Contributions to Discrete Mathematics

Volume 18, Number 2, Pages 176–187
ISSN 1715-0868

# RESOLVABILITY IN HYPERGRAPHS

IMRAN JAVAID, AZEEM HAIDER, MUHAMMAD SALMAN,
AND SADAF MEHTAB

Abstract. This article presents an extension of the study of metric and partition dimension to hypergraphs. We give sharp lower bounds for the metric and partition dimension of hypergraphs in general and give exact values under specified conditions.

## 1. Introduction

A *hypergraph* $H$ is a pair $(V, \mathcal{H})$, where $V$ is a finite non-empty set of vertices and $\mathcal{H}$ is a finite family of distinct non-empty subsets of $V$, called hyperedges, with

$$\bigcup_{E \in \mathcal{H}} E = V.$$

The "order" and the "size" of $H$ is denoted by $n$ and $m$, respectively. A hypergraph $K = (V_1, \mathcal{K})$ is a *subhypergraph* of $H$ if and only if $V_1 \subseteq V$ and $\mathcal{K} \subseteq \mathcal{H}$. A hypergraph $H$ is *linear* if for distinct hyperedges $E_i, E_j \in \mathcal{H}$, $|E_i \cap E_j| \leq 1$, so for a linear hypergraph there are no repeated hyperedges of cardinality greater than one. A hypergraph $H$ such that no hyperedge is a subset of any other is called *Sperner*.

A vertex $v \in V$ is *incident* with a hyperedge $E$ of $H$ if $v \in E$. If $v$ is incident with exactly $d$ hyperedges, then we say that the *degree* of $v$ is $d$; if all the vertices $v \in V$ has degree $r$, then $H$ is $r$*-regular*. Similarly, if there are exactly $k$ vertices incident with a hyperedge $E$, then we say that the size of $E$ is $k$; if all the hyperedges $E \in \mathcal{H}$ have size $k$, then $H$ is $k$*-uniform*. A graph is simply a 2-uniform hypergraph. A hyperedge $E$ of $H$ is called a *pendant hyperedge* if for $E_i, E_j \in \mathcal{H}$, $E \cap E_i \neq \emptyset$ and $E \cap E_j \neq \emptyset$ implies $(E \cap E_i) \cap (E \cap E_j) \neq \emptyset$. A *path* of length $l$ from a

Received by the editors November 9, 2018, and in revised form May 3, 2022.
2000 *Mathematics Subject Classification.* 05C12, 05C65.
*Key words and phrases.* metric dimension, partition dimension, hypergraph.
The authors are thankful to the anonymous referee for his/her valuable comments and suggestions to improve the presentation of this paper.

vertex $v$ to another vertex $u$ in a hypergraph is a finite sequence of the form $v, E_1, w_1, E_2, w_2, \ldots, E_{l-1}, w_{l-1}, E_l, u$ such that $v \in E_1$, $w_i \in E_i \cap E_{i+1}$ for $i = 1, 2, \ldots l-1$ and $u \in E_l$. A hypergraph $H$ is said to be *connected* if there is a path between any two vertices of $H$. All hypergraphs considered in this paper are connected Sperner hypergraphs.

A hypergraph $H$ is said to be a *hyperstar* if there exists a subset $C$ of vertices such that $E_i \cap E_j = C \neq \emptyset$, for any $E_i, E_j \in \mathcal{H}$. Then $C$ is called the center of the hyperstar. If there exists a sequence of hyperedges $E_1, E_2, \ldots, E_m$ in a hypergraph $H$, then $H$ is said to be (1) a *hyperpath* if $E_i \cap E_j \neq \emptyset$ if and only if $|i - j| = 1$; (2) a *hypercycle* if, $E_i \cap E_j \neq \emptyset$ if and only if $i - j \in \{1, -1\}$ (mod $m$) for $m \geq 3$. A connected hypergraph $H$ with no hypercycle is called a *hypertree*. A subhypertree of a hypertree $H$ with edge set, say $\mathcal{E} = \{E_{p_1}, E_{p_2}, \ldots, E_{p_l}\} \subset \mathcal{H}$, is called a *branch* of $H$ if $E_{p_1}$ (say) is the only hyperedge such that, for $E_i, E_j \in \mathcal{H} - \mathcal{E}$, $E_{p_1} \cap E_i \neq \emptyset$ and $E_{p_1} \cap E_j \neq \emptyset$ implies $(E_{p_1} \cap E_i) \cap (E_{p_1} \cap E_j) \neq \emptyset$ as well as $E_{p_i} \cap E_j = \emptyset$ for each $2 \leq i \leq l$ and for all $E_j \in \mathcal{H} - \mathcal{E}$. The hyperedge $E_{p_1}$ is then called the *joint* of the branch.

An ordered set $W$ of vertices of a connected graph $G$ is called a resolving set for $G$ if for every two distinct vertices $u, v \in V(G)$, there is a vertex $w \in W$ such that $d(u, w) \neq d(v, w)$. A resolving set of minimum cardinality is called a basis for $G$ and the number of vertices in a basis is called the metric dimension of $G$, denoted by $\dim(G)$. An ordered $t$-partition $\Pi = \{S_1, S_2, \ldots, S_t\}$ of $V(G)$ is called a resolving partition if for every two distinct vertices $u, v \in V(G)$, there is a set $S_i$ in $\Pi$ such that $d(u, S_i) \neq d(v, S_i)$, where

$$d(v, s) = \min_{s \in S} d(u, s).$$

The minimum $t$ for which there is a resolving $t$-partition of $V(G)$ is called the partition dimension of $G$, denoted by $\mathrm{pd}(G)$. In this article, we consider hypergraphs in the context of metric dimension and partition dimension, which are defined in Sections 2 and 3, respectively. We give sharp lower bounds for the metric and partition dimension of graphs. The metric dimension of some well-known families of hypergraphs such as hyperpaths, hypertrees and $k$-uniform linear hypercycles is investigated. Further, we find the metric and partition dimensions of 3-uniform linear hypercycles as well as the partition dimension of $k$-uniform hyperpath.

## 2. Metric Dimension of Hypergraphs

The metric dimension of a graph was studied by Slater [14] and independently by Harary and Melter [6]. It is a parameter that has appeared in various applications, as diverse as combinatorial optimization, pharmaceutical chemistry, robot navigation and sonar. In recent years, considerable literature has been developed (see [12, 4, 10, 1, 7, 8, 13, 11]). The problem of determining whether $\dim(H) < M$ $(M > 0)$, where $H$ is a simple graph, is an NP-complete problem [5, 13]. The metric dimension of a hypergraph $H$ is defined as follows:

The *distance* between any two vertices $v$ and $u$ of a hypergraph $H$, $d(v,u)$, is the length of the shortest path between them and $d(v,u) = 0$ if and only if $v = u$. The *diameter* of $H$ is the maximum distance between the vertices of $H$, and is denoted by $diam(H)$. Two vertices $u$ and $v$ of $H$ are said to be "diametral" vertices if $d(u,v) = diam(H)$. The *representation*, $r(v|W)$, of a vertex $v$ of $H$ with respect to an ordered set $W = \{w_1, w_2, \ldots, w_q\} \subseteq V$ is the $q$-tuple $r(v|W) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_q))$. The set $W$ is called a *resolving set* for a hypergraph $H$ if $r(v|W) \neq r(u|W)$ for any two different vertices $v, u \in V$. A resolving set with minimum cardinality is called a *basis* for $H$ and that minimum cardinality is called the *metric dimension* of $H$, denoted by $\dim(H)$.

To determine whether a given set $W \subseteq V$ is a resolving set for a hypergraph $H$, $W$ needs only to be verified for the vertices in $V - W$ since every vertex $w \in W$ is the only vertex of $H$ whose distance from $w$ is 0.

For a hypergraph $H$, let the set of hyperedges in $H$ be $\mathcal{H} = \{E_1, E_2, \ldots E_m\}$. Let us denote the set of all the vertices in the set $E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_d}$ having degree $d$ by $\mathcal{C}(i_1, i_2, \ldots, i_d)$, where each $i_j \in \{1, 2, \ldots, m\}$ is distinct. That is

$$\mathcal{C}(i_1, i_2, \ldots, i_d) = \{v \in V |\ v \in E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_d} \text{ and } \deg(v) = d\}.$$

Then, the collection of all such classes defines a partition of the vertex set $V$. Let $\eta(i_1, i_2, \ldots, i_d) = |\mathcal{C}(i_1, i_2, \ldots, i_d)| - 1$ whenever $\mathcal{C}(i_1, i_2, \ldots, i_d) \neq \emptyset$, otherwise we take $\eta(i_1, i_2, \ldots, i_d) = 0$.

**Example.** *The hypergraph $H$ shown in the Figure 1 with set of vertices $V = \{v_1, v_2, v_3, \ldots, v_9\}$ and hyperedges $E_1 = \{v_1, v_2, v_3, v_4\}$, $E_2 = \{v_3, v_4, v_5, v_6, v_7\}$, $E_3 = \{v_6, v_7, v_8, v_9\}$, $E_4 = \{v_2, v_3, v_9\}$.*

*In the graph $H$, $\mathcal{C}(1) = \{v_1\}$, $\mathcal{C}(2) = \{v_5\}$, $\mathcal{C}(3) = \{v_8\}$, $\mathcal{C}(4) = \emptyset$; $\mathcal{C}(1,2) = \{v_4\}$, $\mathcal{C}(1,3) = \emptyset$, $\mathcal{C}(1,4) = \{v_2\}$, $\mathcal{C}(2,3) = \{v_6, v_7\}$, $\mathcal{C}(2,4) = \emptyset$, $\mathcal{C}(3,4) = \{v_9\}$; $\mathcal{C}(1,2,3) = \emptyset$, $\mathcal{C}(1,2,4) = \{v_3\}$, $\mathcal{C}(1,3,4) = \emptyset$, $\mathcal{C}(2,3,4) = \emptyset$ and $\mathcal{C}(1,2,3,4) = \emptyset$. Note that the set of all given classes form a partition for $V$.*

Thus, we have the following straightforward proposition:

**Proposition.** *For any two distinct vertices $u, v \in \mathcal{C}(i_1, i_2, \ldots, i_d)$, we have $d(u,w) = d(v,w)$ for any $w \in V - \{u, v\}$.*

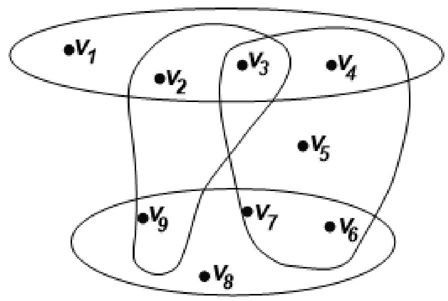


FIGURE 1. Hypergraph $H$

Thus, we extract the following Lemma related to the resolving set for $H$:

**Lemma.** *If $u, v \in \mathcal{C}(i_1, i_2, \ldots, i_d)$ and $W \subseteq V$ resolves $H$, then at least one of the vertices $u$ and $v$ is in $W$. Moreover, if $u \in W$ and $v \notin W$, then $(W - \{u\}) \cup \{v\}$ also resolves $H$.*

Now, we can establish a lower bound for the metric dimension of a hypergraph in the following result.

**Proposition.** *For any hypergraph $H$ with $m$ hyperedges,*

$$\dim(H) \geq \sum_{j=1}^{m} \sum_{i_1 < \cdots < i_j}^{m} \eta(i_1, i_2, \ldots, i_j).$$

*Proof.* It follows from the fact that if there are $|\mathcal{C}(i_1, i_2, \ldots, i_d)|$ number of vertices of degree $d$ in $E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_d}$, then Lemma 2.3 yields that at least $\eta(i_1, i_2, \ldots, i_d)$ vertices should belong to any basis $W$. □

**Remark.** *By Proposition 2.4, it is clear that, in order to obtain a basis of any hypergraph $H$, it suffices to consider only one vertex, say $v_{i_1, i_2, \ldots, i_d}$, from each class $\mathcal{C}(i_1, i_2, \ldots, i_d)$ if $\mathcal{C}(i_1, i_2, \ldots, i_d) \neq \emptyset$. We call this vertex, a representative vertex of the class $\mathcal{C}(i_1, i_2, \ldots, i_d)$. We denote the set of all representative vertices in a hypergraph $H$ by $R(H)$, and hence we always have, $V - R(H) \subseteq W$ for any basis $W$ of $H$.*

Now we discuss some classes of hypergraphs for which the equality holds in the Proposition 2.4.

**Theorem.** *For any hypergraph $H$ with $m$ hyperedges, if $\eta(i) \neq 0$ for all $E_i \in E(H)$, then*

$$\dim(H) = \sum_{j=1}^{m} \sum_{i_1 < \cdots < i_j}^{m} \eta(i_1, i_2, \ldots, i_j).$$

*Moreover, there are*

$$\prod_{j=1}^{m} \prod_{i_1<\cdots<i_j}^{m} (\eta(i_1, i_2, \ldots, i_j) + 1)$$

*basis for $H$.*

*Proof.* Consider $W = V - R(H)$, we have to show that $W$ is a basis for $H$. Take any two different vertices $v, v' \in R(H)$. Since both the vertices $v$ and $v'$ are representative vertices of different classes, there exists a hyperedge $E_j$ such that $v' \in E_j$ and $v \notin E_j$. It follows from $\eta(i) \neq 0$ that there exists a vertex of degree one $w_j \in V$ such that $w_j \in E_j \cap W$. Clearly, $d(v', w_j) = 1$ and $d(v, w_j) \neq 1$, hence $W$ is a basis for $H$. Further, by Lemma 2.3, there are $\prod_{j=1}^{m} \prod_{i_1<\cdots<i_j}^{m} (\eta(i_1, i_2, \ldots, i_j) + 1)$ such $W$. □

If $H$ is a linear hypergraph with $m$ hyperedges, then $\eta(i, j) = 0$ for every $E_i, E_i \in \mathcal{H}$. Thus, we have the following corollary:

**Corollary.** *Let $H$ be a linear hypergraph with $m$ hyperedges and $\eta(i) \neq 0$ for all $E_i \in \mathcal{H}$. Then* $\dim(H) = \sum_{i=1}^{m} \eta(i)$.

We give two examples which show that the condition in Theorem 2.6 cannot be relaxed in general.

**Example.** *Let $H$ be a hypergraph with vertex set $V = \{v_1, v_2, v_3, v_4\}$ and edge set $\mathcal{H} = \{E_1, E_2\}$, where $E_1 = \{v_1, v_2, v_3\}$ and $E_2 = \{v_3, v_4\}$. Clearly, $\eta(2) = 0$ so $H$ does not satisfy the condition of Theorem 2.6. Without loss of generality, we can take the set of representative vertices $R(H) = \{v_1, v_3, v_4\}$, and hence $W = V - R(H) = \{v_2\}$. But, $W$ is not a resolving set for $H$ since $r(v_1|W) = r(v_3|W)$. In fact,* $\dim(H) = 2 > 1$.

**Example.** *Let $H$ be a hypergraph with vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\mathcal{H} = \{E_1, E_2, E_3\}$, where $E_1 = \{v_1, v_2, v_3, v_4\}$, $E_2 = \{v_3, v_4, v_5, v_6\}$ and $E_3 = \{v_1, v_2, v_5, v_6\}$. Clearly, $\eta(i) = 0$ for all $i = 1, 2, 3$ and $\eta(1, 2) = \eta(2, 3) = \eta(3, 1) \neq 0$. Without loss of generality, we can take the set of representative vertices $R(H) = \{v_1, v_3, v_5\}$, and hence $W = V - R(H) = \{v_2, v_4, v_6\}$. But, $W$ is not a resolving set for $H$ since $r(v_1|W) = r(v_3|W) = r(v_5|W)$. In fact,* $\dim(H) = 5 > 3$.

However, the condition in Theorem 2.6 can be reduced in some special cases as shown in the following result.

**Theorem.** *Let $H$ be a hyperpath with $m$ hyperedges $E_1, E_2, \ldots E_m$ in a canonical way. Then*

$$\dim(H) = \sum_{i=1}^{m} \eta(i) + \sum_{i=1}^{m-1} \eta(i, i+1)$$

*if both $\eta(1)$ and $\eta(m)$ are non-zero.*

*Proof.* Let $W = V - R(H)$. Then it follows from the facts $\eta(1) \neq 0$ and $\eta(m) \neq 0$ that there exists a vertex of degree one $w_1 \in E_1 \cap W$ and there exists a vertex of degree one $w_m \in E_m \cap W$. To prove the theorem, we only have to show that the representative vertices are resolved by the set $W$, and it yields from the fact that for any $1 \leq j \leq m$, we have $(d(v_j, w_1), d(v_j, w_m)) = (j, m - j + 1)$, and for any $1 \leq j < m - 1$, we have $(d(v_{j,j+1}, w_1), d(v_{j,j+1}, w_m)) = (j, m - j)$. □

**Theorem.** *Let $H$ be a hypertree with $m$ hyperedges and let $E_{p_1}, E_{p_2}, \ldots, E_{p_t}$ be its pendant hyperedges. Then*

$$\dim(H) = \sum_{j=1}^{m} \sum_{i_1 < \cdots < i_j}^{m} \eta(i_1, i_2, \ldots, i_j)$$

*if $\eta(p_s) \neq 0$ for all $s = 1, 2, \ldots, t$.*

*Proof.* Consider $W = V - R(H)$, similarly as in the proof of Theorem 2.6, again we have to show that $W$ is a basis for $H$. Take any two different vertices $v, v' \in R(H)$, then both vertices are representative of two different classes, and hence there exists a hyperedge $E_j$ such that $v' \in E_j$ but $v \notin E_j$. Now, consider a hyperpath contained in the hypertree $H$ which starts and ends at the pendant hyperedges and contains both $v$ and $E_j$. By using the proof of Theorem 2.10, it can be seen that the vertices $v$ and $v'$ has different representations for $W$, which proves the theorem. □

In a $k$-uniform ($k \geq 3$) linear hyperstar $H$ is a special case of hypertree in which $\eta(i) \neq 0$ for all $E_i \in \mathcal{H}$, so we have the following corollary:

**Corollary.** *If $H$ is a $k$-uniform ($k \geq 3$) linear hyperstar with $m$ ($m \geq 2$) hyperedges, then* $\dim(H) = m(k - 2)$.

Consider a $k$-uniform ($k \geq 4$) linear hypercycle $\mathcal{C}_{m,k}$ with $m$ hyperedges. Then $\eta(i) \neq 0$ for all edges $E_i$ of $\mathcal{C}_{m,k}$. By Corollary 2.7, $\dim(\mathcal{C}_{m,k}) = m(k - 3)$. A 3-uniform linear hypercycle with four hyperedges is shown in Figure 2. For the case $k = 3$, we have $\eta(i) = 0$ for all $E_i \in \mathcal{H}$, hence the lower bound given in Proposition 2.4 is zero and every vertex in $\mathcal{C}_{m,3}$ is the representative vertex. We discuss this case in the following result:

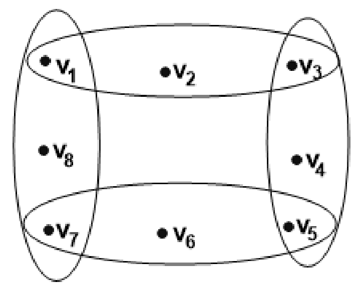


FIGURE 2. A 3-uniform linear hypercycle with 4 hyperedges

**Theorem.** *Let $\mathcal{C}_{m,3}$ be a 3-uniform linear hypercycle with $m$ hyperedges. Then* $\dim(\mathcal{C}_{3,3}) = 2$ *and for all* $m \geq 4$,

$$\dim(\mathcal{C}_{m,3}) = \begin{cases} 2, & \textit{if } k \textit{ is even}, \\ 3, & \textit{if } k \textit{ is odd}. \end{cases}$$

*Proof.* In $\mathcal{C}_{m,3}$, each $v_j \in E_j$ represents a vertex of degree one and $v_{j,j+1} \in E_j \cap E_{j+1}$ with $v_{m,m+1} = v_{m,1}$. Clearly, $\dim(\mathcal{C}_{m,3}) > 1$ for any $m$.

If $m$ is even, then we take $W = \{v_1, v_{\frac{m}{2}}\}$.

For $1 < j < \frac{m}{2}$, we have $r(v_j|W) = (j, \frac{m}{2} - j + 1)$ and for $1 \leq j < \frac{m}{2}$, $r(v_{j,j+1}|W) = (j, \frac{m}{2} - j)$. Now, if $\frac{m}{2} + 1 \leq j < k$, then $r(v_j|W) = (m + 2 - j, j - \frac{m}{2} + 1)$ and $r(v_{j,j+1}|W) = (m + 1 - j, j - \frac{m}{2} + 1)$ with $r(v_m|W) = (2, \frac{m}{2} + 1)$, $r(v_{\frac{m}{2},\frac{m}{2}+1}|W) = (\frac{m}{2}, 1)$ and $r(v_{m,1}|W) = (1, \frac{m}{2})$. It is easy to see that the representations of all the vertices with respect to $W$ are distinct, hence $W$ forms a basis for $\mathcal{C}_{m,3}$ and $\dim(\mathcal{C}_{m,3}) = 2$.

For the special case when $m = 3$, the set $W = \{v_1, v_2\}$ forms a basis for $\mathcal{C}_{3,3}$. Hence $\dim(\mathcal{C}_{3,3}) = 2$.

If $m > 3$ is odd, then we first show that $\dim(\mathcal{C}_{m,3}) > 2$. Suppose to the contrary that $\dim(\mathcal{C}_{m,3}) = 2$ and let $W$ be a basis of $\mathcal{C}_{m,3}$. Let us call the vertices $v_{i,i+1}$, $i \in \{1, 2, \ldots, m\}$, of $\mathcal{C}_{m,3}$, the common vertices. We have the following three possibilities:

(1) $W$ contains two common vertices. Without loss of generality, we may assume that one vertex is $v_{1,2}$ and the second vertex is $v_{j,j+1}$ $(2 \leq j \leq m)$. Then $r(v_{j+1}|W) = r(v_{j+1,j+2}|W)$, for $2 \leq j < \frac{m+1}{2}$; $r(v_2|W) = r(v_{m,1}|W)$, for $j = \frac{m+1}{2}$; $r(v_1|W) = r(v_{2,3}|W)$, for $j = \frac{m+1}{2} + 1$ and $r(v_j|W) = r(v_{j-1,j}|W)$, for $\frac{m+1}{2} + 1 < j \leq m$, a contradiction.

(2) (2) $W$ contains one common vertex. Without loss of generality, we may assume that one vertex is $v_{1,2}$ and the second vertex is

$v_j$ $(1 \leq j \leq m)$. Then $r(v_{j+1}|W) = r(v_{j+1,j+2}|W)$, for $1 \leq j < \frac{m+1}{2}$; $r(v_1|W) = r(v_{m,1}|W)$, for $j = \frac{m+1}{2}$; $r(v_1|W) = r(v_2|W)$, for $j = \frac{m+1}{2} + 1$ and $r(v_2|W) = r(v_{2,3}|W)$, for $\frac{m+1}{2} + 1 < j \leq m$, a contradiction.

(3) $W$ contains no common vertex. Without loss of generality, we may assume that one vertex is $v_1$ and the second vertex is $v_j$ $(2 \leq j \leq m)$. Then it will lead to a contradiction as in (1).

Now, we will show that $\dim(\mathcal{C}_{m,3}) \leq 3$. Take $W = \{v_1, v_2, v_{\frac{m+1}{2}}\}$. We note that, $r(v_{1,2}|W) = (1, 1, \frac{m-1}{2})$ and

$$r(v_j|W) = \begin{cases} (j, j-1, \frac{m+1}{2} - j + 1), & \text{for } 2 < j < \frac{m+1}{2}, \\ (\frac{m+1}{2}, \frac{m+1}{2}, 2), & \text{for } j = \frac{m+1}{2} + 1, \\ (m - j + 2, m - j + 3, j - \frac{m-1}{2}), & \text{for } \frac{m+1}{2} + 1 < j \leq k, \end{cases}$$

$$r(v_{j,j+1}|W) = \begin{cases} (j, j-1, \frac{m+1}{2} - i), & \text{for } 2 \leq j < \frac{m+1}{2}, \\ (\frac{m+1}{2}, \frac{m-1}{2}, 1), & \text{for } j = \frac{m+1}{2}, \\ (m - j + 1, m - j + 2, j - \frac{m-1}{2}), & \text{for } \frac{m+1}{2} < j \leq k. \end{cases}$$

One can see that all the vertices of $V(\mathcal{C}_{m,3}) - W$ have distinct representations. This implies that $\dim(\mathcal{C}_{m,3}) = 3$ when $m > 3$ is odd. □

The primal graph, prim$(H)$, of a hypergraph $H$ is the graph with vertex set $V$ such that vertices $x$ and $y$ of prim$(H)$ are adjacent if and only if $x$ and $y$ are contained in the same hyperedge. A loop on a vertex in prim$(H)$ will exist if it is the only vertex incident with a hyperedge. The *middle graph*, $M(H)$, of $H$ is a subgraph of prim$(H)$ obtained by deleting loops and parallel edges. Since the adjacencies between the vertices in prim$(H)$ are due to the adjacencies in the hypergraph $H$, so determining the length of a path between two vertices $u$ and $v$ in prim$(H)$ is equivalent to determine the length of a path between the vertices $u$ and $v$ in $H$. This fact yields the following result:

**Theorem.** *Let $H$ be a hypergraph. Then*

$$\dim(H) = \dim(\text{prim}(H)) = \dim(M(H)).$$

The *dual* of $H = (\{v_1, v_2, \ldots, v_n\}, \{E_1, E_2, \ldots, E_m\})$, denoted by $H^*$, is the hypergraph whose vertices are $\{E_1, E_2, \ldots, E_m\}$ corresponding to the hyperedges of $H$ and with hyperedges $V_i = \{E_j : v_i \in E_j \text{ in } H\}$, where $i = 1, 2, \ldots, n$. In other words, the dual $H^*$ swaps the vertices and hyperedges of $H$. The primal graph of the dual $H^*$ of a hypergraph $H$ is not a simple graph, in this case, the middle graph of $H^*$ is a simple graph. Since the dual $H^*$ of a hypergraph $H$ is also a hypergraph so $\dim(H^*) = \dim(M(H^*))$. Moreover, the middle graph of dual $H^*$ of $H$ is (1) a simple path $P_n$ if and only if $H$ is a hyperpath; (2) a simple cycle $C_n$ if and only if $H$ is a hypercycle. In [4], all the simple connected graphs having metric dimension one were characterized by proving the result "dim($G$) is one if and only if $G$ is a simple path $P_n$ ($n \geq 1$)". It is straightforward that the metric dimension of the dual of a hypergraph $H^*$ is 1 if and only if $H$ is a hyperpath. In [6], it was shown that the metric dimension of a simple cycle $C_n$ ($n \geq 3$) is 2, So, the metric dimension of the dual $H^*$ of a hypercycle is 2.

## 3. Partition Dimension of Hypergraphs

Possibly to gain insight into the metric dimension, Chartrand et al. introduced the notion of a resolving partition and partition dimension [3, 2]. To define the partition dimension, the distance $d(v, S)$ between a vertex $v$ in $H$ and $S \subseteq V$ is defined as

$$\min_{s \in S} d(v, s).$$

Let $\Pi = \{S_1, S_2, \ldots, S_t\}$ be an ordered $t$-partition of $V$ and $v$ be any vertex of $H$. Then the *representation*, $r(v|\Pi)$, of $v$ with respect $\Pi$ is the $t$-tuple $r(v|\Pi) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_t))$. The partition $\Pi$ is called a *resolving partition* for a hypergraph if $r(v|\Pi) \neq r(u|\Pi)$ for any two distinct vertices $v, u \in V$. The *partition dimension* of a hypergraph $H$ is the cardinality of a minimum resolving partition, denoted by $\mathrm{pd}(H)$.

From the definition of a resolving partition, it can be observed that the property of a given partition $\Pi$ of a hypergraph $H$ to be a resolving partition of $H$ can be verified by investigating the pairs of vertices in the same class. Indeed, $d(x, S_i) = 0$ for every vertex $x \in S_i$ but $d(x, S_j) \neq 0$ with $j \neq i$. It follows that $x \in S_i$ and $y \in S_j$ are resolved either by $S_i$ or $S_j$ for every $i \neq j$. From Proposition 2.2, we have the following lemma:

**Lemma.** *Let $\Pi$ be a resolving partition of $V$. If $u, v \in \mathcal{C}(i_1, i_2, \ldots, i_d)$, then $u$ and $v$ belong to distinct classes of $\Pi$.*

The following result gives the lower bound for the partition dimension of hypergraphs.

**Proposition.** *Let $H$ be a Sperner hypergraph with $m \geq 2$ hyperedges. Then $\mathrm{pd}(H) \geq \lambda + 1$, where $\lambda = \max |\mathcal{C}(i_1, i_2, \ldots, i_d)|$ in $H$.*

*Proof.* Since $\lambda = \max |\mathcal{C}(i_1, i_2, \ldots, i_d)|$ in $H$, by Lemma 3.1, we have at least $\lambda$ disjoint classes in any resolving partition $S_1, S_2, \ldots, S_\lambda$ of $V$. Since $H$ is Sperner, there exists an edge $E$ of $H$ such that $\mathcal{C}(i_1, i_2, \ldots, i_d) \subset E$. If $u \in E - \mathcal{C}(i_1, i_2, \ldots, i_d)$, $v \in \mathcal{C}(i_1, i_2, \ldots, i_d)$ and $u, v \in S_i$ for some $1 \leq i \leq \lambda$, then $r(u|\Pi) = r(v|\Pi)$ which is a contradiction. Thus $\mathrm{pd}(H) \geq \lambda + 1$. □

The lower bound given in Proposition 3.2 is attainable for $k$-uniform linear hyperpaths as proved in Theorem 3.4.

A 2-uniform hypercycle $\mathcal{C}_{m,2}$ is a simple connected cycle on $n$ vertices and it was shown that the partition dimension of a simple connected cycle is 3 [2], so $\text{pd}(\mathcal{C}_{m,2}) = 3$. In the next result, we investigate the partition dimension of a $k$-uniform hypercycle $\mathcal{C}_{m,k}$ for $m \geq 3$ and $k \geq 4$.

**Theorem.** *For $k \geq 4$, let $\mathcal{C}_{m,k}$ be a $k$-uniform linear hypercycle with $m \geq 3$ hyperedges. Then* $\text{pd}(\mathcal{C}_{m,k}) = k$.

*Proof.* Firstly, we discuss the case when $k = 3$. In this case, let $v_j \in E_j$ be a vertex of degree one and $v_{j,j+1} \in E_j \cap E_{j+1}$ with $v_{m,m+1} = v_{m,1}$ be a vertex of 2 in $\mathcal{C}_{m,3}$. If we put all the vertices of $\mathcal{C}_{m,3}$ into two classes $S_1$ and $S_2$, then they do not form a resolving partition $\Pi$ of $V(\mathcal{C}_{m,3})$, because for some hyperedge $E_j \in \mathcal{H}$ such that $E_i \cap S_1 \neq \phi$, $E_j \cap S_2 \neq \phi$ and $u, v \in S_i$, $r(u|\Pi) = r(v|\Pi)$, which is a contradiction. Thus, $\text{pd}(H) \geq 3$. On the other hand, $\text{pd}(\mathcal{C}_{m,3}) \leq 3$, because we have a resolving partition of cardinality 3 for $(\mathcal{C}_{m,3})$ in each of the following cases:

For $m \equiv 0 \pmod 6$, we have a resolving partition for $\text{pd}(\mathcal{C}_{m,3})$ as

$$\Pi = \left\{\left\{v_{m,1}, \ldots, v_{\frac{1}{3}m,\frac{1}{3}m+1}\right\}, \left\{v_{\frac{1}{3}m+1}, \ldots, v_{\frac{2}{3}m,\frac{2}{3}m+1}\right\}, \left\{v_{\frac{2}{3}m+1}, \ldots, v_m\right\}\right\}.$$

For $m \equiv 1, 4 \pmod 6$, we have a resolving partition for $\text{pd}(\mathcal{C}_{m,3})$ as

$$\Pi = \left\{\left\{v_{m,1}, \ldots, v_{\frac{1}{3}(m+2)}\right\}, \left\{v_{\frac{1}{3}(m+2),\frac{1}{3}(m+2)+1}, \ldots, v_{\frac{2}{3}(m+2)-1,\frac{2}{3}(m+2)}\right\}, \left\{v_{\frac{2}{3}(m+2)}, \ldots, v_m\right\}\right\}.$$

For $m \equiv 2 \pmod 6$, we have a resolving partition for $\text{pd}(\mathcal{C}_{m,3})$ as

$$\Pi = \left\{\left\{v_{m,1}, \ldots, v_{\frac{1}{3}(m+1)}\right\}, \left\{v_{\frac{1}{3}(m+1),\frac{1}{3}(m+1)+1}, \ldots, v_{\frac{2}{3}(m+1)-1,\frac{2}{3}(m+1)}\right\}, \left\{v_{\frac{2}{3}(m+1)}, \ldots, v_m\right\}\right\}.$$

For $m \equiv 3 \pmod 6$, we have a resolving partition for $\text{pd}(\mathcal{C}_{m,3})$ as

$$\Pi = \left\{\left\{v_1, \ldots, v_{\frac{1}{3}m+1}\right\}, \left\{v_{\frac{1}{3}m+1,\frac{1}{3}m+2}, \ldots, v_{\frac{2}{3}m+1}\right\}, \left\{v_{\frac{2}{3}m+1,\frac{2}{3}m+2}, \ldots, v_{m,1}\right\}\right\}.$$

For $m \equiv 5 \pmod 6$, we have a resolving partition for $\text{pd}(\mathcal{C}_{m,3})$ as

$$\Pi = \left\{\left\{v_1, \ldots, v_{\frac{1}{3}(m+1),\frac{1}{3}(m+1)+1}\right\}, \left\{v_{\frac{1}{3}(m+1)+1}, \ldots, v_{\frac{2}{3}(m+1)}\right\}, \left\{v_{\frac{2}{3}(m+1),\frac{2}{3}(m+1)+1}, \ldots, v_{m,1}\right\}\right\}.$$

Now, for $k \geq 4$, let us denote the vertices of degree one in $j^{\text{th}}$ hyperedge $E_j$ $(1 \leq j \leq m)$ by $v_1^j, v_2^j, \ldots, v_{k-2}^j$, and let the vertex $v_{j,j+1} \in E_j \cap E_{j+1}$ be of degree 2 in $\mathcal{C}_{m,k}$. Construct a partition $\Pi$ with $k$ parts of the vertex set of $\mathcal{C}_{m,k}$ as follows: Copy the resolving partition for $\mathcal{C}_{m,3}$ and for the other vertices that are in the middle of hyperedges, put $i^{\text{th}}$ vertex (for $i > 1$) in the $(i+2)^{\text{th}}$ part. Then it is an easy exercise to verify, by following Lemma 3.1 and Proposition 3.2, that $\Pi$ is a minimum resolving partition for $\mathcal{C}_{m,3}$, and it completes the proof. □

Every 2-uniform linear hyperpath is a simple path graph whose partition dimension is 2 as shown in [3]. Now, we generalize this result for more than two uniform linear hyperpaths as follows.

**Theorem.** *For $k \geq 3$, let $H$ be a $k$-uniform linear hyperpath with $m$ hyperedges. Then* $\text{pd}(H) = k$.

*Proof.* Let $H$ be a $k$-uniform linear hyperpath. Then it is a routine exercise to verify that a partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V$, where each $S_i$, $1 \leq i \leq k-1$, contains the $i^{\text{th}}$ vertex of every hyperedge of $H$ and $S_k$ contains the $k^{\text{th}}$ vertex of the $m^{\text{th}}$ hyperedge, is a resolving partition for $H$. It follows that $\text{pd}(H) \leq k$.

Since $H$ is Sperner and $\max |C(i_1, i_2, \ldots, i_d)| = k-1$ in $H$, so Proposition 3.2 implies that $\text{pd}(H) \geq k$. □

The *rank* of a hypergraph $H$ is the maximum number of vertices in a hyperedge. One might think that the partition dimension of $H$ is always greater than or equal to the rank of $H$. This is true for a $k$-uniform linear hyperpath and a $k$-uniform linear hypercycle $\mathcal{C}_{m,3}$. But, in general, it is not true as shown in the following example:

**Example.** *Let $H$ be a hypergraph with vertex set $V = \{v_i : \ 1 \leq i \leq 11\}$ and edge set $\mathcal{H} = \{E_1, E_2\}$, where $E_1 = \{v_i;\ 1 \leq i \leq 7\}$ and $E_2 = \{v_i;\ 6 \leq i \leq 11\}$. Clearly, $rank(H) = 7$, $\lambda = 5$ and*

$$\Pi = \{S_i = \{v_i, v_{i+5}\};\ 1 \leq i \leq 5, S_6 = \{v_{11}\}\}$$

*is a minimum resolving partition of V. This implies that* $\text{pd}(H) = 6 \neq$ *$rank(H)$.*

Likewise the results on the metric dimension of the primal, we have the following result on the partition dimension of the primal graph of a hypergraph.

**Theorem.** *Let $H$ be a hypergraph. Then* $\text{pd}(H) = \text{pd}(\text{prim}(H))$.

Let $H^*$ be the dual of a hypergraph $H$, since $H^*$ is also a hypergraph, so $\text{pd}(H^*) = \text{pd}(M(H^*))$. Since it was shown that the simple paths $P_n$ are the only graphs with $\text{pd}(P_n) = 2$ [3] and the partition dimension of the simple cycles $C_n$ is 3. Therefore, partition dimension of dual $H^*$ of a hyperpath is

2 and partition dimension of dual $H^*$ of a hypercycle is 3.

## References


1. I. M. Pelayo C. Seara C. Hernando, M. Mora and D. R. Wood, *Extremal graph theory for metric dimension and diameter*, Elect. Notes in Disc. Math. **29** (2007), 339–343.
2. E. Salehi G. Chartrand and P. Zhang, *On the partition dimension of a graph*, Congr. Numer. **130** (1998), 157–168.
3. ______, *The partition dimension of a graph*, Aequationes Math. **59** (2000), 45–54.
4. M. A. Johnson G. Chartrand, L. Eroh and O. R. Oellermann, *Resolvability in graphs and the metric dimension of a graph*, Disc. Appl. Math. **105** (2000), 99–113.
5. M. R. Garey and D. S. Johnson, *Computers and intractability: A guide to the theory of np-completeness*, Freeman, W H., San Francisco (1979).
6. F. Harary and R. A. Melter, *On the metric dimension of a graph*, Ars Combin. **2** (1976), 191–195.
7. M. A. Chaudhry I. Javaid, M. Salman and S. Shokat, *Fault-tolerance in resolvability*, Utilitas Math. **80** (2009), 263–275.
8. M. T. Rahim I. Javaid and K. Ali, *Families of regular graphs with constant metric dimension*, Utilitas Math. **75** (2008), 21–33.
9. M. Mora I. M. Pelayo M. L. Puertas C. Seara J. Caceres, C. Hernando and D. R. Wood, *On the metric dimension of cartesian product of graphs*, SIAM J. of Disc. Math. **21** (2007), no. 2, 423–441.
10. I. Javaid M. A. Chaudhry and M. Salman, *Fault-tolerant metric and partition dimension of graphs*, Utilitas Math. **83** (2010), 187–199.
11. I. Javaid M. Salman and M. A. Chaudhry, *Resolvability in circulant graphs*, Acta. Math. Sinica, Englich Series. **28** (2012), no. 9, 1851–1864.
12. I. Javaid S. Ahmad, M. A. Chaudhry and M. Salman, *On the metric dimension of generalized petersen graphs*, Quaestiones Mathematicae **36** (2013), 421–435.
13. B. Raghavachari S. Khuller and A. Rosenfeld, *Landmarks in graphs*, Disc. Appl. Math. **70** (1996), 217–229.
14. P. J. Slater, *Leaves of trees*, Congr. Numer. **14** (1975), 549–559.



Center for advanced studies in Pure and Applied Mathematics, Bahauddin Zakariya University Multan, Pakistan
*E-mail address*: ijavaidbzu@gmail.com

Department of Mathematics, College of Science, Jazan University, Jazan, Saudi Arabia
*E-mail address*: aahaider@jazanu.edu.sa

Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, 63100, Pakistan
*E-mail address*: muhammad.salman@iub.edu.pk

Center for advanced studies in Pure and Applied Mathematics, Bahauddin Zakariya University Multan, Pakistan
*E-mail address*: mishi_abi@hotmail.com