n vertices of S^2. In order to compute the canonical numbering related to the triangle $t \in T$, we use the following functions:

- `DEGREE(T,x)`: given a triangulation T and vertex x as inputs, returns the degree $D = \deg(x)$ in triangulation T.
- `ADJACENT(T,p,q,D)`: given vertices p and q of a triangulation T and degree $D = \deg(p)$ as inputs, returns the list `adjacent` of neighbours of p in T.
- `ORDER(T,t,k)`: returns the numbering list `order` of the points of the triangulation T seen from the triangle t. Moreover, if $t = (a, b, c)$, we will replace it by $t = (b, c, a)$ when $k = 1$ or by $t = (c, a, b)$ when $k = 2$ (t will be unchanged if $k = 0$).

The algorithm then has the following overall form in Python (skipping all details for the first two functions):

```
def ORDER(T,t,k):
% the value of k indicates which way you look
% at the triangle t %
   if $k==1$:
      t=[t[1],t[2],t[0]]
   if $k==2$:
      t=[t[2],t[0],t[1]]

% initialisations %
   order=t
   p=0
   q=2
   D=DEGREE(T,t[0])
   T'=copy.deepcopy(T)
   T'.remove(t)
```

```
% order determination %
   while len(order)<n+1:
      P=order(p)
      Q=order(q)
      adj=ADJACENT(T',P,Q,D)
      order=[order.append(x) for x in adj if x not in order]
      p+=1
      q=len(order)-1
      T'=[T'[y] for y in range(len(T')) if p not in T'[y]]
      D=DEGREE(T',order[p])
   return order
```

For instance, let $T = [[0,1,2],[0,2,3],[0,3,4],[0,4,1],[1,4,2],[2,4,3]]$ be a triangulation of S^2 with 5 vertices. We choose $t = [2,4,3]$ and $k = 1$. Then t is changed to $t = [2,3,4]$, and the new order is $order = [2,3,4,1,0]$.

3. Equivalent degree sequences and algorithm

In this section, we fix a triangulation T of the sphere S^2, whose vertices are all in the set V_n. The first triangle of T is called the reference triangle and given labels $(0,1,2)$ in the triangulation's Schlegel diagram. Since T has $2(n-1)$ triangles, each of which can be seen from three different points of view, there are $6(n-1)$ positively oriented triangles. The resulting degree sequences will be said to be equivalent if they differ only from such a change of viewpoint:

Definition 3.1. *Two degree sequences are said to be* equivalent *if they correspond to two different points of view of the same triangulation.*

Remark that this definition, pertaining to equivalent degree sequences, is different from Definition (1.1) regarding positively equivalent triangulations.

We now present an algorithm generating all equivalent degree sequences. For each oriented triangle t of a triangulation T, we number the vertices of T (cf. Section 2) then deduce a permutation of the original order. To that end, we define the function `CLASS(T,n)` which returns the list `EquiCombi` of distinct classes of degree sequences of the triangulation T as seen from the triangles of T. We then exploit a function `COMBINATORICS(T,t,k)` to generate all degree sequences `combi` of the triangulation T viewed from the triangle t following a permutation order determined from integer k. This last step is achieved with the previously defined function `ORDER(T,t,k)`.

```
def COMBINATORICS(T,t,k):
% The only well-oriented triangles are (0,1,2),(1,2,0) and
% (2,0,1). The value of k indicates which way you
%  look at the triangle t %

   if (0 not in t) or (1 not in t) or (2 not in t):
      t=[t[0],t[2],t[1]]
```

```
   combi=[ ]
   combi=[combi.append(DEGREE)(T,x)) for x in ORDER(T,t,k)]
   return combi

def CLASS(T,n):
   EquiCombi=[ ]
   for t in T:
      for k in range(3):
         EquiCombi.append(COMBINATORICS(T,t,k))
  return EquiCombi
```

Note, we omit minor details in the above code: in particular identical degree sequences appearing multiple times are deleted from the former list `EquiCombi`. In order to properly generate *all* nonequivalent degree sequence, we must use this algorithm for all triangulations on the set V_n. We recall ([3]) that these can be found by induction, starting from the triangulations of degree 3 (for the vertex 0) up to the maximal degree n by means of edge flips. In the code, we denote by `FLIP(n)` the set of all triangulations of V_n, seen from triangle $(0, 1, 2)$. The main algorithm thus has the following form (in brief):

```
ClassEqui=[ ]
for T in FLIP(n):
   ClassEqui.append(CLASS(T,n))
```

Implementing everything in Python, we obtain the following results (file `ClassEqui`):

n = 3: 1 class
*[[3,3,3,3]]

n = 4: 1 class
*[[3,4,4,4,3],[4,4,3,4,3],[4,3,4,3,4]]

n = 5: 2 classes
*[[3, 5, 5, 4, 4, 3], [5, 5, 3, 4, 4, 3], [5, 3, 5, 3, 4, 4], [4, 3, 5, 4, 5, 3], [3, 5, 4, 5, 3, 4], [5, 4, 3, 5, 3, 4], [4, 5, 3, 5, 4, 3], [5, 3, 4, 4, 3, 5], [3, 4, 5, 5, 4, 3], [4, 4, 5, 3, 5, 3], [4, 5, 4, 5, 3, 3], [5, 4, 4, 3, 5, 3]]
* [[4, 4, 4, 4, 4, 4]]

n = 6: 6 classes
*[[3, 6, 6, 4, 4, 4, 3], [6, 6, 3, 4, 4, 4, 3], [6, 3, 6, 3, 4, 4, 4], [4, 3, 6, 4, 6, 3, 4], [3, 6, 4, 6, 3, 4, 4], [6, 4, 3, 6, 3, 4, 4], [4, 6, 3, 6, 4, 4, 3], [6, 3, 4, 4, 4, 3, 6], [3, 4, 6, 6, 4, 4, 3], [4, 4, 6, 4, 6, 3, 3], [4, 6, 4, 6, 3, 3, 4], [6, 4, 4, 3, 6, 3, 4], [4, 6, 4, 6, 4, 3, 3], [6, 4, 4, 4, 3, 6, 3], [4, 4, 6, 3, 6, 4, 3]]
*[[3, 6, 5, 4, 5, 3, 4], [6, 5, 3, 4, 5, 3, 4], [5, 3, 6, 4, 5, 4,

```
3], [3, 4, 6, 5, 5, 3, 4], [4, 6, 3, 5, 5, 3, 4], [6, 3, 4, 5, 3, 4,
5], [6, 4, 5, 3, 4, 5, 3], [4, 5, 6, 3, 5, 4, 3], [5, 6, 4, 5, 4, 3,
3], [5, 4, 3, 6, 4, 5, 3], [4, 3, 5, 5, 6, 4, 3], [3, 5, 4, 6, 4, 5,
3], [5, 5, 4, 3, 6, 4, 3], [5, 4, 5, 4, 3, 6, 3], [4, 5, 5, 6, 3, 4,
3]]
*[[3, 5, 6, 4, 5, 4, 3], [5, 6, 3, 4, 5, 4, 3], [6, 3, 5, 4, 3, 5,
4], [3, 6, 4, 5, 4, 3, 5], [6, 4, 3, 5, 4, 3, 5], [4, 3, 6, 5, 5, 4,
3], [5, 4, 6, 3, 4, 5, 3], [4, 6, 5, 5, 3, 4, 3], [6, 5, 4, 3, 5, 4,
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3], [5, 3, 4, 5, 4, 6, 3], [3, 4, 5, 6, 5, 4, 3], [4, 5, 3, 6, 5, 4,
3]]
*[[3, 6, 5, 5, 3, 5, 3], [6, 5, 3, 5, 3, 5, 3], [5, 3, 6, 3, 5, 5,
3], [5, 5, 3, 6, 3, 5, 3], [5, 3, 5, 5, 3, 6, 3], [3, 5, 5, 6, 3, 5,
3], [6, 3, 5, 3, 5, 3, 5], [3, 5, 6, 5, 5, 3, 3], [5, 6, 3, 5, 5, 3,
3], [5, 5, 5, 3, 6, 3, 3]]
*[[3, 5, 5, 5, 4, 4, 4], [5, 5, 3, 5, 4, 4, 4], [5, 3, 5, 4, 4, 5,
4], [4, 4, 5, 5, 4, 5, 3], [4, 5, 4, 4, 5, 3, 5], [5, 4, 4, 5, 3, 5,
4], [5, 4, 5, 3, 5, 4, 4], [4, 5, 5, 4, 4, 5, 3], [5, 5, 4, 4, 5, 3,
4], [4, 4, 4, 5, 5, 5, 3]]
* [[4, 5, 4, 5, 4, 4, 4], [5, 4, 4, 4, 4, 4, 5], [4, 4, 5, 4, 5, 4,
4]]
```

n = 7: 17 equivalent classes.

n = 8: 73 equivalent classes, etc.

For interested readers, a Python program is available at `www.pythonanywhere.com/user/honvault/shares/237465cf2bbe423eb659b2335b07b556/`. The sequence for the number of equivalent classes of degree sequences is therefore: 1, 1, 2, 6, 17, 73, ..., which corresponds to A253882 on the OEIS [5], because it is also the number of 3-connected planar triangulations on n vertices. Remark that degree sequences of different classes may differ only by a permutation, e.g., the second and third classes in the case $n = 6$. These come from two polyhedra that are symmetric to each other about a plane.

The number of distinct triangulations of V_n seen from $(0, 1, 2)$, is known to be ([6]):

$$\frac{2 \cdot (4n-7)!}{(n-1)!(3n-4)!}.$$

Fortunately, the number of equivalent classes appears to be much smaller: for instance, for $n = 8$ there are 2530 triangulations but we get only 73 equivalent classes. Nevertheless, the problem remains that the program must perform an exponentially increasing number of operations, so the results must be kept in a database for further study.

Now, if we restrict our attention to convex polyhedra, it is easy to see that there are only 2 triangulated polyhedra with 6 vertices. It already takes more effort to see that there are six triangulated polyhedra with 7 vertices and seventeen triangulated polyhedra with 8 vertices. Seeing the similarity with the number of equivalent classes of degree sequences, we may conjecture that, if $L(n)$ refers to the length of the list `ClassEqui`, there are at most $L(n)$ convex polyhedra with $n+1$ vertices. We prove this below.

4. Convex realization

As discussed above, the problem of the existence of a convex triangulated polyhedron of $n+1$ vertices ($n \geq 3$) with a given admissible degree sequence is nontrivial. From now on, we refer to *convex polyhedra* for convex triangulated polyhedra which are in the same equivalent class `ClassEqui` of degree sequence and without coplanar faces. We recall that $L(n)$ is the length of the list `ClassEqui`.

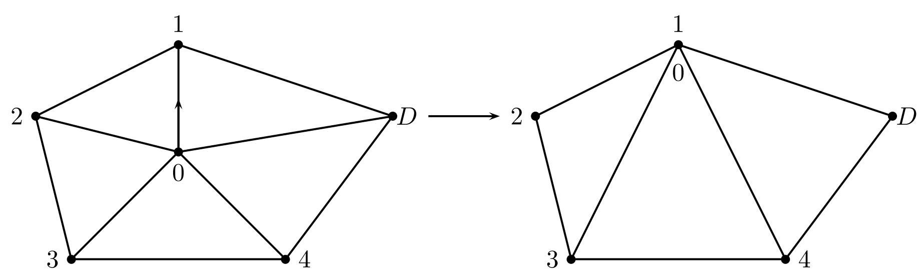

Figure 4. From a triangulation of V_{n+1} to one of V_n.

Theorem 4.1. *For each integer $n \geq 3$, there are exactly $L(n)$ convex triangulated polyhedra of genus 0.*

Proof. By induction on n, the result being straightforward for $n = 3$. Assume that the result is true for an integer $n \geq 3$, and let T be an admissible triangulation of V_{n+1}. We denote by $D := \deg(0)$ the degree of vertex 0, and by $1, \ldots, D$ the vertices adjacent to 0 in T.

We can rearrange vertices $1, \ldots, D$ on the sphere S^2 in order to make the hard spherical polygon $[1, \ldots, D]$ convex. Then, merging vertices 0 and 1 (see Figure 4) yields a triangulation of V_n. By the induction hypothesis, there exists a convex realization P' of this triangulation (cf. Figure 4). Recall now that a polyhedron is convex (in our sense) if and only if its dihedral angles are convex. So, if we now duplicate vertex 1 creating a new vertex 0, and move the latter back down, we obtain a polyhedron P with the initial degree sequence. Moreover, its dihedral angles will be also convex if 0 is sufficiently close to 1 and if the line $(0, 1)$ is well-oriented. □

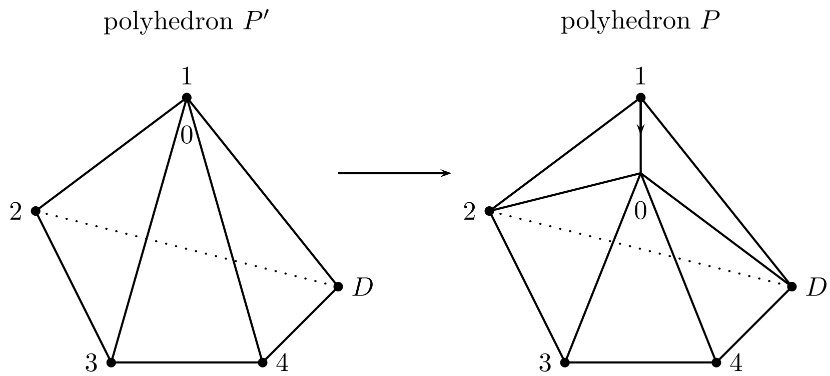

FIGURE 5. From a convex realization in V_n to one in V_{n+1}.

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