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FULLY INDECOMPOSABLE AND NEARLY DECOMPOSABLE GRAPHS

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ABSTRACT. Let A be an n-square non-negative matrix. If A contains no $s \times t$ zero submatrix, where $s + t = n$, then it is called fully indecomposable. Moreover, a graph G is said to be fully indecomposable if its adjacency matrix is fully indecomposable. In this paper we provide some necessary and sufficient conditions for a graph to be fully indecomposable. Among other results we prove that a regular connected graph is fully indecomposable if and only if it is not bipartite.

1. INTRODUCTION

Let G be a graph of order n with the vertex set $V(G) = \{v_1, \ldots, v_n\}$. The *adjacency matrix* of G is an $n \times n$ matrix $A = [a_{ij}]$ indexed by the vertex set, where $a_{ij} = 1$, when there is an edge between v_i and v_j in G and $a_{ij} = 0$, otherwise. By the *permanent* of any $n \times n$ matrix $A = [a_{ij}]$ over a commutative ring we mean

$$\mathrm{per}(A) = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)},$$

where the summation is over all permutations over $\{1, \ldots, n\}$. For a graph G, $i(G)$ denotes the number of isolated vertices of G and $d(v)$ denotes the degree of the vertex $v \in V(G)$. For a vertex v we denote by $N(v)$ the set of all neighbors of vertex v in G; for a subset $S \subseteq V(G)$ we denote by $N(S)$ the set of all neighbors of vertices of S in G. For an $n \times n$ square matrix A and integers $1 \leq i, j \leq n$, $A(i|j)$ denotes the matrix obtained from A by removing the i-th row and the j-th column. A subgraph H of G is called *spanning subgraph* if $V(H) = V(G)$. Furthermore, a spanning subgraph H of G is called a $\{1, 2\}$*-factor* if every component of H is either a path of order 2 or a cycle. We denote by K_n the complete graph of order n that is a graph with n pairwise adjacent vertices. A *transversal* in a matrix is a set D of its entries containing exactly one entry of each row and each column. We say

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that D is non-zero if all of its elements are non-zero. It is not hard to check that every $\{1,2\}$-factor of a graph is equivalent to a non-zero transversal of its adjacency matrix.

Recall that a matrix A of order n is called *partly decomposable* if it contains an $s \times t$ zero submatrix, where $s + t = n$. In other words, a matrix A is partly decomposable if there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where the zero block is of size $s \times (n - s)$. If a matrix contains no $s \times t$ zero submatrix, where $s + t = n$, then it is called *fully indecomposable.* We say that a graph G is *partly decomposable* (*fully indecomposable*) if its adjacency matrix is partly decomposable (fully indecomposable). Finally, a graph G is said to be *nearly decomposable* if it is a fully indecomposable graph but the graph obtained from G by deleting every edge is a partly decomposable graph. There is research on the fully indecomposable and the nearly decomposable non-negative matrices. Naturally, when we restrict our attention to some special kind of matrices, we may find further properties. In this paper we consider the symmetric $(0,1)$-matrices and provide sufficient and necessary conditions for this kind of matrices to be fully indecomposable.

As an observation we have the following:

Remark 1.1. A graph G is partly decomposable if one of the following holds:

(i) G is not connected;
(ii) G is a bipartite graph;
(iii) G has a pendant edge.

Assume that G is a fully indecomposable graph of order n with the adjacency matrix A. By the lower bound for the permanent given in [4, p.57], we have $\mathrm{per}(A) \geq \max\{r_1, \ldots, r_n\}$, where $r_i = \sum_{j=1}^{n} a_{ij}$, for $i = 1, \ldots, n$; thus we have $\mathrm{per}(A) \geq 2$. Therefore every graph with $\mathrm{per}(A) < 2$ is a partly decomposable graph. Now, let G be a fully indecomposable graph with the adjacency matrix A and $\mathrm{per}(A) = 2$. Then $r_i = 2$ for $i = 1, \ldots, n$, i.e., G should be a cycle. It is easy to see that every even cycle is partly decomposable. Thus every fully indecomposable graph with $\mathrm{per}(A) = 2$ is an odd cycle, which is also a nearly decomposable graph.

Assume that G is a graph with the adjacency matrix A. It is well-known that $\mathrm{per}(A)$ counts the number of $\{1,2\}$-factors of G in which every $\{1,2\}$-factor containing r cycles is counted 2^r times. Now, if G is a fully indecomposable graph, then since $\mathrm{per}(A) \geq 2$, we obtain that G contains a $\{1,2\}$-factor.

2. Fully Indecomposable Graphs

The main goal of this section is providing some necessary and sufficient conditions for a graph to be fully indecomposable. We start this section by recalling a theorem of this nature for a non-negative matrix:

Theorem 2.1 ([4, p.38])**.** *Let A be a non-negative $n \times n$ matrix with $n \geq 2$. Then A is fully indecomposable if and only if* $\mathrm{per}(A(i|j)) > 0$ *for all i and j.*

Before stating our first result we need to recall the following useful theorem:

Theorem 2.2. [4, p.31] *Let A be an $n \times n$ non-negative matrix. Then* $\mathrm{per}(A) = 0$ *if and only if A contains an $s \times t$ zero submatrix such that $s + t = n + 1$.*

Now we are in a position to prove the following theorem:

Theorem 2.3. *Let G be a connected graph of order n with the adjacency matrix A. Then for any i, j, with $1 \leq i, j \leq n$,* $\mathrm{per}(A(i|j)) > 0$ *if and only if* $\mathrm{per}(A(i|i)) > 0$.

Proof. One direction is clear. For the other let $\mathrm{per}(A(i|i)) > 0$, for $i = 1, \ldots, n$, and assume there are two indices i and j such that $\mathrm{per}(A(i|j)) = 0$. By Theorem 2.2, $A(i|j)$ has an $s \times t$ zero submatrix C such that $s + t = (n-1) + 1$. Assume that the rows and columns of C are indexed by $X = \{i_1, \ldots, i_s\}$ and $Y = \{j_1, \ldots, j_t\}$, respectively. If $i \notin Y$ or $j \notin X$, then by Theorem 2.2, $\mathrm{per}(A(i|i)) = 0$ or $\mathrm{per}(A(j|j)) = 0$, a contradiction. So assume that $i \in Y$ and $j \in X$. Now, since G is connected we have $X \cap Y \neq \varnothing$. Therefore there exists an integer $1 \leq k \leq n$ such that $k \notin X \cup Y$. Thus $A(k|k)$ contains C as a zero submatrix which together with Theorem 2.2 imply that $\mathrm{per}(A(k|k)) = 0$, a contradiction.This completes the proof of the theorem. □

Remark 2.4. The condition of the connectivity in the previous theorem is not superfluous, as this counterexample will show. Consider the graph G obtained by the disjoint union of K_3 and K_4 and let A be the adjacency matrix of G. It is not hard to verify that A satisfies $\mathrm{per}(A(i|i)) > 0$, for $i = 1, \ldots, 7$. But by the first part of Remark 1.1, since G is not a connected graph, A is a partly decomposable matrix.

Before stating our next result we need to recall two following theorems. The first one is a celebrated theorem due to Tutte.

Theorem 2.5 ([5])**.** *A graph G has a $\{1, 2\}$-factor if and only if $i(G \setminus S) \leq |S|$ for every subset $S \subseteq V(G)$.*

Theorem 2.6 ([3, p.216] and [1, p.253])**.** *Let G be a graph. Then $|N(S)| \geq |S|$ for all independent subset $S \subseteq V(G)$ if and only if G has a $\{1, 2\}$-factor.*

The following theorem provides some necessary and sufficient conditions for a connected graph to be fully indecomposable.

Theorem 2.7. *Let G be a connected graph. Then the following statements are equivalent*

(i) *G is a fully indecomposable graph;*
(ii) *$i(G \setminus S) < |S|$ for all non-empty subset S of vertices;*
(iii) *The graph obtained from G by removing of any vertex contains a $\{1, 2\}$-factor;*
(iv) *$|N(S)| > |S|$ for all non-empty independent subset $S \subset V(G)$.*

Proof. $(i) \Rightarrow (ii)$: Assume that G is a fully indecomposable graph. Then G contains some $\{1, 2\}$-factor, and by Theorem 2.5 we have $i(G \setminus S) \leq |S|$ for every subset $S \subseteq V(G)$. By the contrary suppose that there exists a non-empty subset $S \subseteq V(G)$ such that $i(G \setminus S) = |S|$ and let L denote the set of all isolated vertices of $G \setminus S$. Then the submatrix with rows corresponding to the vertices of L and columns corresponding to $V(G) \setminus S$ is a zero submatrix of size $i(G \setminus S) \times (n - |S|)$, which is a contradiction.

$(ii) \Rightarrow (iii)$: Assume that $i(G \setminus S) < |S|$ for all non-empty $S \subseteq V(G)$. We prove that for any vertex $v \in V(G)$, $G' = G \setminus \{v\}$ contains a $\{1, 2\}$-factor. Using Theorem 2.5, assume that $S' \subseteq V(G')$ is arbitrary. Then by assumption $i(G' \setminus S') = i(G \setminus (S' \cup \{v\}))$ is less than $|S'| + 1$. This yields the result.

$(iii) \Rightarrow (iv)$: See Lemma 6.2.1 in [3, p.217].

$(iv) \Rightarrow (i)$: Assume that for every non-empty independent $S \subset V(G)$, $|N(S)| > |S|$. We will show that G is fully indecomposable. By the contrary assume that G is not fully indecomposable. Then one can find a zero submatrix of size $s \times t$, where $s + t = n$. Thus there exist two subsets $X = \{x_1, \ldots, x_s\}$ and $Y = \{y_1, \ldots, y_t\}$ such that there is no edge between X and Y. Since G is connected, $S = X \cap Y \neq \varnothing$. Clearly S is an independent subset in which $N(S) \subseteq L = V(G) \setminus (X \cup Y)$ and $|L| = |S|$, a contradiction. This implies that G is fully indecomposable. □

Note that by Theorem 2.1 one can see that a $(0, 1)$-matrix A is fully indecomposable if and only if every 1 in A belongs to a non-zero transversal and every 0 of A belongs to a transversal in which all other elements are 1. Therefore adding a new edge to a fully indecomposable graph strictly increases the number of $\{1, 2\}$-factors of G and removing any edge from G strictly decreases the number of $\{1, 2\}$-factors. Thus if G is a fully indecomposable graph, then for any edge e of G there exists a $\{1, 2\}$-factor of G containing e. The following result was proved by S. Zhou and H. Zhang:

Theorem 2.8 ([7]). *Let G be a connected non-bipartite graph. Then every edge of a graph G lies in a $\{1, 2\}$-factor if and only if $i(G \setminus S) < |S|$, for every non-empty subset $S \subseteq V(G)$.*

By Remark 1.1, we know that every fully indecomposable graph is connected and non-bipartite. Thus, after combining Theorem 2.7 with the previous result, we find the following consideration of fully indecomposable graphs.

Theorem 2.9. *Let G be a connected graph. Then G is fully indecomposable if and only if G is non-bipartite and every edge of G lies in a $\{1,2\}$-factor.*

Moreover, we can deduce the following corollary due to Berge [2]:

Corollary 2.10. *Let G be a connected graph. Then $|N(S)| > |S|$ for all non-empty $S \subset V(G)$ if and only if G is non-bipartite and for each edge e of G, there exists a $\{1,2\}$-factor in G containing e.*

The previous corollary is a generalization of the following result:

Theorem 2.11 ([1, p.20])**.** *Let G be a bipartite graph with bipartition (A, B). If $|N(S)| > |S|$ for all $S \subset V(G)$, then for each edge e of G, G has a perfect matching containing e that saturates A.*

The following theorem introduces a family of fully indecomposable graphs:

Theorem 2.12. *Let k be a positive integer. Then a k-regular connected graph G is fully indecomposable if and only if it is not bipartite.*

Proof. One direction is clear, so we provide the other. By Theorem 2.7, G is fully indecomposable if and only if for any vertex $v \in V(G)$, the graph $G' = G \setminus \{v\}$ contains a $\{1,2\}$-factor. Note that G' is a graph of order $n-1$, in which exactly k vertices have degree $k-1$ and $n-1-k$ vertices have degree k. By Theorem 2.5, assume that $S \subseteq V(G')$ and S contains t vertices of degree $k-1$ and denote by L the set of all isolated vertices of $G' \setminus S$. Then $|L| = i(G' \setminus S)$. Let L contain r vertices of degree $k-1$ in G'. It is clear that $0 \leq t, r \leq k$. Thus the number of incident edges with L in G' is

$$k|L| - r \tag{2.1}$$

and the number of incident edges with S in G' is at most $k|S| - t$. Thus we find that

$$k|L| - r \leq k|S| - t,$$

or equivalently,

$$|L| \leq |S| + \frac{r-t}{k}.$$

We claim that $r - t < k$. If not, then $r = k$, $t = 0$, and $N(v) \subseteq L$. Thus we have $|L| \leq |S| + 1$. Suppose now that $|L| = |S| + 1$. In this case, by Equation (2.1) the number of incident edges with L in G' is $k|S|$. Hence S is an independent set. Now, since G is a connected graph, we conclude that $V(G) = S \cup \{v\} \cup L$ and so G is a bipartite graph with two parts: $(S \cup \{v\}, L)$. This is a contradiction, and completes the proof. $\square$

The following result provides some information about the size of some independent set of vertices with certain degrees sequence in partly decomposable graphs:

Theorem 2.13. *Let G be a connected graph of order n. If G is partly decomposable, then G contains an independent set S of size k, for some integer $1 \leq k \leq n/2$ satisfying $k \geq \max_{v\in S} d(v)$.*

Proof. Let A be the adjacency matrix of G. Then A contains an $s \times t$ zero submatrix with $s + t = n$. Thus there are two subsets $X = \{x_1, \ldots, x_s\}$ and $Y = \{y_1, \ldots, y_t\}$, such that there is no edge between X and Y. Since G is connected, $S = X \cap Y \neq \varnothing$. Now suppose that $S = \{z_1, \ldots, z_k\}$. Clearly $1 \leq k \leq n/2$ and S is an independent set of G. Furthermore, $N(S) \subseteq L = V(G) \setminus (X \cup Y)$ and $|L| = k$. This completes the proof. □

Let G be a graph with a fully indecomposable spanning subgraph H. Then G should be a fully indecomposable graph , which gives us the following remark.

Remark 2.14. If G is a graph of order $2n+1$ with minimum degree δ such that $\delta \geq n+1$, then G is a fully indecomposable graph. To see this, note that using Dirac's Theorem [6, p.288] with $\delta \geq n+1$ implies that G has a Hamiltonian cycle. However, every odd cycle is a fully indecomposable graph.

This fact does not hold for the graphs of order $2n$ and $\delta \geq n$. To show this we introduce a graph of order $2n$ with minimum degree n that is partly decomposable. Let $\{v_1, \ldots, v_n\}$ be the vertex set of K_n. Denote by $G = K_n \vee \overline{K_n}$ the graph obtained from K_n by adding new vertices $\{w_1, \ldots, w_n\}$ and edges $\{w_i, v_j\}$ for $i, j = 1, \ldots, n$. Consider the submatrix B of the adjacency matrix of G whose rows and columns are indexed by the vertices $\{w_1, \ldots, w_n\}$. Since $\{w_1, \ldots, w_n\}$ is an independent set of vertices in G, B is a zero submatrix of size $n \times n$. Thus G is a partly decomposable graph with minimum degree n.

3. Nearly Decomposable Graphs

We now study some properties of the nearly decomposable graphs. Zhou and Zhang in [7] studied the graphs G whose every edge lies in a $\{1,2\}$-factor, with the further condition that for all edges e of G, the graph $G-e$ obtained from deleting the edge e of G does not have that every edge lies in a $\{1,2\}$-factor. The graphs with this property are called *minimal* 2*-matching-covered* graphs. The following lemma describes the nearly decomposable graphs in view of minimal 2-matching-covered graphs.

Lemma 3.1. *Let G be a connected graph. Then G is nearly decomposable if and only if G is a non-bipartite minimal* 2*-matching-covered graph.*

Proof. Let G be a nearly decomposable graph. Hence by definition, G is a fully indecomposable graph in which the graph obtained by removing of every edge is partly decomposable. Combining Remark 1.1 and Theorem 2.9, we find that G is a non-bipartite graph whose every edge lies in a $\{1,2\}$-factor. However, for all edges e of G, the graph $G-e$ obtained from G by deleting the edge e does not satisfy this property. Thus G is a non-bipartite minimal 2-matching-covered graph.

The converse is clear. This completes the proof. □

Hence, using the properties of minimal 2-matching-covered graphs and the previous equivalence, we can obtain the following result:

Theorem 3.2. *Let G be a nearly decomposable graph. Then:*

(*i*) $\delta(G) = 2$, *except for* $G = K_4$;
(*ii*) *Removing every perfect matching of an induced* 4*-cycle of* G *yields an elementary bipartite graph.*

Let $\mathcal{F}(n)$ denote the set of all fully indecomposable graphs of order n. Clearly, every graph $G \in \mathcal{F}(n)$ with the minimum number of edges is a nearly decomposable graph. Now consider

$$\mu := \min_{G \in \mathcal{F}(n)} \operatorname{per}(A_G) = \operatorname{per}(A_{G^*}),$$

where A_G is the adjacency matrix of G.

The following theorem states that G^* is a nearly decomposable graph:

Theorem 3.3. *Let G^* be a fully indecomposable graph with the adjacency matrix A_{G^*}. Assume that $\mu = \operatorname{per}(A_{G^*})$, with μ given as above. Then G^* is a nearly decomposable graph.*

Proof. First we show that every fully indecomposable graph contains a spanning nearly decomposable subgraph. Assume that G is a fully indecomposable graph of order n and denote by $\mathcal{F}(G)$ the set of all fully indecomposable spanning subgraphs of G. Let $G' \in \mathcal{F}(G)$ be a graph with the minimum number of edges. Then G' is a nearly decomposable spanning subgraph of G.

Now assume that G is a fully indecomposable graph which is not nearly decomposable. By the above argument we may find a nearly decomposable spanning subgraph of G, say G', whose number of edges is less than the number of edges of G. But by Theorem 2.1, removing any edge of a fully indecomposable graph strictly decreases the number of $\{1, 2\}$-factors. Thus we find $\operatorname{per}(A_{G'}) < \operatorname{per}(A_G)$. This completes the proof. □

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