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A WIDE CLASS OF COMBINATORIAL MATRICES RELATED WITH RECIPROCAL PASCAL AND SUPER CATALAN MATRICES

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Abstract. In this paper, we present a number of combinatorial matrices that are generalizations or variants of the super Catalan matrix and the reciprocal Pascal matrix. We present explicit formulæ for LU-decompositions of all the matrices and their inverses. Alternative derivations using hypergeometric functions are also given.

1. Introduction

Richardson [12] defined the *reciprocal Pascal matrix* M by

$$M_{i,j} = \binom{i+j}{i}^{-1} = \frac{i!j!}{(i+j)!};$$

the indices start for convenience with $(0,0)$, and the matrix is either infinite or has N rows and columns, depending on the context. The author showed that the inverse of the reciprocal Pascal matrix has integer elements which was conjectured already in [5]. For this purpose, he derived the factorization $S = GMG$, where the diagonal matrix G has entries $G_{i,i} = \binom{2i}{i}$, and S is the super Catalan matrix [2, 4] with entries

$$S_{i,j} = \frac{(2i)!\,(2j)!}{i!j!\,(i+j)!}.$$

Prodinger [11] gave an alternative decomposition of M, provided by the LU-decomposition as well as explicit formulæ for the LU-decomposition of its inverse and some related matrices were obtained. For all results, q-analogues are also presented.

Prodinger [10] also studied two matrices A and $\mathcal{A}$ whose entries consist of the super Catalan numbers and their reciprocal analogues defined by

$$A_{ij} = \frac{(2i)!\,(2j)!}{i!j!\,(i+j)!} \quad \text{and} \quad \mathcal{A}_{ij} = \frac{i!j!\,(i+j)!}{(2i)!\,(2j)!}.$$

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He gave explicit formulæ for the LU-decompositions of it and its inverse, and some related matrices were obtained as well as q-analogues related to all these results.

Kılıç et al. [7] defined a variant of the reciprocal super Catalan matrix C with two additional parameters whose entries is given by

$$C_{ij} = \binom{2i+r}{i}^{-1} \binom{2j+s}{j}^{-1} \binom{i+j}{i}^{-1}.$$

Explicit formulæ for its LU-decomposition, LU-decomposition of its inverse and the Cholesky decomposition are obtained. For all results, q-analogues are also presented.

Kılıç et al. [9] also defined two variants of the reciprocal super Catalan matrix with two additional parameters whose entries are given by

$$W_{i,j} = \binom{2i+m}{i} \binom{2j+t}{j}^{-1} \binom{i+j}{i}$$

and

$$H_{i,j} = \binom{2i+m}{i}^{-1} \binom{2j+t}{j} \binom{i+j}{i}^{-1},$$

where m and t are nonnegative integers and all indices of these matrices start at $(0,0)$. Explicit expressions were also presented for LU-decompositions of all the matrices and their inverses. For all results, q-analogues are also presented.

Kılıç and Arıkan [8] defined two generalizations of the reciprocal super Catalan matrix with two additional parameters defined by

$$Y_{k,j} = \binom{k+j}{k} \binom{2k+r}{k}^{-1} \binom{2j+s}{j}^{-1}$$

and

$$T_{k,j} = \binom{2k+r}{k} \binom{2j+s}{j} \binom{k+j}{k}^{-1}$$

for $0 \leq k, j < n$, respectively. Explicit formulæ were given for the LU-decomposition and their inverses, as well as the Cholesky decomposition. For all results, q-analogues are also presented.

Quite recently, for integers $1 \leq q < p$, Richardson [13] defines (q,p)-Patalan numbers

$$b_n := -p^{2n+1} \binom{n - q/p}{n+1}.$$

Here, the general definition of a binomial coefficient, $\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$ is employed. Now, for $q = 1$, $p = 2$ this leads to

$$b_n = -2^{2n+1} \binom{n - 1/2}{n+1} = 2^n \frac{(2n-1)(2n-3)\ldots 3 \cdot 1}{(n+1)!} = \frac{(2n)!}{n!(n+1)!} = C_n,$$

which is a Catalan number.

Richardson [13] has generalized these as well via

$$Q(i,j) := (-1)^j p^{2(i+j)} \binom{i-q/p}{i+j},$$

again for integers $1 \le q < p$. The author gave the factorization $\check{H} = G_{p,q}^{-1} B G_{p,p-q}^{-1}$ for the reciprocal Patalan matrix $\check{H}$, where

$$\check{H}_{i,j} = \left[\frac{1}{Q(i,j)}\right],$$

B is the Pascal matrix defined by $B_{i,j} = \binom{i+j}{i}$ and $G_{p,q}$ is the diagonal matrix with $(G_{p,q})_{i,j} = Q(i,0)$.

Instead of working with the numbers p and q, Kılıç and Prodinger set $x := q/p$, for general x, provided that $0 < x < 1$. They defined the Patalan matrix by

$$\check{M}_{i,j} = -\frac{1}{p^{2(i+j+r)+1}} \binom{i+j+r-x}{i+j+r+1},$$

the reciprocal Patalan matrix by

$$\check{M}_{i,j} = -p^{2(i+j+r)+1} \binom{i+j+r-x}{i+j+r+1}^{-1},$$

the super Patalan matrix by

$$\check{M}_{i,j} = (-1)^{j+s} p^{2(i+r+j+s)} \binom{i+r-x}{i+r+j+s}$$

and the reciprocal super Patalan matrix by

$$\check{M}_{i,j} = (-1)^{j+s} p^{2(i+r+j+s)} \binom{i+r-x}{i+r+j+s}^{-1},$$

with the notion of falling factorials: $x^{\underline{n}} := x(x-1)\dots(x-n+1) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}$. For any matrix $\check{M}$ and its inverse, they gave explicit expressions for the LU-decompositions.

We summarized all combinatorial matrices related to the reciprocal Pascal and super Catalan matrices from the current literature in the above.

The purpose of the present paper is to present a wide class of combinatorial matrices consisting of eight matrices. For any such matrix F, we are interested in *factorizations*, based on the LU-decomposition by the notation $F = LU$ and $F^{-1} = AB$, and present explicit expressions for L, L^{-1}, U, U^{-1}, A, A^{-1}, B, B^{-1}. All our matrices are indexed starting at $(0,0)$ and have N rows (resp. columns), where N might also be infinity, depending on the context. The nonnegative integers t and u are shift parameters.

Except for the result about the Patalan matrices, all earlier results are covered by the present results and are special cases of them.

In the next eight sections, we shall list our results related to each matrix F. We give some proofs for the results of Sections 4, 5, 6, and 8 and leave other (similar) ones to the imagination of the reader.

We provide sample proofs using Zeilberger's algorithm and also using the hypergeometric machinery, as described in [3].

To keep this paper within reasonable length, we refrain from computing q-analogues.

2. Results related to the first matrix

The matrix F has now entries

$$F_{n,k} = (n+k+t)!a_k b_n.$$

Here and in the following, a_k and b_k are arbitrary nonzero numbers.

We list here the formulæ that were found:

$$U_{i,j} = \frac{j!(j+t)!}{(j-i)!}a_j b_i,$$

$$U^{-1}_{i,j} = \frac{(-1)^{i-j}(j+t)!}{i!(i+t)!(j+t)!(j-i)!}\frac{1}{b_j a_i},$$

$$L_{i,j} = \frac{i!(i+t)!}{j!(j+t)!(i-j)!}\frac{b_i}{b_j},$$

$$L^{-1}_{i,j} = \frac{(-1)^{i-j}(i+t)!i!}{(i-j)!j!(j+t)!}\frac{b_i}{b_j},$$

$$B_{i,j} = \frac{(-1)^{i-j}(N+i+t)!}{(N-1-j)!(i+j+t+1)!(j-i)!(2i+t)!}\frac{1}{b_j a_i},$$

$$B^{-1}_{i,j} = \frac{(2j+t+1)!(N-1-i)!(i+j+t)!}{(N+j+t)!(j-i)!}b_i a_j,$$

$$A_{i,j} = \frac{(-1)^{i-j}(N-1-j)!(2j+t+1)!}{(i+j+t+1)!(i-j)!(N-1-i)!}\frac{a_j}{a_i},$$

$$A^{-1}_{i,j} = \frac{(N-1-j)!(i+j+t)!}{(2i+t)!(N-1-i)!(i-j)!}\frac{a_j}{a_i}.$$

Now we present the results related with reciprocal analogue of the first matrix in the next section.

3. Results related to the second matrix

The matrix F has now entries

$$F_{n,k} = \frac{1}{(n+k+t)!}a_k b_n.$$

We list here the formulæ that were found:

$$U_{i,j} = \frac{(-1)^i(i+t-1)!j!}{(2i+t-1)!(j-i)!(i+j+t)!}a_j b_i,$$

$$U^{-1}_{i,j} = \frac{(-1)^i(i+j+t-1)!(2j+t)!}{i!(j-i)!(j+t-1)!}\frac{1}{b_j a_i},$$

$$L_{i,j} = \frac{i!(2j+t)!}{(i-j)!j!(i+j+t)!}\frac{b_i}{b_j},$$

$$L^{-1}_{i,j} = \frac{(-1)^{i-j} i!(i+j+t-1)!}{(2i+t-1)!(i-j)!j!} \frac{b_i}{b_j},$$

$$B_{i,j} = \frac{(-1)^{N-1-j}(N+j+t-1)!}{(N-1-j)!(j-i)!} \frac{1}{b_j a_i},$$

$$B^{-1}_{i,j} = \frac{(-1)^{N-1-j}(N-1-i)!}{(N+i+t-1)!(j-i)!} b_i a_j,$$

$$A_{i,j} = \frac{(-1)^{i-j}(N+i+t-1)!(N-1-j)!}{(N+j+t-1)!(N-1-i)!(i-j)!} \frac{a_j}{a_i},$$

$$A^{-1}_{i,j} = \frac{(N+i+t-1)!(N-1-j)!}{(N+j+t-1)!(N-1-i)!(i-j)!} \frac{a_j}{a_i}.$$

4. Results related to the third matrix

The matrix F has now entries

$$F_{n,k} = \frac{(n+k+t)!}{(2n+2k+2t)!} a_k b_n.$$

We list here the formulæ that were found:

$$U_{i,j} = \frac{(-1)^i j!(i+j+t)!(2i-2+2t)!(2i-1+t)!4^i}{(i-1+t)!(4i-2+2t)!(2i+2j+2t)!(j-i)!} a_j b_i,$$

$$U^{-1}_{i,j} = \frac{(-1)^i (4j+2t)!(2i+2j-2+2t)!(j-1+t)!}{i!(j-i)!(i+j-1+t)!(2j-2+2t)!(2j+t)!4^j} \frac{1}{a_i b_j},$$

$$L_{i,j} = \frac{i!(i+j+t)!(4j+2t)!}{(2i+2j+2t)!(i-j)!(2j+t)!j!} \frac{b_i}{b_j},$$

$$L^{-1}_{i,j} = \frac{(-1)^{i-j} i!(2i-1+t)!(2i+2j-2+2t)!}{(4i-2+2t)!(i+j-1+t)!(i-j)!j!} \frac{b_i}{b_j},$$

$$B_{i,j} = \frac{(-1)^{N-1-j}(2j-2+2N+2t)!}{(j-i)!(N-1-j)!(N-1+j+t)!4^{N-1-i}} \frac{1}{a_i b_j},$$

$$B^{-1}_{i,j} = \frac{(-1)^{N-1-j}(N-1-i)!(N-1+i+t)!4^{N-1-j}}{(2i-2+2N+2t)!(j-i)!} a_j b_i,$$

$$A_{i,j} = \frac{(-1)^{i-j}(2i-2+2N+2t)!(N-1-j)!(j-1+N+t)!}{(N-1-i)!(N-1+i+t)!(i-j)!(2j-2+2N+2t)!} \frac{a_j}{a_i},$$

$$A^{-1}_{i,j} = \frac{(2i-2+2N+2t)!(N-1-j)!(N-1+j+t)!}{(N-1-i)!(N-1+i+t)!(i-j)!(2j-2+2N+2t)!} \frac{a_j}{a_i}.$$

Now we present the results related with reciprocal analogue of the third matrix in the next section.

5. Results related to the fourth matrix

The matrix F has now entries

$$F_{n,k} = \frac{(2n+2k+2t)!}{(n+k+t)!} a_k b_n.$$

We list here the formulæ that were found:

$$U_{i,j} = \frac{j!(2j+2t)!4^i}{(j+t)!(j-i)!} a_j b_i,$$

$$U^{-1}_{i,j} = \frac{(-1)^{i-j}(i+t)!}{i!(2i+2t)!(j-i)!4^j} \frac{1}{a_i b_j},$$

$$L_{i,j} = \frac{i!(2i+2t)!(j+t)!}{(i+t)!(i-j)!(2j+2t)!j!} \frac{b_i}{b_j},$$

$$L^{-1}_{i,j} = \frac{(-1)^{i-j} i!(j+t)!(2i+2t)!}{(i+t)!(i-j)!(2j+2t)!j!} \frac{b_i}{b_j},$$

$$B_{i,j} = \frac{(-1)^{i-j}(i+j+1+t)!}{(j-i)!(2i+2j+2+2t)!(N-1-j)!4^{N-1-i}} \times \frac{(2i+2N+2t)!(2i+t)!}{(N+i+t)!(4i+2t)!} \frac{1}{a_i b_j}.$$

$$B^{-1}_{i,j} = \frac{(4j+2+2t)!(2i+2j+2t)!(N-1-i)!(N+j+t)!4^{N-1-j}}{(i+j+t)!(j-i)!(2j+1+t)!(2j+2N+2t)!} a_j b_i,$$

$$A_{i,j} = \frac{(-1)^{i-j}(i+j+t)!(4j+1+2t)!(N-j-1)!}{(2i+2j+1+2t)!(N-i-1)!(i-j)!(2j+t)!} \frac{a_j}{a_i},$$

$$A^{-1}_{i,j} = \frac{(2i+t)!(2i+2j+2t)!(N-j-1)!}{(4i+2t)!(N-i-1)!(i-j)!(i+j+t)!} \frac{a_j}{a_i}.$$

6. Results related to the fifth matrix

The matrix F has now entries

$$F_{n,k} = \frac{(2n+2k+2t)!}{(n+k+t)!(n+k+t+u)!} a_k b_n.$$

We list here the formulæ that were found:

$$U_{i,j} = \frac{(2i+2u)!(i-1+t+u)!(2j+2t)!j!u!}{(i+u)!(2i-1+t+u)!(i+j+t+u)!(j-i)!(j+t)!(2u)!} a_j b_i,$$

$$U^{-1}_{i,j} = \frac{(-1)^{i-j}(i+j-1+t+u)!(2j+t+u)!(j+u)!(2j+u)!(2u)!}{(j-i)!(2j+u)!(j-1+t+u)!(2j+2u)!j!u!} \times \frac{(i+t)!}{i!(2i+2t)!} \frac{1}{a_i b_j},$$

$$L_{i,j} = \frac{i!(2i+2t)!(j+t)!(2j+t+u)!}{(i+t)!(i+j+t+u)!(i-j)!(2j+2t)!j!} \frac{b_i}{b_j},$$

$$L_{i,j}^{-1} = \frac{(-1)^{i-j} i!(i+j-1+t+u)!(2i+2t)!(j+t)!}{(i+t)!(2i-1+t+u)!(i-j)!(2j+2t)!j!} \frac{b_i}{b_j},$$

$$B_{i,j} = \frac{(-1)^{i-j}(i+j+1+t)!(2i+t)!(2i+2N+2t)!(N-1-i+u)!}{(N+i+t)!(2N-2-2i+2u)!(4i+2t)!} \times \frac{(j+N-1+t+u)!(2u)!}{(2i+2j+2+2t)!(j-i)!(N-1-j)!u!} \frac{1}{a_i b_j},$$

$$B_{i,j}^{-1} = \frac{(N-1-i)!(2i+2j+2t)!(2N-2-2j+2u)!}{(N+i-1+t+u)!(i+j+t)!(2j+2N+2t)!(N-1-j+u)!} \times \frac{(N+j+t)!(4j+2+2t)!u!}{(j-i)!(2j+1+t)!(2u)!} a_j b_i,$$

$$A_{i,j} = \frac{(-1)^{i-j}(i+j+1+t)!(4j+2+2t)!(N-1-j)!}{(2i+2j+2+2t)!(i-j)!(N-1+j+t+u)!(2j+1+t)!} \times \frac{(N-1+i+t+u)!}{(N-1-i)!} \frac{a_j}{a_i},$$

$$A_{i,j}^{-1} = \frac{(2i+t)!(N-1+i+t+u)!(2i+2j+2t)!(N-1-j)!}{(N-1-i)!(4i+2t)!(i-j)!(i+j+t)!(N-1+j+t+u)!} \frac{a_j}{a_i}.$$

Now we present the results related with reciprocal analogue of the fifth matrix in the next section.

7. Results related to the sixth matrix

The matrix F has now entries

$$F_{n,k} = \frac{(n+k+t)!(n+k+t+u)!}{(2n+2k+2t)!} a_k b_n.$$

We list here the formulæ that were found:

$$U_{i,j} = \frac{2^{2i+3-2u}(2i-1+t)!(2i-2+2t)!(i+j+t)!j!(j+t+u)!}{(i-1+t)!(4i-2+2t)!(2i+2j+2t)!(j-i)!} \times \frac{\sqrt{\pi}(-1)^i(2u+1)!}{\Gamma(\frac{3}{2}-i+u)u!} a_j b_i,$$

$$U_{i,j}^{-1} = \frac{(-1)^i(4j+2t)!(2i+2j-2+2t)!(j-1+t)!}{(i+t+u)!i!(i+j-1+t)!(2j+t)!(j-i)!(2j-2+2t)!} \times \frac{\Gamma(\frac{3}{2}-j+u)u!}{(2u+1)!2^{2j-2u+3}\sqrt{\pi}} \frac{1}{a_i b_j},$$

$$L_{i,j} = \frac{(i+t+u)!i!(i+j+t)!(4j+2t)!}{(2i+2j+2t)!(i-j)!(j+t+u)!j!(2j+t)!} \frac{b_i}{b_j},$$

$$L_{i,j}^{-1} = \frac{(-1)^{i-j} i!(i+t+u)!(2i-1+t)!(2i+2j-2+2t)!}{(4i-2+2t)!(i-j)!(i+j-1+t)!j!(j+t+u)!} \frac{b_i}{b_j},$$

$$B_{i,j} = \frac{(-1)^j(2j-2+2N+2t)!}{(j-1+N+t)!(N-1-j)!(j-i)!(i+j+1+t+u)!} \\ \times \frac{(N-i-3)!(i+N+t+u)!}{(2i+t+u)!(2N-2i-6)!} \frac{u!\Gamma(i-N+\frac{5}{2}+u)}{2^{2N-2i-2u+1}\sqrt{\pi}(2u+1)!} \frac{1}{a_i b_j},$$

$$B^{-1}_{i,j} = \frac{(N-1-i)!(N-1+i+t)!(2j+1+t+u)!(i+j+t+u)!}{(2i-2+2N+2t)!(j-i)!(N+j+t+u)!} \\ \times \frac{2^{2N-2j-2u+1}\sqrt{\pi}(2u+1)!}{u!\Gamma(j-N+\frac{5}{2}+u)} a_j b_i,$$

$$A_{i,j} = \frac{(-1)^{i-j}(2j+1+t+u)!(N-1-j)!(N-1+j+t)!}{(i-j)!(i+j+1+t+u)!(2j-2+2N+2t)!} \\ \times \frac{(2i-2+2N+2t)!}{(N-1-i)!(N-1+i+t)!} \frac{a_j}{a_i},$$

$$A^{-1}_{i,j} = \frac{(i+j+t+u)!(N-1-j)!(N-1+j+t)!}{(N-1+i+t)!(i-j)!(2j-2+2N+2t)!} \\ \times \frac{(2i-2+2N+2t)!}{(2i+t+u)!(N-1-i)!} \frac{a_j}{a_i}.$$

In the next two sections, we will present two special cases of Sections 6 and 7 for the reader's convenience.

8. Results related to the seventh matrix

The matrix F has now entries

$$F_{n,k} = \frac{(2n+2k+2t)!}{(n+k+t)!(n+k+1+t)!} a_k b_n.$$

We list here the formulæ that were found:

$$U_{i,j} = \frac{(2i+1)!(i+t)!(2j+2t)!j!}{i!(2i+t)!(i+j+1+t)!(j-i)!(j+t)!} a_j b_i,$$

$$U^{-1}_{i,j} = \frac{(-1)^{i-j}(i+t)!(i+j+t)!(2j+1+t)!j!}{i!(2i+2t)!(j-i)!(2j+1)!(j+t)!} \frac{1}{a_i b_j},$$

$$L_{i,j} = \frac{i!(2i+2t)!(j+t)!(2j+1+t)!}{(i+t)!(i+j+1+t)!(i-j)!(2j+2t)!j!} \frac{b_i}{b_j},$$

$$L^{-1}_{i,j} = \frac{(-1)^{i-j}i!(i+j+t)!(2i+2t)!(j+t)!}{(i+t)!(2i+t)!(i-j)!(2j+2t)!j!} \frac{b_i}{b_j},$$

$$B_{i,j} = \frac{(-1)^{i-j}(i+j+1+t)!(j+N+t)!}{(2i+2j+2+2t)!(j-i)!(N-1-j)!} \\ \times \frac{(2i+t)!(2i+2N+2t)!(N-1-i)!}{(N+i+t)!(2N-1-2i)!(4i+2t)!} \frac{1}{a_i b_j},$$

$$B^{-1}_{i,j} = \frac{(2i+2j+2t)!(2N-1-2j)!(N+j+t)!(4j+2+2t)!}{(i+j+t)!(2j+2N+2t)!(N-1-j)!(j-i)!(2j+1+t)!} \times \frac{(N-1-i)!}{(N+i+t)!}a_jb_i.$$

$$A_{i,j} = \frac{(-1)^{i-j}(N+i+t)!(i+j+1+t)!(4j+2+2t)!(N-1-j)!}{(N-1-i)!(2i+2j+2+2t)!(i-j)!(N+j+t)!(2j+1+t)!}\frac{a_j}{a_i},$$

$$A^{-1}_{i,j} = \frac{(2i+t)!(N+i+t)!(2i+2j+2t)!(N-1-j)!}{(N-1-i)!(4i+2t)!(i-j)!(i+j+t)!(N+j+t)!}\frac{a_j}{a_i}.$$

9. Results related to the eighth matrix

The matrix F has now entries

$$F_{n,k} = \frac{(n+k+t)!(n+k+1+t)!}{(2n+2k+2t)!}a_kb_n.$$

We list here the formulæ that were found:

$$U_{i,j} = \frac{12(2i-4)!(2i-1+t)!(2i-2+2t)!(i+j+t)!j!(j+1+t)!}{(i-2)!(i-1+t)!(4i-2+2t)!(2i+2j+2t)!(j-i)!}a_jb_i,$$

$$U^{-1}_{i,j} = \frac{(-1)^{i-j}(4j+2t)!(j-2)!(2j-2+2t)!(j-1+t)!}{12(i+1+t)!i!(2j+t)!(2j-4)!(j-i)!} \times \frac{(2i+2j-2+2t)!}{(i+j-1+t)!}\frac{1}{a_ib_j},$$

$$L_{i,j} = \frac{(i+1+t)!i!(i+j+t)!(4j+2t)!}{(2i+2j+2t)!(i-j)!(j+1+t)!j!(2j+t)!}\frac{b_i}{b_j},$$

$$L^{-1}_{i,j} = \frac{(-1)^{i-j}i!(i+1+t)!(2i-1+t)!(2i+2j-2+2t)!}{(4i-2+2t)!(i-j)!(i+j-1+t)!j!(j+1+t)!}\frac{b_i}{b_j},$$

$$B_{i,j} = \frac{(-1)^{i-j}(2j-2+2N+2t)!}{12(j-1+N+t)!(N-1-j)!(j-i)!(i+j+2+t)!} \times \frac{(N-i-3)!(i+1+N+t)!}{(2i+1+t)!(2N-2i-6)!}\frac{1}{a_ib_j},$$

$$B^{-1}_{i,j} = \frac{12(2N-2j-6)!(2j+2+t)!(i+j+1+t)!}{(j-i)!(N+1+j+t)!(N-j-3)!} \times \frac{(N-1-i)!(N-1+i+t)!}{(2i-2+2N+2t)!}a_jb_i,$$

$$A_{i,j} = \frac{(-1)^{i-j}(2j+2+t)!(N-1-j)!(N-1+j+t)!}{(i-j)!(i+j+2+t)!(2j-2+2N+2t)!} \times \frac{(2i-2+2N+2t)!}{(N-1-i)!(N-1+i+t)!}\frac{a_j}{a_i},$$

$$A^{-1}_{i,j} = \frac{(i+j+1+t)!(N-1-j)!(N-1+j+t)!}{(i-j)!(2j-2+2N+2t)!} \times \frac{(2i-2+2N+2t)!}{(2i+1+t)!(N-1-i)!(N-1+i+t)!}\frac{a_j}{a_i}.$$

10. Proofs for the Results related to the third matrix

For L and L^{-1},

$$\begin{aligned}
&\sum_{j\le d\le k} L_{kd}L^{-1}_{dj} \\
&= \sum_{j\le d\le k} \frac{k!(k+d+t)!(4d+2t)!}{(2k+2d+2t)!(k-d)!(2d+t)!d!}\frac{b_k}{b_d} \\
&\times \frac{(-1)^{d-j}d!(2d-1+t)!(2d+2j-2+2t)!}{(4d-2+2t)!(d+j-1+t)!(d-j)!j!}\frac{b_d}{b_j} \\
&= 2\,(-1)^j\,\frac{k!}{j!}\frac{b_k}{b_j} \\
&\times \sum_{j\le d\le k} \frac{(-1)^d(k+d+t)!(2d+2j-2+2t)!(4d+2t-1)}{(2k+2d+2t)!(k-d)!(d+j-1+t)!(d-j)!}.
\end{aligned}$$

The Zeilberger algorithm computes the sum on the RHS of the last equation as 0 when $k \neq j$. If $k = j$, it is obvious that $L_{jj}L^{-1}_{jj} = 1$. Thus

$$\sum_{j\le d\le k} L_{kd}L^{-1}_{dj} = \delta_{k,j},$$

as claimed.

For U and U^{-1},

$$\begin{aligned}
&\sum_{k\le d\le j} U_{kd}U^{-1}_{dj}\\
&=\sum_{k\le d\le j}\frac{(-1)^k d!(k+d+t)!(2k-2+2t)!(2k-1+t)!4^k}{(k-1+t)!(4k-2+2t)!(2k+2d+2t)!(d-k)!}a_db_k\\
&\times\frac{(-1)^d(4j+2t)!(2d+2j-2+2t)!(j-1+t)!}{d!(j-d)!(d+j-1+t)!(2j-2+2t)!(2j+t)!4^j}\frac{1}{a_db_j}\\
&=(-1)^k\frac{4^{k-j}(2k-2+2t)!(2k-1+t)!(4j+2t)!(j-1+t)!}{(k-1+t)!(4k-2+2t)!(2j-2+2t)!(2j+t)!}\frac{b_k}{b_j}\\
&\times\sum_{k\le d\le j}\frac{(-1)^d(k+d+t)!(2d+2j-2+2t)!}{(2k+2d+2t)!(d-k)!(j-d)!(d+j-1+t)!}.
\end{aligned}$$

The Zeilberger algorithm computes the sum on the RHS of the last equation as 0 when $k\neq j$. If $k=j$, it is obvious that $U_{kk}U^{-1}_{kk}=1$. Thus

$$\sum_{j\le d\le k} U_{kd}U^{-1}_{dj}=\delta_{k,j},$$

as claimed.

For LU-decomposition, we have to prove that

$$\sum_{0\le d\le \min\{i,j\}} L_{id}U_{dj}=F_{ij}.$$

Consider

$$\begin{aligned}
\sum_{0\le d\le \min\{i,j\}} L_{id}U_{dj}=a_jb_i\sum_{0\le d\le \min\{i,j\}}&\frac{i!(i+d+t)!(4d+2t)!}{(2i+2d+2t)!(i-d)!(2d+t)!d!}\\
\times&\frac{(-1)^d j!(d+j+t)!(2d-2+2t)!(2d-1+t)!4^d}{(d-1+t)!(4d-2+2t)!(2d+2j+2t)!(j-d)!}.
\end{aligned}$$

Denote the sum on the RHS of the last equation by $\mathtt{SUM}_i$. The Zeilberger algorithm gives the recurrence

$$\mathtt{SUM}_{i+1}=\frac{1}{2(2i+2j+2t+1)}\mathtt{SUM}_{i-1},$$

where the initial value, $\mathtt{SUM}_0=\frac{(j+t)!}{(2j+2t)!}$. Solving the recursion gives the claim.

For A and A^{-1}, consider

$$\begin{aligned}
&\sum_{j\le d\le k} A_{kd}A^{-1}_{dj}\\
&=\sum_{j\le d\le k}\frac{(-1)^{k-d}(2k-2+2N+2t)!(N-1-d)!(d-1+N+t)!}{(N-1-k)!(N-1+k+t)!(k-d)!(2d-2+2N+2t)!}\frac{a_d}{a_k}\\
&\times\frac{(2d-2+2N+2t)!(N-1-j)!(N-1+j+t)!}{(N-1-d)!(N-1+d+t)!(d-j)!(2j-2+2N+2t)!}\frac{a_j}{a_d}\\
&=\frac{(2k-2+2N+2t)!(N-1-j)!(N-1+j+t)!}{(N-1-k)!(N-1+k+t)!(2j-2+2N+2t)!}\frac{a_j}{a_k}\\
&\times\sum_{j\le d\le k}\frac{(-1)^{k-d}(2d-2+2N+2t)!}{(k-d)!(2d-2+2N+2t)!(d-j)!}.
\end{aligned}$$

The algorithm evaluates the second sum on the RHS of the last equation as 0 provided that $k\neq j$. If $k=j$, it is obvious that $A_{k,k}A^{-1}_{k,k}=1$. Thus

$$\sum_{k\le d\le j} A_{kd}A^{-1}_{dj}=\delta_{k,j},$$

as claimed.

For B and B^{-1}, consider

$$\begin{aligned}
\sum_{i\le d\le j} B_{id}B^{-1}_{dj}&=\sum_{i\le d\le j}\frac{(-1)^{N-1-d}(2d-2+2N+2t)!}{(d-i)!(N-1-d)!(N-1+d+t)!4^{N-1-i}}\frac{1}{a_ib_d}\\
&\times\frac{(-1)^{N-1-j}(N-1-d)!(N-1+d+t)!4^{N-1-j}}{(2d-2+2N+2t)!(j-d)!}a_jb_d\\
&=(-1)^j4^{i-j}\frac{a_j}{a_i}\sum_{i\le d\le j}\frac{(-1)^d}{(d-i)!(j-d)!}.
\end{aligned}$$

The algorithm evaluates the sum on the RHS of the last equation as 0 provided that $i\neq j$. For the case $j=i$, the sum is equal to 1. Thus

$$\sum_{k\le d\le j} B_{kd}B^{-1}_{dj}=\delta_{k,j},$$

as claimed.

For the LU factorization of the inverse matrix, we should show that $F = B^{-1}A^{-1}$ instead of $F^{-1} = AB$. So, we have

$$\begin{aligned}
&\sum_{\max\{i,j\}\le d\le N-1} B^{-1}_{i,d}A^{-1}_{d,j} \\
&= \sum_{\max\{i,j\}\le d\le N-1} \frac{(-1)^{N-1-d}(N-1-i)!(N-1+i+t)!4^{N-1-d}}{(2i-2+2N+2t)!(d-i)!} a_d b_i \\
&\times \frac{(2d-2+2N+2t)!(N-1-j)!(N-1+j+t)!}{(N-1-d)!(N-1+d+t)!(d-j)!(2j-2+2N+2t)!} \frac{a_j}{a_d}.
\end{aligned}$$

Denote the sum on the RHS of the last equation by $\mathtt{SUM}_N$ after taking N instead of $N-1$. That is,

$$\begin{aligned}
\mathtt{SUM}_N = &\sum_{\max\{i,j\}\le d\le N} \frac{(-1)^{N-d}4^{N-d}(N-j)!(N+j+t)!}{(d-j)!(2j+2N+2t)!} \\
&\times \frac{(N-i)!(N+i+t)!(2d+2N+2t)!}{(2i+2N+2t)!(d-i)!(N-d)!(N+d+t)!}.
\end{aligned}$$

The algorithm gives the recursion

$$\mathtt{SUM}_{N+1} = \mathtt{SUM}_N.$$

We obtain

$$\mathtt{SUM}_{N+1} = \mathtt{SUM}_j = \frac{(j+i+t)!}{(2i+2j+2t)!},$$

which gives the claim

$$\sum_{\max\{i,j\}\le d\le N-1} B^{-1}_{i,d}A^{-1}_{d,j} = \frac{(j+i+t)!}{(2i+2j+2t)!} a_j b_i = F_{i,j}.$$

11. Proofs related to the fourth matrix

For L and L^{-1},

$$\begin{aligned}
\sum_{j\le d\le k} L_{kd}L^{-1}_{dj} &= \sum_{j\le d\le k} \frac{k!(2k+2t)!(d+t)!}{(k+t)!(k-d)!(2d+2t)!d!} \frac{b_k}{b_d} \\
&\times \frac{(-1)^{d-j}d!(j+t)!(2d+2t)!}{(d+t)!(d-j)!(2j+2t)!j!} \frac{b_d}{b_j} \\
&= \frac{k!(2k+2t)!(j+t)!}{(k+t)!(2j+2t)!j!} \frac{b_k}{b_j} \sum_{j\le d\le k} \frac{(-1)^{d-j}}{(k-d)!(d-j)!}.
\end{aligned}$$

In this instance, one can immediately see without any computer help that the sum is (apart from a factor)

$$\sum_{0\le d\le k-j} \binom{k-j}{d}(-1)^d = \delta_{k,j}$$

and it follows readily that $L_{jj}L^{-1}_{jj} = 1$.

For U and U^{-1},

$$\sum_{k\le d\le j} U_{kd}U^{-1}_{dj} = \sum_{k\le d\le j} \frac{d!(2d+2t)!4^k}{(d+t)!(d-k)!} a_d b_k \frac{(-1)^{d-j}(d+t)!}{d!(2d+2t)!(j-d)!4^j} \frac{1}{a_d b_j}$$

$$= 4^{k-j}\frac{b_k}{b_j}\sum_{k\le d\le j} \frac{(-1)^{d-j}}{(d-k)!(j-d)!}.$$

Again, the sum is essentially an alternating sum over a line in Pascal's triangle, and the result is immediate.

For LU-decomposition, we have to prove that

$$\sum_{0\le d\le \min\{i,j\}} L_{id}U_{dj} = F_{ij}.$$

Consider

$$\sum_{0\le d\le \min\{i,j\}} L_{id}U_{dj}$$

$$= a_j b_i \sum_{0\le d\le \min\{i,j\}} \frac{i!(2i+2t)!(d+t)!j!(2j+2t)!4^d}{(i+t)!(i-d)!(2d+2t)!d!(j+t)!(j-d)!}.$$

Denote the sum on the RHS of the last equation by $\mathtt{SUM}_i$. The Zeilberger algorithm gives the recurrence

$$\mathtt{SUM}_i = 2(2i+2j+2t-1)\mathtt{SUM}_{i-1},$$

where the initial value, $\mathtt{SUM}_0 = \frac{(2j+2t)!}{(j+t)!}$. Solving the recursion gives the claim.

For A and A^{-1}, consider

$$\sum_{j\le d\le k} A_{kd}A^{-1}_{dj}$$

$$= \sum_{j\le d\le k} \frac{(-1)^{k-d}(k+d+t)!(4d+1+2t)!(N-d-1)!}{(2k+2d+1+2t)!(N-k-1)!(k-d)!(2d+t)!} \frac{a_d}{a_k}$$

$$\times \frac{(2d+t)!(2d+2j+2t)!(N-j-1)!}{(4d+2t)!(N-d-1)!(d-j)!(d+j+t)!} \frac{a_j}{a_d}$$

$$= \frac{(N-j-1)!}{(N-k-1)!}\frac{a_j}{a_k}$$

$$\times \sum_{j\le d\le k} \frac{(-1)^{k-d}(k+d+t)!(4d+1+2t)!(2d+2j+2t)!}{(2k+2d+1+2t)!(k-d)!(4d+2t)!(d-j)!(d+j+t)!}.$$

The algorithm evaluates the second sum on the RHS of the last equation as 0 provided that $k \neq j$. If $k = j$, it is obvious that $A_{k,k}A^{-1}_{k,k} = 1$. Thus

$$\sum_{k\le d\le j} A_{kd}A^{-1}_{dj} = \delta_{k,j},$$

as claimed.

For B and B^{-1},

$$\begin{aligned}
&\sum_{k\le d\le j} B_{kd}B^{-1}_{dj}\\
&=\sum_{k\le d\le j}\frac{(-1)^{k-d}(k+d+1+t)!}{(d-k)!(2k+2d+2+2t)!(N-1-d)!}\\
&\times\frac{(2k+2N+2t)!(2k+t)!}{(N+k+t)!(4k+2t)!4^{N-1-k}}\frac{1}{a_kb_d}\\
&\times\frac{(4j+2+2t)!(2d+2j+2t)!(N-1-d)!(N+j+t)!4^{N-1-j}}{(d+j+t)!(j-d)!(2j+1+t)!(2j+2N+2t)!}a_jb_d\\
&=(-1)^k\frac{4^{k-j}(2k+2N+2t)!(2k+t)!(N+j+t)!(4j+2+2t)!}{(2j+2N+2t)!(4k+2t)!(N+k+t)!(2j+1+t)!}\frac{a_j}{a_k}\\
&\times\sum_{k\le d\le j}\frac{(-1)^d(k+d+1+t)!(2d+2j+2t)!}{(d-k)!(2k+2d+2+2t)!(d+j+t)!(j-d)!}.
\end{aligned}$$

The algorithm evaluates the sum on the RHS of the last equation as 0 provided that $k\neq j$. For the case $j=k$, the sum is equal to 1. Thus

$$\sum_{k\le d\le j} B_{kd}B^{-1}_{dj}=\delta_{k,j},$$

as claimed.

For the LU factorization of the inverse matrix, we should show that $F=B^{-1}A^{-1}$ instead of $F^{-1}=AB$. So, we have

$$\begin{aligned}
&\sum_{\max\{k,j\}\le d\le N-1} B^{-1}_{k,d}A^{-1}_{d,j}\\
&=\sum_{\max\{k,j\}\le d\le N-1}\frac{(4d+2+2t)!(2k+2d+2t)!(N+d+t)!4^{N-1-d}}{(k+d+t)!(d-k)!(2d+1+t)!(2d+2N+2t)!}a_db_k\\
&\times\frac{(N-1-k)!(2d+t)!(2d+2j+2t)!(N-j-1)!}{(4d+2t)!(N-d-1)!(d-j)!(d+j+t)!}\frac{a_j}{a_d}\\
&=2a_jb_k\sum_{\max\{k,j\}\le d\le N-1}\frac{(4d+2t+1)\,(2k+2d+2t)!(N+d+t)!}{(k+d+t)!(d-k)!(2d+2N+2t)!}\\
&\times\frac{4^{N-1-d}(N-1-k)!(2d+2j+2t)!(N-j-1)!}{(N-d-1)!(d-j)!(d+j+t)!}.
\end{aligned}$$

Denote the sum on the RHS of the last equation by $\mathtt{SUM}_N$ after taking N instead of $N-1$. That is,

$$\begin{aligned}
&\mathtt{SUM}_N \\
&= \sum_{j\le d\le N} \frac{(4d+2t+1)\,(2k+2d+2t)!(N+1+d+t)!(2d+2j+2t)!}{(k+d+t)!(2d+2N+2t+2)!(d+j+t)!} \\
&\times \frac{(N-k)!(N-j)!4^{N-d}}{(d-k)!(d-j)!(N-d)!}.
\end{aligned}$$

The algorithm gives the recursion

$$\mathtt{SUM}_{N+1} = \mathtt{SUM}_N.$$

We obtain

$$\mathtt{SUM}_{N+1} = \mathtt{SUM}_j = \frac{(2k+2j+2t)!}{2(k+j+t)!}.$$

Finally we derive

$$\sum_{\max\{k,j\}\le d\le N-1} B^{-1}_{k,d}A^{-1}_{d,j} = \frac{(2k+2j+2t)!}{(k+j+t)!}a_j b_k = F_{k,j},$$

as claimed.

12. Proofs for the fifth matrix

For L and L^{-1}:

$$\begin{aligned}
\sum_{j\le d\le i} L_{id}L^{-1}_{dj} &= \sum_{j\le d\le i} \frac{i!(2i+2t)!(d+t)!(2d+t+u)!}{(i+t)!(i+d+t+u)!(i-d)!(2d+2t)!d!}\frac{b_i}{b_d} \\
&\times \frac{(-1)^{d-j}d!(d+j-1+t+u)!(2d+2t)!(j+t)!}{(d+t)!(2d-1+t+u)!(d-j)!(2j+2t)!j!}\frac{b_d}{b_j} \\
&= \frac{b_i}{b_j}\,(-1)^j\,\frac{i!(2i+2t)!(j+t)!}{(i+t)!(2j+2t)!j!} \\
&\times \sum_{j\le d\le i}(-1)^d\,\frac{(2d+t+u)(d+j-1+t+u)!}{(i+d+t+u)!(i-d)!(d-j)!}.
\end{aligned}$$

The Zeilberger algorithm evaluates the sum on the RHS of the last equation as 0 for $i\ne j$. If $i=j$, it can be easily seen $L_{ii}L^{-1}_{ii}=1$. So we have

$$\sum_{j\le d\le i} L_{id}L^{-1}_{dj} = \delta_{ij},$$

as desired.

For the LU factorization

$$\begin{aligned}
&\sum_{0\le d\le \min\{i,j\}} L_{id}U_{dj}\\
&= \sum_{0\le d\le \min\{i,j\}} \frac{i!(2i+2t)!(d+t)!(2d+t+u)!}{(i+t)!(i+d+t+u)!(i-d)!(2d+2t)!d!}\frac{b_i}{b_d}\\
&\times \frac{(2d+2u)!(d-1+t+u)!(2j+2t)!j!u!}{(d+u)!(2d-1+t+u)!(d+j+t+u)!(j-d)!(j+t)!(2u)!}a_j b_d\\
&= b_i a_j \sum_{0\le d\le \min\{i,j\}} \frac{i!(2i+2t)!(2j+2t)!j!u!}{(i+t)!(j+t)!(2u)!}\\
&\times \frac{(2d+t+u)(d+t)!(2d+2u)!(d-1+t+u)!}{(i+d+t+u)!(i-d)!(2d+2t)!d!(d+u)!(d+j+t+u)!(j-d)!}.
\end{aligned}$$

Denote the sum on the RHS of the last equation by $\mathtt{SUM}$, the Zeilberger algorithm produces the recursion

$$\mathtt{SUM}_i = \frac{2\,(2t+2j+2i-1)}{(i+j+t+u)}\mathtt{SUM}_{i-1},$$

with initial value $\mathtt{SUM}_0 = \frac{(2j+2t)!}{(j+t)!(j+t+u)!}$.

Solving it, we obtain

$$\mathtt{SUM}_i = \frac{2^i\,(2t+2j+2i-1)^{\underline{i}}}{(i+j+t+u)^{\underline{i}}}\mathtt{SUM}_0 = \frac{2^i\,(2t+2j+2i-1)^{\underline{i}}\,(2j+2t)!}{(i+j+t+u)^{\underline{i}}\,(j+t)!\,(j+t+u)!}.$$

By multiplying both denominator and numerator by $(i+j+t)^{\underline{i}}$ and after some arrangements we get

$$\mathtt{SUM}_i = \frac{(2t+2j+2i)!}{(i+j+t)!\,(i+j+t+u)!} = F_{i,j},$$

as claimed.

For B and B^{-1}:

$$\begin{aligned}
\sum_{i\le d\le j} B_{id}B^{-1}_{dj} &= (-1)^i\frac{a_j}{a_i}\frac{(2i+t)!(2i+2N+2t)!(N-1-i+u)!}{(N+i+t)!(2N-2-2i+2u)!}\\
&\times \frac{(2N-2-2j+2u)!(N+j+t)!(4j+2+2t)!}{(4i+2t)!(2j+2N+2t)!(N-1-j+u)!(2j+1+t)!}\\
&\times \sum_{i\le d\le j}(-1)^d\frac{(i+d+1+t)!(2d+2j+2t)!}{(2i+2d+2+2t)!(d-i)!(d+j+t)!(j-d)!}.
\end{aligned}$$

By the Zeilberger algorithm, the sum on RHS is equal to 0 provided that $i \ne j$. When $i = j$, $B_{ii}B^{-1}_{ii} = 1$.

For the LU factorization of the inverse matrix, we should show that $F = B^{-1}A^{-1}$ instead of $F^{-1} = AB$. So, we have

$$\begin{aligned}
&\sum_{\max\{i,j\}\le d\le N-1} B^{-1}_{i,d}A^{-1}_{d,j} \\
&= \sum_{\max\{i,j\}\le d\le N-1} \frac{(N-1-i)!(2i+2d+2t)!(2N-2-2d+2u)!}{(N+i-1+t+u)!(i+d+t)!(2d+2N+2t)!} \\
&\times \frac{(N+d+t)!(4d+2+2t)!u!}{(N-1-d+u)!(d-i)!(2d+1+t)!(2u)!} \\
&\times \frac{(2d+t)!(N-1+d+t+u)!(2d+2j+2t)!(N-1-j)!}{(N-1-d)!(4d+2t)!(d-j)!(d+j+t)!(N-1+j+t+u)!}
\end{aligned}$$

and replacing $(N-1)$ with N,

$$\begin{aligned}
&\sum_{\max\{i,j\}\le d\le N} B^{-1}_{i,d}A^{-1}_{d,j} \\
&= \frac{u!}{(2u)!}\sum_{j\le d\le N} \frac{(N-i)!(2i+2d+2t)!}{(N+i+t+u)!(2d+2N+2+2t)!} \\
&\times \frac{(2N-2d+2u)!(N+1+d+t)!(4d+2+2t)!}{(i+d+t)!(N-d+u)!(d-i)!(2d+1+t)!} \\
&\times \frac{(2d+t)!(N+d+t+u)!(2d+2j+2t)!(N-j)!}{(N-d)!(4d+2t)!(d-j)!(d+j+t)!(N+j+t+u)!}.
\end{aligned}$$

The Zeilberger algorithm produces the recursion

$$\mathtt{SUM}_{N+1} = \mathtt{SUM}_N.$$

Then we write

$$\mathtt{SUM}_{N+1} = \mathtt{SUM}_j = \frac{(2i+2j+2t)!(2u)!}{(i+j+t)!u!(i+j+t+u)!}.$$

Finally we get

$$\sum_{\max\{i,j\}\le d\le N-1} B^{-1}_{i,d}A^{-1}_{d,j} = \frac{u!}{(2u)!}\frac{(2i+2j+2t)!(2u)!}{(i+j+t)!u!(i+j+t+u)!} = F_{i,j},$$

which completes the proof.

13. Proofs related to the sixth matrix

For L and L^{-1}:

$$\begin{aligned}
&\sum_{j\le d\le i} L_{id}L^{-1}_{dj}\\
&=\sum_{j\le d\le i}\frac{(i+t+u)!i!(i+d+t)!(4d+2t)!}{(2i+2d+2t)!(i-d)!(d+t+u)!d!(2d+t)!}\frac{b_i}{b_d}\\
&\times\frac{(-1)^{d-j}d!(d+t+u)!(2d-1+t)!(2d+2j-2+2t)!}{(4d-2+2t)!(d-j)!(d+j-1+t)!j!(j+t+u)!}\frac{b_d}{b_j}\\
&=\frac{2b_i(i+t+u)!i!}{b_j(j+t+u)!j!}\\
&\times\sum_{j\le d\le i}\frac{(-1)^{d-j}(i+d+t)!(2d+2j-2+2t)!(4d+2t-1)}{(2i+2d+2t)!(i-d)!(d-j)!(d+j-1+t)!}.
\end{aligned}$$

The Zeilberger algorithm computes the sum on the RHS of the last equation as 0 when $i\neq j$. If $i=j$, it is obvious that $L_{jj}L^{-1}_{jj}=1$. Thus

$$\sum_{j\le d\le k} L_{kd}L^{-1}_{dj}=\delta_{k,j},$$

as claimed.

For U and U^{-1},

$$\begin{aligned}
&\sum_{i\le d\le j} U_{id}U^{-1}_{dj}\\
&=\sum_{i\le d\le j}\frac{2^{2i+3-2u}(2i-1+t)!(2i-2+2t)!(i+d+t)!d!(d+t+u)!}{(i-1+t)!(4i-2+2t)!(2i+2d+2t)!(d-i)!}\\
&\times\frac{\sqrt{\pi}(-1)^i(2u+1)!}{\Gamma(\frac32-i+u)u!}a_db_i\frac{\Gamma(\frac32-j+u)u!}{(2u+1)!2^{2j-2u+3}\sqrt{\pi}}\frac{1}{a_db_j}\\
&\times\frac{(-1)^d(4j+2t)!(2d+2j-2+2t)!(j-1+t)!}{(d+t+u)!d!(d+j-1+t)!(2j+t)!(j-d)!(2j-2+2t)!}\\
&=\frac{2^{2i-2j}(2i-1+t)!(2i-2+2t)!(-1)^i(4j+2t)!(j-1+t)!}{(i-1+t)!(4i-2+2t)!(2j+t)!}\\
&\times\frac{b_i}{b_j}\frac{\Gamma(\frac32-j+u)}{\Gamma(\frac32-i+u)}\\
&\times\sum_{i\le d\le j}\frac{(-1)^d(i+d+t)!(2d+2j-2+2t)!}{(2i+2d+2t)!(d-i)!(d+j-1+t)!(j-d)!(2j-2+2t)!}.
\end{aligned}$$

The Zeilberger algorithm computes the sum on the RHS of the last equation as 0 when $i \neq j$. If $i = j$, it is obvious that $U_{ii}U_{ii}^{-1} = 1$. Thus

$$\sum_{j \le d \le k} U_{id}U_{dj}^{-1} = \delta_{i,j},$$

as claimed.

For A and A^{-1}, consider

$$\begin{aligned}
&\sum_{j \le d \le k} A_{kd}A_{dj}^{-1} \\
&= \sum_{j \le d \le k} \frac{(-1)^{k-d}(2d+1+t+u)!(N-1-d)!(N-1+d+t)!}{(k-d)!(k+d+1+t+u)!(2d-2+2N+2t)!} \\
&\times \frac{(2k-2+2N+2t)!}{(N-1-k)!(N-1+k+t)!}\frac{a_d}{a_k} \\
&\times \frac{(2d-2+2N+2t)!(d+j+t+u)!}{(2d+t+u)!(N-1-d)!(N-1+d+t)!(d-j)!} \\
&\times \frac{(N-1-j)!(N-1+j+t)!}{(2j-2+2N+2t)!}\frac{a_j}{a_d} \\
&= \frac{(2k-2+2N+2t)!(N-1-j)!(N-1+j+t)!}{(N-1-k)!(N-1+k+t)!(2j-2+2N+2t)!}\frac{a_j}{a_k} \\
&\times \sum_{j \le d \le k} \frac{(-1)^{k-d}(d+j+t+u)!(2d+1+t+u)}{(k-d)!(k+d+1+t+u)!(d-j)!}.
\end{aligned}$$

By the Zeilberger algorithm, for the second sum in the last equation, we obtain that it is equal to 0 provided that $k \neq j$. If $k = j$, it is obvious that $A_{k,k}A_{k,k}^{-1} = 1$. Thus

$$\sum_{k \le d \le j} A_{kd}A_{dj}^{-1} = \delta_{k,j},$$

as claimed.

By a similar argument, one can obtain that

$$\sum_{j \le d \le k} B_{id}B_{dj}^{-1} = \delta_{i,j}.$$

For LU-decomposition, we have to prove that

$$\sum_{0 \le d \le \min\{i,j\}} L_{id}U_{dj} = F_{ij}.$$

Now consider

$$\begin{aligned}
&\sum_{0\le d\le \min\{i,j\}} L_{id}U_{dj}\\
&= \sum_{0\le d\le \min\{i,j\}} \frac{(i+t+u)!i!(i+d+t)!(4d+2t)!}{(2i+2d+2t)!(i-d)!(d+t+u)!d!(2d+t)!}\frac{b_i}{b_d}\\
&\times \frac{2^{2d+3-2u}(2d-1+t)!(2d-2+2t)!(d+j+t)!j!(j+t+u)!}{(d-1+t)!(4d-2+2t)!(2d+2j+2t)!(j-d)!}\\
&\times \frac{\sqrt{\pi}(-1)^d(2u+1)!}{\Gamma(\frac32-d+u)u!}a_jb_d\\
&= a_jb_i \sum_{0\le d\le \min\{i,j\}} \frac{(i+t+u)!i!(i+d+t)!\,(4d+2t-1)}{(2i+2d+2t)!(i-d)!(d+t+u)!d!}\\
&\times \frac{2^{2d-2u+3}(2d-2+2t)!(d+j+t)!j!(j+t+u)!}{(d-1+t)!(2d+2j+2t)!(j-d)!}\\
&\times \frac{\sqrt{\pi}(-1)^d(2u+1)!}{\Gamma(\frac32-d+u)u!}.
\end{aligned}$$

The algorithm gives the recursion

$$\texttt{SUM}_{i+1} = \frac{(1+i+j+t+u)}{2(1+2i+2j+2t)}\texttt{SUM}_i$$

and we can compute the initial value

$$\texttt{SUM}_0 = \frac{(j+t)!(j+t+u)!}{(2j+2t)!}\frac{2^{3-2u}\sqrt{\pi}(2u+1)!}{\Gamma(\frac32+u)u!}.$$

By the duplication formula we write

$$\begin{aligned}
\Gamma(\tfrac32+u) &= \frac{\Gamma(\frac12+(u+1))\Gamma(u+1)}{\Gamma(u+1)}\\
&= \frac{\sqrt{\pi}2^{1-2u+2}\Gamma(2u+2)}{\Gamma(u+1)} = \frac{\sqrt{\pi}2^{3-2u}(2u+1)!}{u!}
\end{aligned}$$

and so

$$\texttt{SUM}_0 = \frac{(j+t)!(j+t+u)!}{(2j+2t)!}.$$

Consequently,

$$\begin{aligned}
\texttt{SUM}_i &= \frac{(i+j+t+u)}{2(2i+2j+2t-1)}\texttt{SUM}_{i-1} = \frac{(i+j+t+u)^{\underline{i}}}{2^i(2i+2j+2t-1)^{\underline{i}}}\texttt{SUM}_0\\
&= \frac{(i+j+t+u)^{\underline{i}}}{2^i(2i+2j+2t-1)^{\underline{i}}}\frac{(j+t)!(j+t+u)!}{(2j+2t)!}\\
&= \frac{(i+j+t)!\,(i+j+t+u)!}{(2i+2j+2t)!},
\end{aligned}$$

as claimed.

For the LU-decomposition of F^{-1}, we should show that $F^{-1} = AB$ which is the same as $F = B^{-1}A^{-1}$. So it is sufficient to show that

$$\sum_{\max\{i,j\}\le d\le n-1} B^{-1}_{id}A^{-1}_{dj} = F_{i,j}.$$

Consider

$$\sum_{\max\{i,j\}\le d\le N-1} B^{-1}_{id}A^{-1}_{d,j} = \sum_{j\le d\le N-1} \frac{2^{2N-2d-2u+1}\sqrt{\pi}(2u+1)!}{u!\Gamma(d-N+\frac{5}{2}+u)}\frac{a_j}{a_d}a_d b_i$$
$$\times \frac{(N-1-i)!(N-1+i+t)!(2d+1+t+u)!(i+d+t+u)!(-1)^d}{(2i-2+2N+2t)!(d-i)!(N+d+t+u)!}$$
$$\times \frac{(2d-2+2N+2t)!(d+j+t+u)!(N-1-j)!(N-1+j+t)!}{(2d+t+u)!(N-1-d)!(N-1+d+t)!(d-j)!(2j-2+2N+2t)!}$$
$$= a_j b_i \sum_{j\le d\le N} \frac{2^{2N-2d-2u+3}\sqrt{\pi}(2u+1)!}{u!\Gamma(d-N+\frac{3}{2}+u)}$$
$$\times \frac{(N-i)!(N+i+t)!(2d+1+t+u)!(i+d+t+u)!(-1)^d}{(2i+2N+2t)!(d-i)!(N+d+t+u+1)!}$$
$$\times \frac{(2d+2N+2t)!(d+j+t+u)!(N-j)!(N+j+t)!}{(2d+t+u)!(N-d)!(N+d+t)!(d-j)!(2j+2N+2t)!}.$$

The Zeilberger algorithm gives the recursion for the sum on the RHS of the last equation

$$\mathtt{SUM}_N = \mathtt{SUM}_{N-1}.$$

So we write

$$\mathtt{SUM}_N = \mathtt{SUM}_j = \frac{2^{-2u+3}\sqrt{\pi}(2u+1)!(j+i+t)!(i+j+t+u)!}{u!\Gamma(\frac{3}{2}+u)(2i+2j+2t)!},$$

which, since $\Gamma(\frac{3}{2}+u) = \frac{\sqrt{\pi}2^{3-2u}(2u+1)!}{u!}$, equals

$$\frac{2^{-2u+3}\sqrt{\pi}(2u+1)!}{u!\frac{\sqrt{\pi}2^{3-2u}(2u+1)!}{u!}}\frac{(j+i+t)!(i+j+t+u)!}{(2i+2j+2t)!} = \frac{(j+i+t)!(i+j+t+u)!}{(2i+2j+2t)!},$$

as claimed.

14. Hypergeometric proofs related to the seventh matrix

As one reviewer remarked, instead of relying on the machinery of automated proofs (Zeilberger's algorithm), one can also use hypergeometric functions. A user-friendly introduction to the subject can be found in [3].

A note on notation: The most common notation today is the one used in [1]:

$${}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_p\end{matrix};z\right)$$

with $(x)_n := x(x+1)\ldots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x+n)}$.

Knuth comments in [3] that instead of '${}_pF_q$' one could simply write 'F' as the numbers of upper (resp. lower) parameters can be simply read off is certainly true. However, most people still use the notation with p and q.

Slater's influential book [14] uses the notation ${}_pF_q[a_1,\dots,a_p;b_1,\dots,b_q;z]$. This book appeared at a time when the typesetting of fractions was very complicated and costly and thus avoided. Nowadays people prefer the two-line notation, as it is so much more readable.

To show that the matrices

$$U_{i,j}=\frac{(2i+1)!(i+t)!(2j+2t)!j!}{i!(2i+t)!(i+j+1+t)!(j-i)!(j+t)!}a_jb_i$$

and

$$U^{-1}_{j,k}=\frac{(-1)^{j-k}(j+t)!(j+k+t)!(2k+1+t)!k!}{j!(2j+2t)!(k-j)!(2k+1)!(k+t)!}\frac{1}{a_jb_k}$$

are indeed inverses, we need to evaluate

$$\sum_j U_{i,j}U^{-1}_{j,k}=\frac{b_i(2i+1)!(i+t)!(2k+1+t)!k!(-1)^{i+k}}{b_ki!(2i+t)!(2k+1)!(k+t)!}\\ \times\sum_{0\le j\le k-i}\frac{(-1)^j(j+i+k+t)!}{(2i+j+1+t)!j!(k-i-j)!}.$$

Calling the term in the sum T_j, we compute the quotient ("term ratio")

$$\frac{T_{j+1}}{T_j}=-\frac{(j+1+i+k+t)(k-i-j)}{(2i+j+2+t)(j+1)}.$$

The sum (without the extra factors) is then given by

$$T_0\cdot{}_2F_1\Bigl(\begin{matrix}1+i+k+t,i-k\\2i+2+t\end{matrix};1\Bigr)=T_0\frac{\Gamma(1)\Gamma(2i+2+t)}{\Gamma(i+1-k)\Gamma(i+2+t+k)},$$

where the ${}_2F_1$ summation of Gauss was employed. Note that the special (=terminating) case of the summation is the celebrated Chu–Vandermonde identity (=summation). Because of the factor $\Gamma(i+1-k)$ in the denominator, the only nonzero term of the sum occurs for $i=k$, and it is

$$\frac{b_i(2i+1)!(i+t)!(2i+1+t)!i!(-1)^{2i}}{b_ii!(2i+t)!(2i+1)!(i+t)!}\frac{(2i+t)!}{(2i+1+t)!}\frac{\Gamma(2i+2+t)}{\Gamma(2i+2+t)}=1,$$

as it should.

Here is a second example, worked out in full detail, namely to show that B and B^{-1} are inverse:

$$(BB^{-1})_{ik} = \frac{a_k(2i+t)!(2i+2N+2t)!(N-1-i)!}{a_i(N+i+t)!(2N-1-2i)!(4i+2t)!} \times \frac{(2N-1-2k)!(N+k+t)!(4k+2+2t)!}{(2k+2N+2t)!(N-1-k)!(2k+1+t)!} \times \sum_{0\le j\le k-i} \frac{(-1)^j(2i+j+1+t)!(2j+2i+2k+2t)!}{(4i+2j+2+2t)!j!(j+i+k+t)!(k-i-j)!}.$$

Let us again compute the term ratio of two consecutive summands:

$$\frac{T_{j+1}}{T_j} = \frac{(j+\frac{1}{2}+i+k+t)(j+i-k)}{(j+\frac{3}{2}+2i+t)(j+1)}.$$

Consequently we have to evaluate

$${}_2F_1\left(\begin{matrix}\frac{1}{2}+i+k+t, i-k\\ \frac{3}{2}+2i+t\end{matrix};1\right) = \frac{\Gamma(1)\Gamma(\frac{3}{2}+2i+t)}{\Gamma(1+i-k)\Gamma(\frac{3}{2}+i+t+k)},$$

again by the Gauss ${}_2F_1$ summation. For $i<k$, this evaluates to 0, and we get

$$(BB^{-1})_{ii} = \frac{a_i(2i+t)!(2i+2N+2t)!(N+i+t)!(4i+2+2t)!}{a_i(N+i+t)!(4i+2t)!(2k+2N+2t)!(2i+1+t)!} \times \frac{(N-1-i)!(2N-1-2i)!(2i+1+t)!(4i+2t)!}{(2N-1-2i)!(N-1-i)!(4i+2+2t)!(2i+t)!} = 1.$$

Other proofs can also be done in this style, but require heavy human interaction (as Krattenthaler [6] calls it: "*Do it yourself!*") While it is fun to work through a few of the proofs using hypergeometric summations, to do all of them would be a time consuming procedure that might be a good project for a graduate student.

One referee suggested saying something about possible generalizations. The first thought in this direction is finding so-called q-analogues. Nothing about this has been done, but in any case, we would like to stress the fact that the important and challenging part of such an enterprise is to *find* the relevant formulæ (by inspired guessing), while proofs are somehow routine, using the well-oiled machinery of the q-Zeilberger algorithm (resp. q-hypergeometric functions and identities).

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