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NEW COMBINATORIAL INTERPRETATIONS OF SOME ROGERS-RAMANUJAN TYPE IDENTITIES

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Abstract. In present paper, three Rogers–Ramanujan type identities are interpreted combinatorially in terms of certain associated lattice path functions. Out of these three identities, two are further explored using the Bender–Knuth matrices. These results give new combinatorial interpretations of these basic series identities. Using two bijections, first between the associated lattice path functions and the $(n+t)$-color partitions and second between the associated lattice path functions and the weighted lattice path functions, we extend the recent work of Sareen and Rana to three new 5-way combinatorial identities. By using the bijection between Bender–Knuth matrices and the n-color partitions, we further extend their work to two new 6-way combinatorial identities.

1. Introduction and Definitions

Among the famous discoveries in the world of mathematics are the celebrated Rogers–Ramanujan identities [15, 16]:

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2}}{(q;q)_\lambda}=\prod_{\lambda=1}^{\infty}\frac{1}{(1-q^{5\lambda-1})(1-q^{5\lambda-4})},$$

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2+\lambda}}{(q;q)_\lambda}=\prod_{\lambda=1}^{\infty}\frac{1}{(1-q^{5\lambda-2})(1-q^{5\lambda-3})},$$

where $(a;q)_0=1$ and $(a;q)_\lambda=(1-a)(1-aq)\cdots(1-aq^{\lambda-1})$.

In 1916, MacMahon [14] interpreted these identities combinatorially in terms of ordinary partitions. In 1985, Agarwal [2] introduced n-color partitions and in 1987, Agarwal and Andrews [7] generalized these partitions to $(n+t)$-color partitions. Several identities of Rogers–Ramanujan type had been interpreted combinatorially using $(n+t)$-color partitions, see for instance [3, 7, 13, 12]. In 1989, Agarwal and Bressoud [8] established a bijection between the appropriate class of lattice paths of weight μ and a set of $(n+t)$-color partitions of μ. This bijection provides new combinatorial interpretations of the basic series identities which had already been interpreted in

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terms of $(n+t)$-color partitions such as in [4, 9]. By establishing a bijection between certain class of F-partitions and between $(n+t)$-color partitions, many more Rogers–Ramanujan type identities are further explored combinatorially, see [5, 19]. Using the bijection between the appropriate classes of these combinatorial tools, recently Sareen and Rana [17] gave combinatorial interpretations of three Rogers–Ramanujan type identities that are listed in the Slater's compendium [18]. These results lead to three 4-way combinatorial identities. The objective of this paper is to further extend these results using associated lattice paths [10] and the Bender–Knuth matrices [11]. We do this by establishing a bijection between the associated lattice path functions and the $(n+t)$-color partitions and a bijection between the associated lattice path functions and the weighted lattice path functions. We will use the bijection between n-color partitions and the Bender–Knuth matrices established in [6]. Before we recall the main results of [17] and state our main results, let us first have a look at some definitions:

Definition 1.1 ([2])**.** *A partition with* "n copies of n" *is a partition in which a part of size n, $n \geq 0$, can come in n different colors denoted by subscripts: $n_1, n_2, \cdots, n_n$.*

Definition 1.2 ([7])**.** *A partition with* "$(n+t)$ copies of n", *$t \geq 0$, is a partition in which a part of size n, $n \geq 0$, can come in $(n+t)$ different colors denoted by subscripts: $n_1, n_2, ..., n_{n+t}$. Note that zeros are permitted if and only if t is greater than or equal to one. Furthermore, zeros are not permitted to repeat in any partition.*

Remark: We note that if we take $t = 0$, then these are nothing but the n-color partitions.

Definition 1.4. *The* weighted difference *of two parts g_k, h_l, $g \geq h$, is defined by $g - h - k - l$ and is denoted by $((g_k - h_l))$.*

In [8] the *lattice paths* are described as follows:

Definition 1.5. *All paths will be of finite length lying in the first quadrant. They will begin on the y-axis and terminate on the x-axis. Only three moves are allowed at each step:*

> Northeast*: from (i, j) to $(i+1, j+1)$;*
> Southeast*: from (i, j) to $(i+1, j-1)$, only allowed if $j > 0$;*
> Horizontal*: from $(i, 0)$ to $(i+1, 0)$, only allowed along x-axis.*

All our lattice paths are either empty or terminate with a southeast step: from $(i, 1)$ to $(i+1, 0)$. Furthermore, when describing lattice paths the following terminology is used:

> Peak*: Either a vertex on the y-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.*

Valley*: A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.*
Mountain*: A section of the path which starts on either the x-axis or y-axis, which ends on the x-axis and which does not touch the x-axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.*
Plain*: A section of the path consisting of only horizontal steps which starts either on the y-axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.*
Height*: Height of a vertex is its y-coordinate.*
Weight*: Weight of a vertex is its x-coordinate.*
Weight of a Path*: It is the sum of the weights of its peaks.*

Definition 1.6. *A two rowed array of nonnegative integers*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_l \\ b_1 & b_2 & \cdots & b_l \end{pmatrix}$$

where, $a_1 \geq a_2 \geq \cdots \geq a_l \geq 0$, $b_1 \geq b_2 \geq \cdots \geq b_l \geq 0$, is known as a generalized Frobenius partition *or simply an F*-partition of μ *if*

$$\mu = l + \sum_{k=1}^{l} a_k + \sum_{k=1}^{l} b_k.$$

Anand and Agarwal [10] gave the following description of *associated lattice paths*:

Definition 1.7. *All paths will be of finite length lying in the first quadrant. They will begin on the y-axis and terminate on the x-axis. Only three moves are allowed at each step:*

Northeast*: from (i, j) to $(i+1, j+1)$;*
Southeast*: from (i, j) to $(i+1, j-1)$, only allowed if $j > 0$.*
Horizontal*: from $(i, 0)$ to $(i+1, 0)$, only allowed when the first step is preceded by a northeast step and the last is followed by a southeast step.*

The following terminology is used in describing associated lattice paths:

Truncated Isosceles Trapezoidal Section (TITS)*: A section of the path which starts on the x-axis with northeast steps followed by horizontal steps and then followed by southeast steps ending on the x-axis forms a Truncated Isosceles Trapezoidal Section. Since the lower base lies on x-axis and is not a part of the path, we use the term truncated.*
Slant Section (SS)*: A section of the path consisting of only southeast steps which starts on the y-axis (origin not included) and ends on the x-axis.*

Height of a slant section*: It is 't' if it starts from* $(0, t)$*. Clearly, a path can have an SS only in the beginning of the path. An associated lattice path can have at most one SS.*

Weight of a TITS*: This is defined by representing every TITS by an ordered pair* $\{a, b\}$ *where* a *denotes its altitude and* b *the length of the upper base. For instance, the weight of a TITS with ordered pair* $\{a, b\}$ *is* a *units.*

Weight of a Path*: It is the sum of weights of its TITSs.*

Note. *A Slant Section is assigned weight zero.*

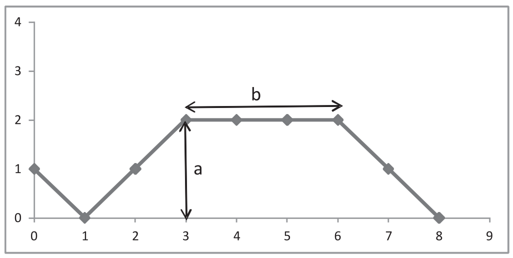

FIGURE 1. TITS with ordered pair $\{2, 3\}$.

Example 1.8. *In Figure-1, the associated lattice path has one SS of height 1 and one TITS with ordered pair* $\{2, 3\}$ *and its weight is 2 units.*

Definition 1.9. *A plane partition* δ *of a positive integer* μ *is an array*

$$\begin{array}{|cccc} \hline a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

of nonnegative integers for which $\sum_{i,j} a_{i,j} = \mu$ *and rows and column are arranged in nonincreasing order. The nonzero entries* $a_{i,j}$ *are called the parts of* δ*.*

Remark: In [7], it is observed that the number of n-color partitions of μ is equivalent to the number of plane partitions of μ.

E.A. Bender and D. E. Knuth proved the following theorem in [11]. For the definition and other details of the 1-1 correspondence of this theorem, which is denoted by ϕ, the reader is referred to [11].

Theorem 1.11 ([11])**.** *There is a 1-1 correspondence between plane partitions of* μ*, on one hand, and infinite matrices* $\gamma_{u,v}$ *with* $u, v \geq 1$*, of nonnegative integer entries such that*

$$\sum_{t \geq 1} t \left(\sum_{u+v=t+1} \gamma_{u,v} \right) = \mu,$$

on the other.

Corresponding to every nonnegative integer μ the matrices of the above theorem were called BK_μ-matrices by Agarwal in [6], where BK stands for Bender and Knuth. These are infinite matrices, but will be represented in the sequel by the largest possible square matrices whose last row (or column) is nonzero. Thus, for example, the six BK_3-matrices are represented by

$$(3)\,,\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}.$$

We state here three more definitions:

Definition 1.12 ([6]). *We define a matrix $U_{i,j}$ as an infinite matrix whose $(i,j)^{th}$ entry is 1 and all other entries are zeros. We call the $U_{i,j}$ the units of the BK_μ-matrices.*

Definition 1.13 ([1]). *We define the following order on the set of all units of BK_μ-matrices: If $k+l<g+h$ then $U_{k,l}<U_{g,h}$, and if $k+l=g+h$, then $U_{k,l}<U_{g,h}$ when $k<g$. Thus, the units satisfy the order:*

$$U_{1,1}<U_{1,2}<U_{2,1}<U_{1,3}<U_{2,2}<U_{3,1}<U_{1,4}<U_{2,3}<U_{3,2}<\cdots$$

Definition 1.14 ([1]). *The* order difference *of two units $U_{g,h}$, $U_{k,l}$, where $g+h\geq k+l$ is defined by $h-l-2k$ and is denoted by $[[U_{g,h}-U_{k,l}]]$.*

Note. *The representation of a BK_μ-matrix as the linear combination of the units $U_{i,j}$ is called the standard factorization of that BK_μ-matrix.*

The following are the three basic series identities which appear in Slater's compendium [18]:

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2}}{(q^4;q^4)_\lambda(q;q^2)_\lambda}=\frac{(q^3,q^{11},q^{14};q^{14})_\infty(q^8,q^{20};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}, \quad (1.1)$$

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda(\lambda+2)}}{(q^4;q^4)_\lambda(q;q^2)_\lambda}=\frac{(q,q^{13},q^{14};q^{14})_\infty(q^{12},q^{16};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}, \quad (1.2)$$

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda(\lambda+2)}}{(q^4;q^4)_\lambda(q;q^2)_{\lambda+1}}=\frac{(q^5,q^9,q^{14};q^{14})_\infty(q^4,q^{24};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}, \quad (1.3)$$

where $(\alpha_1,\alpha_2,\cdots\alpha_k;z)_\infty=\prod_{l=1}^{k}(\alpha_l;z)_\infty$.

The combinatorial interpretations of basic series identities (1.1)-(1.3) are given in [17] in the form of following theorems.

Theorem 1.15. *Let $G_1(\mu)$ denote the number of n-color partitions of μ into parts such that if m_i is the smallest or the only part in the partition, then $m\equiv i \pmod 4$ and the weighted difference of any two consecutive parts is nonnegative and is congruent to $0 \pmod 4$. Let $H_1(\mu)$ denote the number of lattice paths of weight μ which start from $(0,0)$, have no valley above height 0, the length of plains, if any, are congruent to $0 \pmod 4$. Let $I_1(\mu)$ denote* *the number of F-partitions of μ such that $a_l \equiv 0 \pmod 2$, $a_k \leq b_k$ and $a_k > b_{k+1}$, with a_k and b_{k+1} of opposite parity. Let*

$$J_1(\mu) = \sum_{t=0}^{\mu} M_1(\mu - t) N_1(t),$$

where $M_1(\mu)$ is the number of partitions of μ into parts congruent to $\pm 2, \pm 4, \pm 10, \pm 12 \pmod{28}$ and $N_1(\mu)$ denotes the number of partitions of μ into distinct parts congruent to $\pm 1, \pm 5, 7 \pmod{14}$. Then

$$G_1(\mu) = H_1(\mu) = I_1(\mu) = J_1(\mu),$$

for all μ.

Theorem 1.16. *Let $G_2(\mu)$ denote the number of n-color partitions of μ into parts greater than or equal to 3 such that if m_i is the smallest or the only part in the partition, then $m - i \equiv 2 \pmod 4$ and the weighted difference of any two consecutive parts is nonnegative and congruent to $0 \pmod 4$. Let $H_2(\mu)$ denote the number of lattice paths of weight μ which start from $(0, 0)$, have no valley above height 0, have a plain of length congruent to $2 \pmod 4$ in the beginning of the path, and the length of other plains, if any, are congruent to $0 \pmod 4$. Let $I_2(\mu)$ denote the number of F-partitions of μ such that $a_l \equiv 1 \pmod 2$, $a_k \leq b_k$, and $a_k > b_{k+1}$ with a_k and b_{k+1} of opposite parity. Let*

$$J_2(\mu) = \sum_{t=0}^{\mu} M_2(\mu - t) N_2(t),$$

where $M_2(\mu)$ is the number of partitions of μ into parts congruent to $\pm 4, \pm 6, \pm 8, \pm 10 \pmod{28}$ and $N_2(\mu)$ denotes the number of partitions of μ into distinct parts congruent to $\pm 3, \pm 5, 7 \pmod{14}$. Then $G_2(\mu) = H_2(\mu) = I_2(\mu) = J_2(\mu)$, for all μ.

Theorem 1.17. *Let $G_3(\mu)$ denote the number of partitions of μ with $(n+2)$ copies of n into parts such that for some i, i_{i+2} is a part and the weighted difference of any two consecutive parts is nonnegative and is congruent to $0 \pmod 4$. Let $H_3(\mu)$ denote the number of lattice paths of weight μ which start from $(0, 2)$, have no valley above height 0, the length of plains, if any, are congruent to $0 \pmod 4$. Let $I_3(\mu)$ denote the number of F-partitions of μ such that $a_l = 0$, $a_k \leq b_k + 2$, and $a_k > b_{k+1} + 2$ with a_k and b_{k+1} of opposite parity. Let*

$$J_3(\mu) = \sum_{t=0}^{\mu} M_3(\mu - t) N_3(t),$$

where $M_3(\mu)$ is the number of partitions of μ into parts congruent to $\pm 2, \pm 6, \pm 8, \pm 12 \pmod{28}$ and $N_3(\mu)$ denotes the number of partitions of μ into distinct parts congruent to $\pm 1, \pm 3, 7 \pmod{14}$. Then $G_3(\mu) = H_3(\mu) = I_3(\mu) = J_3(\mu)$ for all μ.

Our aim is to further extend these results by means of associated lattice paths and Bender–Knuth matrices. In Section 2, we will extend Theorems 1.15-1.17 by the aid of associated lattice paths and in Section 3, Theorems 1.15 and 1.16 are further explored by using Bender–Knuth matrices.

2. Combinatorial interpretation in terms of associated lattice paths

In this section our main objective is to interpret identities (1.1)-(1.3) by means of associated lattice paths. These results will yield three new combinatorial identities which will extend Theorems 1.15-1.17 to three new 5-way combinatorial identities. We shall prove the following theorems.

Theorem 2.1. *Let $K_1(\mu)$ denote the number of associated lattice paths of weight μ such that:*

(i) *for any TITS with ordered pair $\{a, b\}$, b does not exceed a;*
(ii) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with same altitude are ordered by the length of their upper base;*
(iii) *for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$, $a_1 \leq a_2$, $a_2 - b_2 = a_1 + b_1 + h$, where h is a nonnegative multiple of 4.*

Then

$$G_1(\mu) = H_1(\mu) = I_1(\mu) = J_1(\mu) = K_1(\mu),$$

for all μ.

Example 2.2. $G_1(5) = 3$. *The relevant partitions are:* $5_1,\ 5_5,\ 4_2 + 1_1$ *and* $H_1(5) = 3$. *The relevant Lattice paths are as follows:*

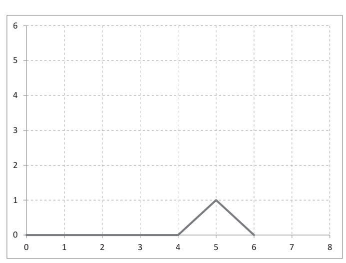

Figure 2. Peak with height 1 and weight 5.

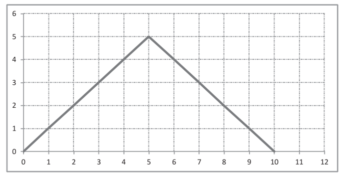

FIGURE 3. Peak with height 5 and weight 5.

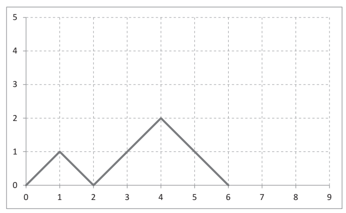

FIGURE 4. Two Peaks with heights 1, 2 and weights 1, 4 respectively.

Furthermore, $I_1(5) = 3$.
The relevant F*-partitions are:*

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix},$$

while

$$\begin{aligned} J_1(5) &= \sum_{t=0}^{5} M_1(5-t) N_1(t) \\ &= 0(1) + 2(1) + 0(0) + 1(0) + 0(0) + 1(1) = 3, \end{aligned}$$

and $K_1(5) = 3$.
The relevant Associated lattice paths are:

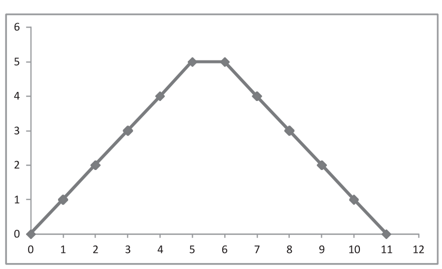

FIGURE 5. TITS with ordered pair $\{5,1\}$.

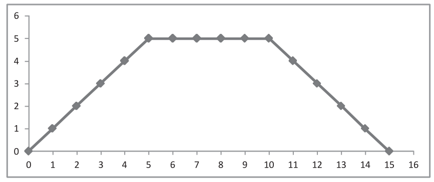

FIGURE 6. TITS with ordered pair {5,5}.

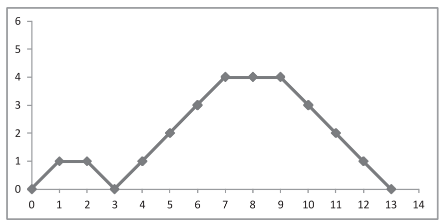

FIGURE 7. Two TITSs with ordered pairs {1,1} and {4,2}.

Theorem 2.3. *Let $K_2(\mu)$ denote the number of associated lattice paths of weight μ such that:*

(i) *for any TITS with ordered pair $\{a, b\}$, b does not exceed a;*
(ii) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with the same altitude are ordered by the length of their upper base;*
(iii) *the altitude of each TITS is greater than or equal to 3;*
(iv) *for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$ such that $a_1 \leq a_2$, $a_2 - b_2 = a_1 + b_1 + h$ where h is a nonnegative multiple of 4.*

Then

$$G_2(\mu) = H_2(\mu) = I_2(\mu) = J_2(\mu) = K_2(\mu), \text{ for all } \mu.$$

Theorem 2.4. *Let $K_3(\mu)$ denote the number of associated lattice paths of weight μ such that:*

(i) *for any TITS with ordered pair $\{a, b\}$, b does not exceed $(a + 2)$;*
(ii) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with the same altitude are ordered by the length of their upper base;*
(iii) *there is an SS of height 2 or a TITS with ordered pair $\{a, a + 2\}$;*
(iv) *for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$ such that $a_1 \leq a_2$, $a_2 - b_2 = a_1 + b_1 + h$, where h is a nonnegative multiple of 4.*

Then

$$G_3(\mu) = H_3(\mu) = I_3(\mu) = J_3(\mu) = K_3(\mu), \text{ for all } \mu.$$

We will discuss the detailed proof of Theorem-2.1 and give the outline of the proofs of the remaining theorems.

Proof of Theorem 2.1. We will prove this theorem in three steps. First we will show that the L.H.S. of Equation (1.1) also generates the associated lattice paths enumerated by $K_1(\mu)$. Then we will show a bijection between n-color partitions enumerated by $G_1(\mu)$ and the associated lattice paths enumerated by $K_1(\mu)$. Finally we will establish a bijection between weighted lattice paths enumerated by $H_1(\mu)$ and the associated lattice paths enumerated by $K_1(\mu)$.

Step I. We shall prove that

$$\sum_{\mu=0}^{\infty} K_1(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}}{(q;q^2)_\lambda (q^4;q^4)_\lambda}. \tag{2.1}$$

In $q^{\lambda^2}/(q;q^2)_\lambda(q^4;q^4)_\lambda$, the factor q^{λ^2} generates an associated lattice path having λ TITSs such that i^{th} TITS has the ordered pair $\{2i-1,1\}$. For $\lambda=3$, the path begins as:

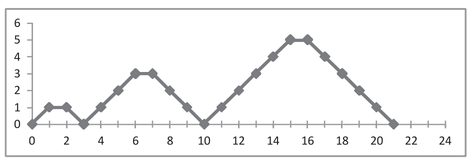

FIGURE 8. TITSs when $\lambda = 3$.

In the above Figure we consider two successive TITSs, say, the i^{th} and $(i+1)^{th}$. Their corresponding ordered pairs are $\{2i-1,1\}$ and $\{2i+1,1\}$ respectively.

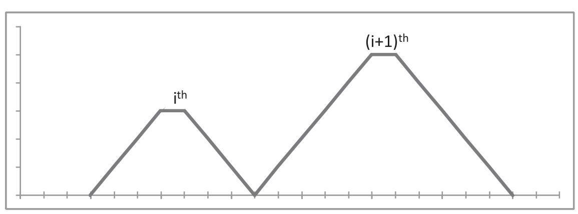

FIGURE 9. i^{th} and $(i+1)^{th}$ TITSs.

The factor $1/(q^4;q^4)_\lambda$ generates λ nonnegative multiples of 4, say $\alpha_1 \geq \alpha_2 \geq \cdots \alpha_\lambda \geq 0$, which are encoded by increasing the altitude of the i^{th} TITS by $\alpha_{\lambda-i+1}$, $1 \leq i \leq \lambda$. Thus the ordered pair associated with the i^{th} TITS becomes $\{(2i-1)+\alpha_{\lambda-i+1}, 1\}$.

Figure-9 now becomes Figure-10.

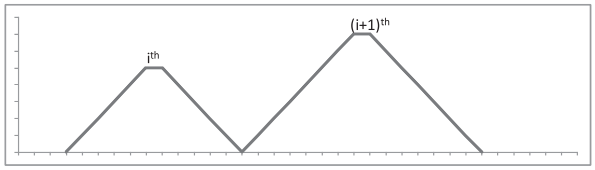

FIGURE 10. i^{th} and $(i+1)^{th}$ TITSs.

The factor $1/(q;q^2)_\lambda$ generates λ nonnegative multiples of $(2i-1)$, $1 \le i \le \lambda$, say, $\beta_1 \times 1,\ \beta_2 \times 3, \cdots,\ \beta_\lambda \times (2\lambda-1)$. This is encoded by increasing the altitude of i^{th} TITS by $2(\beta_\lambda + \beta_{\lambda-1} + \cdots + \beta_{\lambda-i+2}) + \beta_{\lambda-i+1}$ and the length of the upper base by $\beta_{\lambda-i+1}$. So the associated ordered pair becomes $\{2i-1+\alpha_{\lambda-i+1}+2(\beta_\lambda+\beta_{\lambda-1}+\cdots+\beta_{\lambda-i+2})+\beta_{\lambda-i+1}, 1+\beta_{\lambda-i+1}\}$.

Figure-10 now changes to Figure-11.

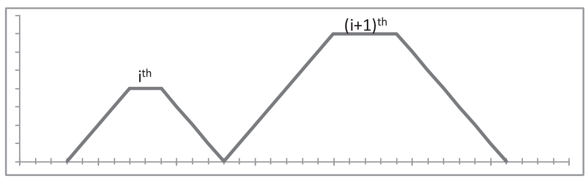

FIGURE 11. i^{th} and $(i+1)^{th}$ TITSs.

Every associated lattice path enumerated by $K_1(\mu)$ is uniquely generated in this manner; this proves (2.1).

Step II. We now establish a $1-1$ correspondence between the associated lattice paths enumerated by $K_1(\mu)$ and the n-color partitions enumerated by $G_1(\mu)$.

We do this by encoding each associated lattice path as the sequence of weights of TITSs with each altitude of the TITS subscripted by the length of the respective upper base. Thus, if we denote the two TITS's in Figure-11 by P_r and Q_s respectively, then

$$\begin{aligned} P &= (2i-1)+\alpha_{\lambda-i+1}+2(\beta_\lambda+\beta_{\lambda-1}+\cdots+\beta_{\lambda-i+2})+\beta_{\lambda-i+1};\\ r &= \beta_{\lambda-i+1}+1;\\ Q &= (2i+1)+\alpha_{\lambda-i}+2(\beta_\lambda+\beta_{\lambda-1}+\cdots+\beta_{\lambda-i+1})+\beta_{\lambda-i};\\ s &= \beta_{\lambda-i}+1. \end{aligned}$$

The weighted difference of these two parts is $((Q_s-P_r)) = Q-P-r-s = \alpha_{\lambda-i}-\alpha_{\lambda-i+1}$ which is nonnegative and is a multiple of 4.

Obviously, if $\{P,r\}$ is the ordered pair of first TITS in the associated lattice path then it will correspond to the smallest part in the corresponding n-color partition, or to the singleton part if the n-color partition has only one part; in both cases $P-r=\alpha_\lambda \equiv 0 \pmod 4$.

To see the reverse implication, we consider two n-color parts of a partition enumerated by $G_1(\mu)$, say, P_r and Q_s with $Q \geq P$. Clearly $r \leq P$ and $s \leq Q$.

Since P_r and Q_s are the parts of n-color partition enumerated by $G_1(\mu)$, the weighted difference is equal to $((Q_s - P_r)) \equiv 0 \pmod 4$. This implies that $Q - P - r - s = h$, where h is nonnegative multiple of 4, which in turn implies that $Q - s = P + r + h$, where h is nonnegative multiple of 4.

Step III. Finally, we establish a bijection between the weighted lattice paths enumerated by $H_1(\mu)$ and the associated lattice paths enumerated by $K_1(\mu)$. We do this by mapping each peak of weight a and height b of a weighted lattice path enumerated by $H_1(\mu)$ to a TITS with ordered pair $\{a, b\}$ of an associated lattice path enumerated by $K_1(\mu)$,and conversely. Under this mapping, all the conditions on the weighted lattice paths enumerated by $H_1(\mu)$ are translated to the conditions on the associated lattice paths enumerated by $K_1(\mu)$ and vice-versa. Hence this completes the bijection between the weighted lattice paths enumerated by $H_1(\mu)$ and the associated lattice paths enumerated by $K_1(\mu)$. □

Outline of the Proofs of Theorems 2.3-2.4. Here, the changes required to prove the remaining theorems are discussed briefly.

Theorem-2.3 : An appeal to Theorem-2.1, the extra factor $q^{2\lambda}$ causes an increase by 2 in the altitude of each of the TITSs. Thus the altitude of each TITS is ≥ 3.

Theorem-2.4 : An appeal to Theorem-2.1, the extra factor $q^{2\lambda}/(1 - q^{2\lambda+1})$ puts an SS of height 2 in the beginning of the path or a TITS with ordered pair $\{a, a + 2\}$. Clearly, it will correspond to a_{a+2} or we can say i_{i+2} part of the corresponding colored partition.

3. Combinatorial interpretation in terms of Bender–Knuth matrices

This section is fully devoted to interpret identities (1.1) and (1.2) by means of the Bender–Knuth matrices. This will further extend Theorems 1.15 and 1.16 to two new 6-way combinatorial identities.

Theorem 3.1. *Let $L_1(\mu)$ denote the number of BK_μ-matrices X such that, in the standard factorization of X, the order difference between any two consecutive units $U_{g,h}$ and $U_{k,l}$ is nonnegative and is congruent to* $0 \pmod 4$ *where if $U_{i,j}$ is the only or least unit then $j \equiv 1 \pmod 4$. Then*

$$G_1(\mu) = H_1(\mu) = I_1(\mu) = J_1(\mu) = K_1(\mu) = L_1(\mu)$$

for all μ.

Example 3.2. *For $L_1(5) = 3$, the relevant BK_μ-matrices are*

$$\begin{pmatrix} 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&0&0 \end{pmatrix}.$$

Theorem 3.3. *Let $L_2(\mu)$ denote the number of BK_μ-matrices X such that, in the standard factorization of X, the order difference between any two consecutive units $U_{g,h}$ and $U_{k,l}$ is nonnegative and is congruent to $0 \pmod 4$ where if $U_{i,j}$ is the only or least unit, then $j \equiv 3 \pmod 4$. Then*

$$G_2(\mu) = H_2(\mu) = I_2(\mu) = J_2(\mu) = K_2(\mu) = L_2(\mu)$$

for all μ.

We will discuss the proof of Theorem 3.1 in detail, and since the proof of Theorem 3.3 is similar, we will give an outline of its proof.

Proof of Theorem 3.1. We shall prove that if X is a BK_μ-matrix enumerated by $L_1(\mu)$, then the n-color partition $\chi(X)$ is enumerated by $G_1(\mu)$. Here the mapping is given by

$$\chi: \ U_{g,h} \mapsto (g+h-1)_g$$

and the inverse mapping is

$$\chi^{-1}: \ m_i \mapsto U_{i,m-i+1}.$$

Conversely, if π is an n-color partition enumerated by $G_1(\mu)$, then the BK_μ-matrix $\chi^{-1}(\pi)$ is enumerated by $L_1(\mu)$.

Let $X = a_{1,1}U_{1,1} + a_{1,2}U_{1,2} + \cdots + a_{2,1}U_{2,1} + a_{2,2}U_{2,2} \cdots$ be a BK_μ-matrix enumerated by $L_1(\mu)$, where $a_{i,j}$ are nonnegative integers which denote the multiplicities of $U_{i,j}$. Now in view of the condition on ordered difference, i.e., the order difference is nonnegative, the entries in X can't exceed 1, i.e. each $a_{i,j} = 1$ or 0.

Let $U_{g,h},\ U_{k,l}$ $(g+h \geq k+l)$ be two consecutive units of a BK_μ-matrix X enumerated by $L_1(\mu)$ which correspond to two consecutive n-color parts $m_i,\ n_j$ of $\chi(X)$. Then $m_i = (g+h-1)_g$ and $n_j = (k+l-1)_k$ since $g+h \geq k+l$ implies $m \geq n$ and

$$\begin{aligned} ((m_i - n_j)) &= (g+h-1) - g - (k+l-1) - k = h - l - 2k \\ &= [[U_{g,h} - U_{k,l}]] \equiv 0 \pmod 4. \end{aligned}$$

This shows that in $\chi(X)$, the weighted difference between any two consecutive parts is congruent to $0 \pmod 4$ and is nonnegative.

Furthermore, if $U_{g,h}$ is the only or the least unit of X then $\chi(U_{g,h}) = m_i$ will be the only, or the least, part of $\chi(X)$, and since in $U_{g,h},\ h \equiv 1 \pmod 4$, we see that $m_i = \chi(U_{g,h}) = (g+h-1)_g = (g+4t)_g$ for some positive integer t. This implies $m - i = (g+4t) - g = 4t \equiv 0 \pmod 4$. Thus $\chi(X)$ is enumerated by $G_1(\mu)$.

To see the reverse implication, let π be an n-color partition of μ enumerated by $G_1(\mu)$. We shall prove that the BK_μ-matrix $\chi^{-1}(\pi)$ is enumerated by $L_1(\mu)$.

Let $m_i,\ n_j,\ m \geq n$ be two consecutive parts of π such that $\chi^{-1}(m_i) = U_{g,h}$ and $\chi^{-1}(n_j) = U_{k,l}$. Then $U_{g,h} = U_{i,m-i+1}$ and $U_{k,l} = U_{j,n-j+1}$. Since $m \geq n$, we have $g+h = m+1 \geq n+1 = k+l$, which implies $(g+h) \geq (k+l)$ and

$$\begin{aligned}[[U_{g,h} - U_{k,l}]] &= [[U_{i,m-i+1} - U_{j,n-j+1}]] = (m-i+1) - (n-j+1) - 2j \\ &= m - n - i - j = ((m_i - n_j)) \equiv 0 \pmod 4,\end{aligned}$$

which is nonnegative.

Now, if m_i is the only, or the least, part of π then $\chi^{-1}(\pi) = U_{g,h}$ will be the only, or the least, unit in $\chi^{-1}(\pi)$. Moreover, since $m-i \equiv 0 \pmod 4$, we have $\chi^{-1}(m_i) = U_{i,m-i+1} = U_{i,4t+1}$ for some positive integer t. This implies $h = 4t+1 \equiv 1 \pmod 4$ and completes the proof of Theorem 3.1. □

Outline of the Proof of Theorem 3.3 An appeal to Theorem 1.16, with the condition $m-i \equiv 2 \pmod 4$, causes j to be congruent to $3 \pmod 4$.

4. Conclusion

A fine connection between different combinatorial objects is observed in this paper. Theorems 2.1-2.4, in conjunction with Theorems 1.15-1.17, yield three new 5-way combinatorial identities in terms of five different combinatorial objects viz. ordinary partitions, $(n+t)$-color partitions, weighted lattice paths, F-partitions, and associated lattice paths. Each of these 5-way combinatorial identities yield ten identities in the usual sense. Out of these ten combinatorial identities four are totally new. Theorems 3.1-3.3, along with Theorems 1.15 and 1.16 produce two new 6-way combinatorial identities in terms of six different combinatorial objects viz. ordinary partitions, n-color partitions, weighted lattice paths, F-partitions, associated lattice paths, and Bender–Knuth matrices. Each of these 6-way combinatorial identities yield fifteen identities in the usual sense. Out of these fifteen combinatorial identities, five are entirely new. It would be of interest if more basic series identities can be interpreted combinatorially using associated lattice paths and Bender–Knuth matrices.

References

1. A. K. Agarwal, *New combinatorial interpretations of some analytic identities*, Proceedings of the Third International Conference of the Society for Special Functions and their Applications, 41–47, Soc. Spec. Funct. Appl., Chennai, 2002.
2. ______, *Partitions with "n copies of n"*, Proc. of the colloque De Combinatoire Enumerative (Montreal, Quebec), Lecture Notes in Math, Springer, Berlin (1985), no. 1234, 1–4.
3. ______, *Rogers-Ramanujan identities for n-color partitions*, J. Number Theory **28** (1988), 299–305.
4. ______, *Lattice paths and n-color partitions*, Utilitas Mathematica **53** (1998), 71–80.

5. ———, *Identities and generating functions for certain classes of F-partitions*, Ars Combin. **57** (2000), 65–75.
6. ———, *n-color analogues of Gaussian polynomials*, Ars Combin. **61** (2001), 97–117.
7. A. K. Agarwal and G. E. Andrews, *Rogers-Ramanujan identies for paritions with "n copies of n"*, J. Combin. Theory, Ser. A **45(1)** (1987), 40–49.
8. A. K. Agarwal and D. M. Bressoud, *Lattice paths and multiple basic hypergeometric series*, Pacific J. Math. **136(2)** (1989), 209–228.
9. A. K. Agarwal and M. Goyal, *Lattice paths and Rogers identities*, Open J. of Discrete Mathematics **1** (2011), 89–95.
10. S. Anand and A. K. Agarwal, *A new class of lattice paths and partitions with n copies of n*, Proc. Indian Acad. Sci. (Math. Sci.) **122(1)** (2012), 23–39.
11. E. A. Bender and D. E. Knuth, *Enumeration of plane partitions*, J. Combin. Theory, Ser. A **13** (1972), 40–54.
12. M. Goyal and A. K. Agarwal, *Further Rogers-Ramanujan identities for n-color partitions*, Utilitas Mathematica **95** (2014), 141–148.
13. ———, *On a new class of combinatorial identities*, Ars Combin. **127** (2016), 65–77.
14. P. A. MacMahon, *Combinatorial analysis*, vol. 2, Cambridge University Press, Cambridge, reprinted by Chalesa, New York, 1916.
15. S. Ramanujan and L. J. Rogers, *Proof of certain identities in combinatory analysis*, Proc. Cambridge Philosophical Soc. **XIX** (1919), 211–216.
16. L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25** (1894), 318–343.
17. J. K. Sareen and M. Rana, *Four-way combinatorial interpretations of some Rogers-Ramanujan type identities*, Ars Combin. (to appear).
18. L. J. Slater, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc **54(2)** (1952), 147–167.
19. G. Sood and A. K. Agarwal, *Frobenius partition theoretic interpretations of some basic series identities*, Contributions to Discrete Mathematics **7(2)** (2012), 54–65.

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