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FACTORIZATIONS OF COMPLETE GRAPHS INTO CYCLES AND 1–FACTORS

UĞUR ODABAŞI

ABSTRACT. In this paper, we consider factorizations of complete graph K_v into cycles and 1–factors. We will focus on the existence of factorizations of K_v containing two nonisomorphic factors. We obtain all possible solutions for uniform factors involving m–cycles and 1–factors with a few possible exceptions when m is odd.

1. INTRODUCTION

In this paper, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of a graph G, respectively. Also, we denote by $K_{a\times b}$ a *complete equipartite graph* having a parts of size b each. In particular, $K_{2\times a}$ is called a *complete bipartite graph* and denoted by $K_{a,a}$ as well.

Given two graphs G and H, an H*–decomposition* of G is a set $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ of edge-disjoint subgraphs of G such that $\bigcup_{i=1}^{k} E(H_i) = E(G)$ and $H_i \cong H$ for all $H_i \in \mathcal{H}$.

A *factor* in a graph G is a spanning subgraph of G. An $\{F_1^{k_1}, F_2^{k_2}, \ldots, F_l^{k_l}\}$ *–factorization* of a graph G is a decomposition which consists precisely of k_i factors isomorphic to F_i. If every component of a factor is isomorphic to the same graph, then the factor is said to be *uniform*. A factorization of G is also known as a *resolvable decomposition* of G and a factor can be called a *parallel class* of G.

The case where $G \cong K_v$ (or $G \cong K_v - I$, where I is a 1–factor of K_v and v is even) and $F_i \cong F$ for all $1 \leq i \leq l$ is known as the *Oberwolfach problem.* If F consists of k_i m_i–cycles, $1 \leq i \leq t$, then the corresponding Oberwolfach problem is denoted by $OP(m_1^{k_1}, m_2^{k_2}, \ldots, m_t^{k_t})$. It is known that the solutions to the cases $OP(3^2), OP(3^4), OP(4,5)$, and $OP(3^2,5)$ do not exist [3, 16, 21]. The Oberwolfach problem for a single cycle size $OP(m^k)$ for all $m \geq 3$ has been solved in two separate cases: odd cycles in [3] and the even cycle case in [15].

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A generalization of the Oberwolfach problem is the *Hamilton–Waterloo problem* which asks for an $\{F_1^{k_1}, F_2^{k_2}\}$–factorization of K_v(or $K_v - I$ for even v) where F_1 and F_2 are nonisomorphic.

When all of the parallel classes in the decomposition are uniform we have a uniformly resolvable decomposition of K_v(or $K_v - I$ for even v) and we use the notation to denote such a decomposition with r_i F_i–factors by $\mathrm{URD}(v; F_1^{r_1}, F_2^{r_2})$. We denote the decomposition by $\mathrm{URD}(v; K_2^{r_1}, C_m^{r_2})$, while if F_1 is a 1–factor of K_v and F_2 contains only copies of cycles C_m. We will denote such a decomposition also as a $\{K_2^{r_1}, C_m^{r_2}\}$–factorization. Moreover, if each F_i is composed of m_i–cycles, then we use the notation to denote such a decomposition by $\mathrm{URD}(v; m_1^{r_1}, m_2^{r_2})$.

The first results addressing the Hamilton–Waterloo problem [1] settled the problem for all $v \leq 17$ and in addition considered the cases $(m_1, m_2) \in \{(4,6),(4,8),\ (4,16),(8,16),(3,5),(3,15),(5,15)\}$. In [7], Bryant et al. settled the Hamilton–Waterloo problem for bipartite 2–factors, and in [9] Buratti and Rinaldi studied regular 2–factorizations leading to some cyclic solutions to Oberwolfach and Hamilton–Waterloo problems, and also in [8], an infinite class of cyclic solutions to the Hamilton–Waterloo problem is given. El-Zanati et al. [13] have considered the problem for $m_i = p^i$ and $v = p^n$ where p is an odd prime and $1 \leq i \leq n$. In [19], the problem has been solved for 4–cycles and odd cycle factors with a few possible exceptions. In a recent paper [10], Burgess et al. solve almost completely the Hamilton–Waterloo problem for odd cycles. For more recent results we refer the reader to [6, 11, 12, 24, 25].

In this paper, we seek factorizations of K_v into 1–factors and C_m–factors. While doing this we also get new solutions for the Hamilton–Waterloo problem regarding cycles with different parity.

Problem 1.1. *For which values of r (or s) and m does there exist a $\{K_2^r, C_m^s\}$–factorization of K_v?*

It is not hard to verify that if K_v has an C_m–factor, then m must divide v and v must be even since it has a 1–factor. Also note that by counting the degree of any fixed vertex in the factors, we have $r + 2s = v - 1$. In the following lemma, we summarize the obvious necessary conditions for the existence of a $\{K_2^r, C_m^s\}$–factorization of K_v.

Lemma 1.2. *Let v, m, r, and s be nonnegative integers with $m \geq 3$. If there exists a solution to* $\mathrm{URD}(v; K_2^r, C_m^s)$*, then*

- *v is even;*
- *$m|v$;*
- *$r + 2s = v - 1$.*

If m is also even, then we have the desired factorizations since K_v has a $\{C_m^{(v-2)/2}, K_2\}$–factorization (a solution to the Oberwolfach problem [15]) and each C_m–factor in the factorization has a 1–factorization. Thus we restrict our attention to the case when m is odd. In [23], Rees considered the problem for the case $m = 3$ and proved that the obvious necessary conditions are sufficient also with exceptions $(v, r, s) = (6, 1, 2)$ and $(v, r, s) = (12, 1, 5)$ which correspond to the nonexistence of solutions to $OP(3^2)$ and $OP(3^4)$, respectively. Also, Adams et al. [2] solved the problem completely in the case of $m = 5$.

2. Preliminaries

In [17], the Oberwolfach problem is considered for complete equipartite graphs where all cycles have the same length and we will use this result in our main construction.

Theorem 2.1 ([17])**.** *The complete equipartite graph $K_{a\times b}$ has a C_l–factorization for $l \geq 3$ and $b \geq 2$ if and only if $l|ab$, $b(a-1)$ is even, l is even if $a = 2$, and $(a, b, l) \neq (3, 2, 3), (3, 6, 3), (6, 2, 3), (2, 6, 6)$.*

Let H be a finite additive group and let S be a subset of $H - \{0\}$ such that the negative of every element of S also belongs to S. The *Cayley graph* on H with connection set S, denoted by $\mathrm{Cay}(H, S)$, is the graph with vertex set H and edge set $E(\mathrm{Cay}(H, S)) = \{(a, b)|a, b \in H, a - b \in S\}$. We will make use of the following theorem.

Theorem 2.2 ([5])**.** *Any connected 4-regular Cayley graph on a finite Abelian group has a Hamilton cycle decomposition.*

Let G be a graph and $G_0, G_1, \dots, G_{k-1}$ be k vertex disjoint copies of G with $v_i \in V(G_i)$ for each $v \in V(G)$. Let $G[k]$ denote the graph with vertex set $V(G[k]) = V(G_0)\cup V(G_1)\cup\cdots\cup V(G_{k-1})$ and edge set $E(G[k]) = \{u_iv_j : uv \in E(G) \quad \text{and} \quad 0 \leq i, j \leq k-1\}$. For example $K_m[2] \cong K_{2m} - I$ and $K_2[m] \cong K_{m,m}$ where I is a 1–factor of K_{2m}.

It is easy to see that if a graph G has an H–decomposition, then there exists an $H[k]$–decomposition of $G[k]$. Moreover if a graph G has an H–factorization, then there exists an $H[k]$–factorization of $G[k]$.

In fact, this graph operation is a generalization of Häggkvist's *doubling construction* and it coincides with a special case of a graph product called the lexicographic product. Häggkvist [14] constructed 2–factorizations containing even cycles using $G[2]$.

Lemma 2.3. [14] *Let G be a path or a cycle with m edges and let H be a 2–regular graph on $2m$ vertices where each component of H is a cycle of even length. Then $G[2]$ has an H–decomposition.*

Baranyai and Szasz [4] have shown that if a graph G can be decomposed into x Hamilton cycles and if H is a graph with y vertices and can be decomposed into z Hamilton cycles then their lexicographic product is decomposable into $xy+z$ Hamilton cycles. So, $C_m[n]$ has a C_{mn}–factorization. Also Alspach et al. [3] have shown that for an odd integer m and a prime p with $3 \leq m \leq p$, $C_m[p]$ has a C_p–factorization.

In [19, 20], the authors decomposed $C_m[4]$ into 2–factors involving cycles of lengths 4, m, $2m$, and $4m$. Burgess et al. have recently shown the following result in [10].

Theorem 2.4 ([10])**.** *For all odd integers $a \geq b \geq 3$, $C_b[a]$ has a C_a–factorization.*

The following result can be found in [3] and will be used to improve the main result of this paper.

Theorem 2.5 ([3])**.** *Let v be a positive integer with $v \notin \{1, 2, 4, 6, 7, 11, 12\}$. Then there is a 2–factorization of K_v (v odd) or $K_v - I$ (v even) such that each cycle in each 2–factor is either a 3–cycle or a 5–cycle.*

The *ring-sum* $G_1 \oplus G_2$ of two graphs $G_1 = (V_1, E_1)$, and $G_2 = (V_2, E_2)$, is the graph $G_1 \oplus G_2 = ((V_1 \cup V_2), (E_1 \cup E_2) - (E_1 \cap E_2))$. The *union* of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Also αG will denote the vertex disjoint union of α copies of G.

3. Preliminary Decompositions

First we will give two well-known results of Walecki [18] for Hamilton cycle decompositions of complete graph of odd order, or complete graph of even order minus a 1–factor, then by using these results we will obtain solutions when $v = 2m$ which will be generalized in Section 4.

Lemma 3.1 ([18])**.** *For all odd $m \geq 3$, K_m has a Hamilton decomposition with prescribed cycles $\{C^*, \rho(C^*), \rho^2(C^*), \ldots, \rho^{\frac{m-3}{2}}(C^*)\}$ for the permutation $\rho = (0)(2, 4, 6 \ldots, m-1, m-2, \ldots, 5, 3, 1)$ where $C^* = (0, 1, 2, \ldots, m-1)$.*

Lemma 3.2 ([18])**.** *For all even $m \geq 4$, $K_m - I^*$ has a Hamilton decomposition with prescribed cycles $\{C^*, \sigma(C^*), \sigma^2(C^*) \ldots, \sigma^{\frac{m-4}{2}}(C^*)\}$ for some permutation σ of $\{0, 1, \ldots, m-1\}$ where $C^* = (0, 1, \ldots, m-1)$ and $E(I^*) = \{(0, m/2), (i, m-i) : 1 \leq i \leq (m/2) - 1\}$.*

For the sake of brevity, we use C^* and ρ to denote the m–cycle of K_m and the permutation, respectively, as described in Lemma 3.1.

As we noted before, $K_{2m} - I \cong K_m[2]$ where $V(K_{2m}) = V(K_m[2])$. Also, by Lemma 3.1, $K_m[2]$ has a decomposition into graphs of the form $C^*[2]$ for odd m. In [22], Piotrowski showed that, when m is odd, the double of any m–cycle cannot be decomposed into m–cycle factors; that is, $C^*[2] \ncong 2C_m \oplus 2C_m$. However, by the following lemma, we will be able to decompose $K_m[2]$ into m–cycle factors and 1–factors via switching some edges of each $C^*[2]$ in $K_m[2]$ with some edges of $\rho(C^*)[2]$. Also, for brevity, we use Γ to denote $(C^* \oplus \rho(C^*))[2]$.

Lemma 3.3. *Let m be an integer with $m \geq 4$. Then Γ has an m–cycle factorization.*

Proof. If m is even, the result follows from Lemma 2.3 since $\Gamma = C^*[2] \oplus \rho(C^*)[2]$. So we may assume m is odd.

Let the vertex set of Γ be $\mathbb{Z}_2 \times \mathbb{Z}_m$, and define two m–cycles in Γ as follows: $C^* = (v_0, v_1, \ldots, v_{m-1})$ where $v_i = (0, i)$ and $C^*_{(0)} = (v'_0, v'_1, \ldots, v'_{m-1})$ where $v'_0 = (0, 0)$ and

$$v'_i = \begin{cases} (1, i+1), & \text{if } i \text{ is odd} \\ (0, i-1), & \text{if } i \text{ is even} \end{cases}$$

for $1 \leq i \leq m-1$ and when $m = 7$, $v'_5 = (0, 5)$, $v'_6 = (1, 6)$. Then

$$\begin{aligned} F_1 &= C^* \cup (C^* + (1, 0)) \\ F_2 &= \rho'(F_1) \\ F_3 &= C^*_{(0)} \cup (C^*_{(0)} + (1, 0)) \\ F_4 &= \Gamma - (F_1 \oplus F_2 \oplus F_3) \end{aligned}$$

are m–cycle factors of Γ where $\rho'(i, j) = (i, \rho(j))$ for $0 \leq j \leq m-2$ and $\rho'(i, m-1) = (i+1, \rho(m-1))$. It can be checked that

$$\mathcal{F} = \{F_1, F_2, F_3, F_4\}$$

is an m–cycle factorization of Γ. □

We will also make use of the following lemma which will be very useful in proving the main result of this paper.

Lemma 3.4. *Let m be an integer with $m \geq 3$. Then $C_m[2]$ has a $\{C^r_m, C^s_{2m}\}$-factorization for nonnegative integers r and s with $r + s = 2$ except when m is odd and $r = 2$, and except possibly when m is even and $r = 1$.*

Proof. When m is an even integer, the required decompositions exist by Lemma 2.3. Now we may assume that m is an odd integer. We can represent $C_m[2]$ as the Cayley graph over $\mathbb{Z}_2 \times \mathbb{Z}_m$ with the connection set $\mathbb{Z}_2 \times \{1, -1\}$. Let $C = (v_0, v_1, \ldots, v_{m-1})$ and $C' = (u_0, u_1, \ldots, u_{2m-1})$ be cycles of $C_m[2]$ where $v_i = (0, i)$ for $0 \leq i \leq m-1$ and

$$u_i = \begin{cases} (0, i), & \text{if } i \text{ is even} \\ (1, i), & \text{if } i \text{ is odd} \end{cases}$$

for $0 \leq i \leq 2m-1$. It can be checked that $F_1 = C \cup (C + (1, 0))$ and $F_2 = C'$ are edge disjoint m–cycle and $2m$–cycle factors of $C_m[2]$, respectively. Thus $\{F_1, F_2\}$ is a 2–factorization of $C_m[2]$ for $r = 1$. As noted before, there is no m–cycle factorization of $C_m[2]$. For $r = 0$, since $C_m[2]$ is a connected 4–regular Cayley graph, by Theorem 2.2, $C_m[2]$ can be decomposed into two C_{2m}, which are Hamilton cycles and this completes the proof. □

Combining the results of Lemma 3.3 and 3.4, we now obtain the following corollary.

Corollary 3.5. *Let $r, s \in \{0,1,2,3,4\}$ with $r+s=4$. Then for each integer $k \geq 0$ and $m \geq 4$, $\rho^k(\Gamma)$ has a $\{C_m^r, C_{2m}^s\}$–factorization with a possible exception $r=3$ when m is odd.*

Proof. First, we will prove the corollary for $k=0$, then state that for all $k \geq 0$ the graph has the required decomposition.

When $r=4$, the corollary follows from Lemma 3.3. By Lemma 3.4, Γ has a $\{C_m^r, C_{2m}^s\}$–factorization for $r=0,1,2$ and also for $r=3$ when m is even since $\Gamma \cong C^*[2] \oplus \rho(C^*)[2]$. Moreover, since $\Gamma \cong \rho^k(\Gamma)$, for all $k \geq 0$, the graph $\rho^k(\Gamma)$ has the required decompositions. □

For even $m \geq 4$, since K_m has a $\{C_m^{(m-4)/2}, C^* \oplus I^*\}$–factorization by Lemma 3.2, $K_m[2]$ has a $\{(C_m[2])^{(m-4)/2}, (C^* \oplus I^*)[2]\}$–factorization where I^* is described as in Lemma 3.2. Also, since $I^*[2]$ does not contain any m or $2m$–cycle for $m > 4$, we will use edge-disjoint union of $I^*[2]$ and $C^*[2]$. Now we give 2–factorizations of $(C^* \oplus I^*)[2]$ in the following lemma.

Lemma 3.6. *Let $m \geq 4$ be an even integer and $G = C^* \oplus I^*$ where $C^* = (0,1,\ldots,m-1)$ is an m–cycle and I^* is a 1–factor of K_m with $E(I^*) = \{(0,m/2),(i,m-i) : 1 \leq i \leq (m/2)-1\}$. Then $G[2]$ has a*

(i) *C_{2m}–factorization,*
(ii) *C_m–factorization when $m \equiv 0 \pmod 4$, and*
(iii) *$\{C_m^2, C_{2m}^1\}$–factorization when $m \equiv 2 \pmod 4$.*

Proof. In [20], it is shown that the graph G has a C_{2m}–factorization. Let the vertex set of G be $\mathbb{Z}_2 \times \mathbb{Z}_m$, and define two cycles in G as follows: $C = (v_0, v_1, \ldots, v_{m-1})$ where $v_i = (0,i)$ for $0 \leq i \leq m-1$ and $C' = (u_0, u_1, \ldots, u_{m-1})$ where $u_0 = (0,0)$ and for $1 \leq i \leq m-1$,

$$u_i = \begin{cases} (\frac{1-(-1)^i}{2}, \frac{m}{2} - \lfloor \frac{i}{2} \rfloor), & \text{for } i \equiv 1,2 \pmod 4 \\ (\frac{1-(-1)^i}{2}, \frac{m}{2} + \lfloor \frac{i}{2} \rfloor), & \text{for } i \equiv 0,3 \pmod 4 . \end{cases}$$

Then $F_1 = C \cup (C + (1,0))$ and $F_2 = C' \cup (C' + (1,0))$ are two edge-disjoint m–cycle factors in $G[2]$. Also it can be checked that $F_3 \cong G - (F_1 \oplus F_2)$ is a C_m–factor in $G[2]$ when $m \equiv 0 \pmod 4$ or a C_{2m}–factor in $G[2]$ when $m \equiv 2 \pmod 4$. Then $\{F_1, F_2, F_3\}$ is a C_m–factorization of $G[2]$ when $m \equiv 0 \pmod 4$ or $\{C_m^2, C_{2m}^1\}$–factorization of $G[2]$ when $m \equiv 2 \pmod 4$. □

Now we can give new solutions to the Hamilton–Waterloo problem for the case of $v = 2m$.

Lemma 3.7. *Let m be an integer with $m \geq 3$. Then there exist a* URD$(2m; m^r, (2m)^s)$ *for all nonnegative integers r and s such that $r+s=m-1$ except when $m=3$ and $r=2$.*

Proof. Since the problem has a solution for $s=0$ in [15], we may assume that $s \geq 1$. Note that $K_{2m} - I \cong K_m[2]$ where I is a 1–factor in K_{2m}. When m is even, $K_m[2]$ has a $\{(C_m[2])^{(m-4)/2}, (C^* \oplus I^*)[2]\}$–factorization by Lemma 3.2. Let r_i and s_i be nonnegative integers for $i = 1, 2$ with $r_1 + s_1 = (m-4)/2$ and $r_2 + s_2 = 3$ where $r_2 = 0$ or

$$r_2 = \begin{cases} 3, & \text{for } m \equiv 0 \pmod 4 \\ 2, & \text{for } m \equiv 2 \pmod 4 . \end{cases}$$

Then, decomposing $(m-4)/2$ many $C_m[2]$–factors of K_{2m} into $\{C_m^{r_1}, C_{2m}^{s_1}\}$–factors by Lemma 3.4 and $(C^* \oplus I^*)[2]$ into $\{C_m^{r_2}, C_{2m}^{s_2}\}$–factors by Lemma 3.6 gives us a $\{C_m^r, C_{2m}^s\}$–factorization of $K_{2m} - I$ where $r = r_1 + r_2$ and $s = s_1 + s_2$ satisfying $r + s = m - 1$ with $1 \le r, s \le m - 1$ and this completes the proof for even m. So we may assume m is odd.

Let r and s be nonnegative integers and m be an odd integer such that $r + 2s = 2m - 1$. It is well-known that $OP(3^2)$ has no solution, thus we may assume $(m, r) \neq (3, 1)$. It is also clear that the cases $r = 1$ and $r = 2m - 1$ correspond to $OP(m^2)$ which has a solution, [3], and well-known 1–factorization of K_{2m}, [18], respectively.

By Lemma 2.3, K_m has a decomposition into prescribed cycles $\rho^k(C^*)$ for $0 \le k \le (m-3)/2$. Also, since $K_{2m} \cong K_m[2] \oplus K_2$, K_{2m} has a decomposition into a K_2–factor and $(m-1)/2$ factors isomorphic to $C^*[2]$.

We will prove the theorem in two cases; $m \equiv 1$ or $3 \pmod 4$.

Case 1: $m \equiv 1 \pmod 4$.

By pairing up consecutive graphs of the form $\rho^k(C^*[2])$ in the decomposition of $K_m[2]$, we can obtain a $\{\Gamma^{\frac{m-1}{4}}, K_2\}$–decomposition of K_{2m}. Now, let I be a 1–factor in K_{2m}, and r_i's be nonnegative integers for $i = 0, 1, 2$, and 4 with $\sum_{i=1(i\neq 3)}^{4} r_i = (m-1)/4$. Placing a $\{C_m^i, C_{2m}^{4-i}\}$–factorization r_i of the Γ's by Corollary 3.5, gives us a $\{C_m^r, C_{2m}^s\}$–factorization of $K_{2m} - I$ where $r = \sum_{i=1(i\neq 3)}^{4} i r_i$ and $r + s = m - 1$. Then, since any nonnegative integer can be written as $r = \sum_{i=1(i\neq 3)}^{4} i r_i$ and $r+s = m-1$ for nonnegative integers r_i $(0 \le i \neq 3 \le 4)$, a solution to $\mathrm{URD}(2m; m^r, (2m)^s)$ exists for any r satisfying $r+s = m-1$.

Case 2: $m \equiv 3 \pmod 4$.

Similarly, by pairing up the consecutive graphs $\rho^k(C^*[2])$ in the decomposition of $K_m[2]$, we can obtain a $\{\Gamma^{\frac{m-3}{4}}, C^*[2]\}$–decomposition of $K_{2m} - I$. Now, let r_i be nonnegative integer with $\sum_{i=1(i\neq 3)}^{4} r_i = (m-3)/4$ and $(x, y) \in \{(0, 2), (1, 1)\}$. Decomposing $(m-3)/4$ Γ's into $\{C_m^i, C_{2m}^{4-i}\}$–factors by Lemma 3.3 and $C^*[2]$ into a $\{C_m^x, C_{2m}^y\}$–factor by Lemma 3.4 gives us a $\{C_m^r, C_{2m}^s\}$-factorization of $K_{2m} - I$ where $r = \sum_{i=1(i\neq 3)}^{4} i r_i + x$ and $r + s = m - 1$. Thus the result now follows. □

4. Conclusions

In this section, we will combine our results to give general solutions to our problem.

Theorem 4.1. *For every positive integer* $v \equiv 0 \pmod 4$ *and for all nonnegative integers* r, s, *and odd* m *with* $r+2s = v-1$, *there exists a solution to* $\mathrm{URD}(v; K_2^r, C_m^s)$.

Proof. In Theorem 14 of [19], for all possible integers r_1, s, and $m \geq 3$, a solution to $\mathrm{URD}(v; C_4^{r_1}, C_m^s)$ has been given with the possible exception when $r_1 = 2$ and $v = 8m$ for $m \geq 5$. It is easy to see that every C_4–factor in the factorization can be decomposed into two 1–factor. Thus, it remains to show that the problem has a solution when $r = 5$ and $v = 8m$. It is obvious that K_{8m} has a $\{K_{2m}, K_{4\times(2m)}\}$–factorization. Now, decomposing the K_{2m} into a $\{K_2^5, C_m^{m-3}\}$–factor by Lemma 3.7 and $K_{4\times(2m)}$ into $3m$ C_m–factor by Theorem 2.1 completes the proof. □

We now consider the cases when $v \equiv 2 \pmod 4$ and $m|v$. Thus, there exists an odd $t \in \mathbb{Z}^+$ such that $v = 2mt$. Also note that

$$K_{2mt} \cong (K_{2t} - I)[m] \oplus tK_{2m}. \tag{4.1}$$

Theorem 4.2. *For all nonnegative integers* r, s, *and odd integers* m, t *with* $r + 2s = 2mt - 1$, *there exists a solution to* $\mathrm{URD}(2mt; K_2^r, C_m^s)$ *except possibly when* $7 \leq m \leq t-4$, $(m-1)t < s < m(t-1)$ *and* t *is not divisible by* 3 *or* 5.

Proof. First we assume that $1 \leq t \leq m$. Also let $0 \leq r_1, s_1 \leq t-1$, and $0 \leq r_2, s_2 \leq m-1$ be integers with $r_1 + s_1 = t-1$ and $r_2 + s_2 = m-1$. By Lemma 3.7, $K_{2t} - I$ and K_{2m} has a $\{C_t^{s_1}, C_{2t}^{r_1}\}$ and $\{C_m^{s_2}, C_{2m}^{r_2}, K_2\}$–factorization, respectively, except when $(t, s_1) = (3, 2)$. So we have a decomposition of K_{2mt} into uniform factors including $C_t[m]$, $C_{2t}[m]$, C_m, C_{2m}, and K_2. By Theorem 2.4, $C_t[m]$ has a C_m–factorization. Moreover each $C_{2t}[m]$ has a Hamilton cycle decomposition by Theorem 2.2, and hence has a 1–factorization. Similarly, we can think of each C_{2m}–factor of K_{2mt} as a union of two edge disjoint 1–factors. Thus, placing these decompositions of $C_t[m]$, $C_{2t}[m]$, and C_{2m} on $K_{2t} - I$ and K_{2m} in the equivalence (4.1), gives the required decomposition of K_{2mt} for $r = 2r_1 m + 2r_2 + 1$ and $s = ms_1 + s_2$ except when $t = 3$ and $s_1 = 2$. In the case when $t = 3$, we can decompose K_{6m} into a $K_3[2m]$ and a K_{2m} factor. Decomposing $K_3[2m]$ into $2m$ C_m–factors by Theorem 2.1 and K_{2m} into a $\{C_m^{s_1}, K_2^r\}$–factor by Lemma 3.7 gives us a $\{C_m^{s'}, K_2^r\}$–factorization of K_{6m} where $s = 2m + s'$ with $2m \leq s \leq 3m-1$.

Now, we assume that $t > m \geq 7$. By Theorem 2.5, $K_{2t} - I$ has a 2–factorization where every component of each 2–factor is either a 3–cycle or a 5–cycle. Also by Theorem 2.4, both $C_3[m]$ and $C_5[m]$ have a C_m–factorization. Thus, placing these decompositions on $(K_{2t} - I)[m]$ and decomposing K_{2m} into a $\{C_m^{s_1}, C_{2m}^{r_1}\}$–factor by Lemma 3.7 yields a solution to the problem for $m(t-1) \leq s = m(t-1) + s_1 \leq mt - 1$ and $0 \leq r = 2r_1 + 1 \leq 2m - 1$. On the other hand, we can decompose K_{2mt} into a K_{2t} and a $K_m[2t]$ factors. Here we decompose K_{2t} into $2t-1$ 1–factors and $K_m[2t]$ into $(m-1)/2$ $C_m[2t]$ factors. By Theorem 3.5 of [11], we have also a $\{C_m^{2t-r_i}, K_2^{2r_i}\}$–factorization of $C_m[2t]$ whenever $0 \leq i \leq (m-1)/2$ and $0 \leq r_i \neq 2 \leq 2t$. Taking

$r = \sum_{i=1}^{\frac{m-1}{2}} r_i$ and $s = \sum_{i=1}^{\frac{m-1}{2}} (2t - r_i)$ yields a solution to our problem for $2t - 1 \leq r \leq 2mt - 1$ and $0 \leq s \leq (m-1)t$.

Finally, if t is divisible by 3, then, in (4.1), $K_{2t} - I$ has a $\{C_3^{s_1}, K_2^{2r_1}\}$–factorization for $0 \leq r_1, s_1 \leq t-1$ with $r_1 + s_1 = t - 1$ by [23]. Each $C_3[m]$ has a C_m–factorization by Theorem 2.4. So, in (4.1), decomposing K_{2m} into a $\{C_3^{s_2}, K_2^{2r_2}\}$–factorization for $0 \leq r_2, s_2 \leq m-1$ with $r_2 + s_2 = m - 1$, gives us the required decomposition of K_{2mt} for $r = 2r_1 m + 2r_2 + 1$ and $s = ms_1 + s_2$. In a similar manner, when t is divisible by 5, we may obtain solution to the problem from the result of Adams et al. in [2]. □

Combining the these results it is now possible to obtain the following main result.

Theorem 4.3. *For all nonnegative integers r and s with $2m|v$ and $r + 2s = v - 1$, there exists a solution to* $\mathrm{URD}(v; K_2^r, C_m^s)$ *except possibly when all the following conditions hold:*

- $v \equiv 2 \pmod 4$*;*
- *m is odd;*
- $7 \leq m \leq \frac{v}{2m} - 4$*;*
- $\frac{v}{2} - \frac{v}{2m} + 1 \leq s \leq \frac{v}{2} - m - 1$*;*
- *$\frac{v}{2m}$ is not divisible by 3 or 5.*

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Department of Engineering Sciences, Faculty of Engineering, Istanbul University, Avcilar - ISTANBUL, 34320 TURKEY
Current address: Illinois State University, Normal, IL 61790-4520, USA
E-mail address: ugur.odabasi@istanbul.edu.tr