Contributions to Discrete Mathematics

Volume 13, Number 1, Pages 23–34
ISSN 1715-0868

UNIFORMLY RESOLVABLE $(C_4, K_{1,3})$-DESIGNS OF INDEX 2

MARIO GIONFRIDDO, SELDA KÜÇÜKÇİFÇİ, SALVATORE MILICI,
AND E. ŞULE YAZICI

Abstract. In this paper we consider the uniformly resolvable decompositions of the complete graph λK_v into subgraphs where each resolution class contains only blocks isomorphic to the same graph. We consider the cases in which all the resolution classes are either C_4 or $K_{1,3}$. We prove that this type of system does not exist for λ odd and determine completely the spectrum for $\lambda = 2$.

1. Introduction

Given a collection of simple graphs $\mathcal{H}$, an *$\mathcal{H}$-decomposition* of the complete graph λK_v is a decomposition of the edges of λK_v into isomorphic copies of graphs in $\mathcal{H}$. The copies of $H \in \mathcal{H}$ in the decomposition are called *blocks*. When $\mathcal{H} = \{G\}$ such a decomposition is also called a *G-design* of order v and index λ. Such a decomposition is called *resolvable* if it is possible to partition the blocks into *classes* $\mathcal{P}_i$ such that every point of K_v appears exactly once in some block of each $\mathcal{P}_i$ [12].

A resolvable $\mathcal{H}$-decomposition of λK_v is sometimes also referred to as an *$\mathcal{H}$-factorization of* λK_v and a resolution class is called an *$\mathcal{H}$-factor of* λK_v. The case where $\mathcal{H}$ is a single edge (K_2) is known as a *1-factorization of* λK_v and it is well known to exist if and only if v is even. A single class of a 1-factorization, a pairing of all points, is also known as a *1-factor* or a *perfect matching*.

In many cases we wish to impose further constraints on the resolution classes of an $\mathcal{H}$-decomposition. For example, a resolution class is called *uniform* if every block of the resolution class is isomorphic to the same graph

Received by the editors August 10, 2015, and in revised form March 17, 2017.

2000 *Mathematics Subject Classification.* 05C51, 05C38, 05C70.

Key words and phrases. Resolvable graph decomposition; uniform resolutions; 3-stars; 4-cycles.

The research of M. Gionfriddo is supported by MIUR-PRIN 2012 and INDAM-GNSAGA, Italy.

The research of S. Küçükçifçi is supported by Scientific and Technological Research Council of Turkey Grant Number: 114F505.

The research of S. Milici is supported by INDAM-GNSAGA, Italy.

in $\mathcal{H}$. Uniformly resolvable decompositions of K_v have also been studied in [1], [2], [4], [5], [6], [8]–[10], [13]–[30].

In this paper we study the existence of a uniformly resolvable decomposition of λK_v having the following type: r resolution classes containing only copies of 4-cycles and s resolution classes containing only copies of 3-stars.

We prove that, for such a system to exist, the index λ is necessarily always even. Therefore, we consider the case $\lambda = 2$, the minimum possible value for the existence. We will use the notation $(C_4, K_{1,3})$-URD$(v, 2; r, s)$ for such a uniformly resolvable decomposition of $2K_v$. Let

> $J(2K_v; C_4, K_{1,3}) = \{(r, s)$: there exists a uniformly resolvable decomposition of $2K_v$ into r resolution classes consisting of C_4's and s resolution classes consisting of $K_{1,3}$'s$\}$.

For $v \equiv 0 \pmod 4$, define $I(v)$ as in Table 1 below:

v	$I(v)$
$0 \pmod{12}$	$\{(v-1-3x, 4x), x = 0, 1, \ldots, \frac{v-3}{3}\}$
$4 \pmod{12}$	$\{(v-1-3x, 4x), x = 0, 1, \ldots, \frac{v-1}{3}\}$
$8 \pmod{12}$	$\{(v-1-3x, 4x), x = 0, 1, \ldots, \frac{v-2}{3}\}$

TABLE 1. The set $I(v)$.

In this paper we completely solve the spectrum problem for such systems; that is, characterize the existence of uniformly resolvable decompositions of $2K_v$ into r resolution classes of 4-cycles and s resolution classes of 3-stars, by proving the following result:

Main Theorem. *For every integer $v \geq 4$, divisible by 4, the set $J(2K_v; C_4, K_{1,3})$ is identical to the set $I(v)$ given in Table 1.*

Now let us introduce some useful definitions, notations, and results then discuss constructions we will use in proving the main theorem. For missing terms or results that are not explicitly explained in the paper, the reader is referred to [7] and its online updates.

For any four vertices a_1, a_2, a_3, a_4, let the 3-*star*, $K_{1,3}$, be the simple graph with the vertex set $\{a_1, a_2, a_3, a_4\}$ and the edge set $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}\}$ and the 4-cycle C_4 be the simple graph with the vertex set $\{a_1, a_2, a_3, a_4\}$ and the edge set $\{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_4, a_1\}\}$. In what follows, we will denote a 3-star by $(a_1; a_2, a_3, a_4)$ and a 4-cycle by (a_1, a_2, a_3, a_4), (a_4, a_3, a_2, a_1) or any cyclic shift of these.

A resolvable $\mathcal{H}$-decomposition of the complete multipartite graph with u parts each of size g is known as a resolvable group divisible design $\mathcal{H}$-RGDD of type g^u, the parts of size g are called the groups of the design. When $\mathcal{H} = K_n$ we will call it an n-RGDD.

A $(C_4, K_{1,3})$-URGDD $(2; r, s)$ of type g^u is a uniformly resolvable decomposition of the complete multipartite graph of index 2 with u parts each of size g into r resolution classes containing only copies of 4-cycles and s resolution classes containing only copies of 3-stars.

If the blocks of an $\mathcal{H}$-GDD of type g^u can be partitioned into partial resolution classes, each of which contain all points except those of one group, we refer to the decomposition as a *frame*. When $\mathcal{H} = K_n$ we will call it an *n-frame* and it can be deduced that the number of partial resolution classes missing a specified group G is $|G|/(n-1)$.

An incomplete resolvable $(C_4, K_{1,3})$-decomposition of $2K_v$ with a hole of size h is a $(C_4, K_{1,3})$-decomposition of $2K_{v+h} - 2K_h$ in which there are two types of resolution classes, full resolution classes and partial resolution classes which cover every point except those in the hole (the points of $2K_h$ are referred to as the hole). Specifically a $(C_4, K_{1,3})$-IURD$(2K_{v+h} - 2K_h; [r, s], [\bar{r}, \bar{s}])$ is a uniformly resolvable $(C_4, K_{1,3})$-decomposition of $2K_{v+h}$ $-2K_h$ with r partial resolution classes of 4-cycles which cover only the points not in the hole, s partial resolution classes of 3-stars which cover only the points not in the hole, $\bar{r}$ full resolution classes of 4-cycles which cover every point of $2K_{v+h}$ and $\bar{s}$ full resolution classes of 3-stars which cover every point of $2K_{v+h}$.

We also need the following definitions. Let (s_1, t_1) and (s_2, t_2) be two pairs of nonnegative integers. Define $(s_1, t_1) + (s_2, t_2) = (s_1 + s_2, t_1 + t_2)$. If X and Y are two sets of pairs of nonnegative integers, then $X + Y$ denotes the set $\{(s_1, t_1) + (s_2, t_2) : (s_1, t_1) \in X, (s_2, t_2) \in Y\}$. If X is a set of pairs of nonnegative integers and h is a positive integer, then $h * X$ denotes the set of all pairs of nonnegative integers which can be obtained by adding any h elements of X together (repetitions of elements of X are allowed).

The following results can be proven in a similar manner to [18].

Lemma 1.1. *If there exists a* $(C_4, K_{1,3})$-URD$(v, 2; r, s)$ *of* $2K_v$ *with* $r > 0$ *and* $s > 0$ *then* $v \equiv 0 \pmod 4$ *and* $(r, s) \in I(v)$.

The following lemma will be very useful in proving the main results of this paper.

Lemma 1.2. *Let v, g, and u be nonnegative integers such that $v = gu$. If there exists*

(1) *a* 4-RGDD *of type* g^u;
(2) *a* $(C_4, K_{1,3})$-URD$(4, 2; r_1, s_1)$ *with* $(r_1, s_1) \in J_1 = \{(3, 0), (0, 4)\}$;
(3) *a* $(C_4, K_{1,3})$-URD$(2K_g; r_2, s_2)$, *with* $(r_2, s_2) \in J_2$, *where* $J_2 = \{(r_2, s_2) :$ *there exists a* $(C_4, K_{1,3})$-URD$(2K_g; r_2, s_2)\}$;

then there exists a $(C_4, K_{1,3})$-URD$(2K_v; r, s)$ *for each* $(r, s) \in J_2 + t * J_1$, *where* $t = g(u-1)/3$ *is the number of resolution classes of the* 4-RGDD *of type* g^u .

2. Nonexistence for λ odd

In this section we prove that $(C_4, K_{1,3})$-URD$(\lambda K_v; r, s)$ exists only for λ even. In what follows, we consider *balanced* and *strongly balanced* G-designs. We recall the definitions in [3],[11], and [12].

Let G be a graph and let $A_1, A_2, ..., A_h$ be the orbits of the automorphism group of G on its vertex set. If $\Sigma = (X, \mathcal{B})$ is a G-design, the *degree* $d_{A_i}(x)$ of a vertex $x \in X$ is the number of blocks of Σ containing x as an element of A_i. We say that $\Sigma = (X, \mathcal{B})$ is *strongly balanced* if, for every $i = 1, 2, ..., h$, there exists a constant C_i such that $d_{A_i}(x) = C_i$, for every $x \in X$.

Clearly, since for each vertex $x \in X$ the relation $d(x) = \sum_{i=1}^{h} d_{A_i}(x)$ holds, we have that all the vertices of Σ have the same degree, that is to say they are contained in the same number of blocks. This means that *strongly balanced designs are always balanced.*

In the case of C_4, since the cycle C_4 admits exactly one automorphism class, all C_4-designs are *balanced* and also *strongly balanced.* In the case of $K_{1,3}$, the automorphism group admits two orbits $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively formed by the centers and by the pendant edges of the stars $K_{1,3}$. This implies that all *balanced* $K_{1,3}$-designs are *strongly balanced* ([12], pp. 134–135).

Theorem 2.1. *There does not exist a* $(C_4, K_{1,3})$-URD$(\lambda K_v; r, s)$ *where* λ *is odd.*

Proof. Let $\Sigma = (X, \mathcal{B})$ be a $(C_4, K_{1,3})$-URD$(\lambda K_v; r, s)$ and suppose λ odd. Necessarily $v = 4k$. Further, if Π is a uniform resolution of Σ, Π can be partitioned into $\mathcal{C} = \{C_1, ..., C_r\}$, a collection of all the resolution classes containing only cycles C_4, and $\mathcal{D} = \{D_1, ..., D_s\}$, a collection of all the resolution classes containing only stars $K_{1,3}$. Observe that:

(1) since $v \equiv 0 \pmod 4$, necessarily $s > 0$;
(2) $|C_1| = |C_2| = ... = |C_r| = |D_1| = |D_2| = ... = |D_s| = k$;
(3) in all the resolution classes of $\mathcal{C}$ every vertex x is incident to two edges: this implies that there are $2r$ pairs containing x in the blocks C_4 and $\lambda(v-1) - 2r$ pairs containing x in the blocks $K_{1,3}$ of $\mathcal{D}$;
(4) if Γ indicates the partial $K_{1,3}$-design generated by $\mathcal{D}$, then we can verify that Γ is *strongly balanced*: indeed, from (2), every vertex x in $\mathcal{D}$ is contained in exactly $\lambda(v-1) - 2r$ edges shared in exactly s blocks $K_{1,3}$, in which x can have degree three or one.

At this point, we see that s is necessarily multiple of 4. Indeed, if h indicates the number of blocks $K_{1,3}$ containing any vertex x in the center, then the number of blocks in $\mathcal{D}$, which is ks, is also equal to vh for the reason that Γ is strongly balanced. It follows that $ks = vh$, from which $s = vh/k = 4h$. This means that the number of pairs containing a fixed vertex x, $\lambda(v-1)$, can be calculated as follows:

$$\lambda(v-1) = 2r + 3h + (s-h) = 2r + 6h,$$

which is a contradiction, for λ odd and $v = 4k$. $\square$

Following the result of Theorem 2.1, we will examine the spectrum of $(C_4, K_{1,3})$-URD$(\lambda K_v; r, s)$s, for $\lambda = 2$, which is the minimum index for which their existence is possible.

3. Small cases

Lemma 3.1. $J(2K_4; C_4, K_{1,3}) = \{(3,0), (0,4)\}$.

Proof. Let $V(K_4)=\mathbb{Z}_4$.

- $(3,0)$
  The three resolution classes of 4-cycles are: $\{(0,1,2,3)\}$, $\{(0,2,3,1)\}$ and
  $\{(0,2,1,3)\}$.
- $(0,4)$
  The four resolution classes of 3-stars can be obtained from the base block $\{\{0; 1, 2, 3\}\}$.

$\square$

Lemma 3.2. $J(2K_8; C_4, K_{1,3}) = \{(7,0), (4,4), (1,8)\}$.

Proof. Let $V(K_8)=\mathbb{Z}_8$.

- $(7,0), (4,4)$
  Take a $(C_4, K_{1,3})$-URGDD$(2; 4, 0)$ of type 4^2 [8] and replace each group of size 4 with the same $(C_4, K_{1,3})$-URD$(2K_4; r, s)$, with $(r, s) \in \{(3,0),\ (0,4)\}$ which exists by Lemma 3.1.
- $(1,8)$
  Eight resolution classes of 3-stars and one resolution class of 4-cycles are

$$\{(0;2,4,6),(1;3,5,7)\},\{(2;4,1,6),(3;5,0,7)\},\{(5;2,0,7),(4;1,3,6)\},$$
$$\{(0;2,1,3),(4;6,5,7)\},\{(2;4,1,3),(6;5,0,7)\},\{(5;2,0,7),(1;4,3,6)\},$$
$$\{(7;0,1,2),(3;4,5,6)\},\{(7;2,0,4),(6;1,3,5)\},\{(1,5,4,0),(2,6,7,3)\}.$$

$\square$

Lemma 3.3. $J(2K_{12}; C_4, K_{1,3}) = \{(11,0), (8,4), (5,8), (2,12)\}$.

Proof. Let $V(K_{12})=\mathbb{Z}_{12}$.

- $(11,0), (8,4)$
  Take a $(C_4, K_{1,3})$-URGDD$(2; 8, 0)$ of type 4^3 [8] and replace each group of size 4 with the same $(C_4, K_{1,3})$-URD$(2K_4; r, s)$, with $(r, s) \in \{(3,0),\ (0,4)\}$, which exists by Lemma 3.1.
- $(5,8)$

Five resolution classes of 4-cycles and eight resolution classes of 3-stars are:

$\{(0,1,4,7),(2,3,6,5),(8,11,9,10)\},\{(0,11,10,3),(1,2,9,8),(4,6,7,5)\},$
$\{(3,1,4,8),(2,0,6,10),(7,11,9,5)\},\{(3,11,5,0),(1,2,9,7),(4,6,8,10)\},$
$\{(1,3,2,0),(4,8,11,7),(6,10,9,5)\},$
$\{(0;4,5,6),(7;8,9,10),(11;1,2,3)\},\{(1;5,6,7),(4;9,10,11),(8;0,2,3)\},$
$\{(2;4,6,7),(5;8,10,11),(9;0,1,3)\},\{(3;4,5,7),(6;8,9,11),(10;0,1,2)\},$
$\{(3;4,10,6),(8;7,9,5),(11;1,2,0)\},\{(1;10,6,8),(4;9,5,11),(7;0,2,3)\},$
$\{(2;4,6,8),(10;7,5,11),(9;0,1,3)\},\{(0;4,10,8),(6;7,9,11),(5;3,1,2)\}.$

- $(2,12)$
  Two resolution classes of 4-cycles are: $\{(0,5,6,1),(2,7,8,3),(4,9,10,11)\}$, $\{(0,7,6,11),(1,8,9,2)$ and $(3,4,5,10)\}$. Twelve resolution classes of 3-stars can be obtained from the base blocks: $\{(4;10,1,6),(9;2,5,7),(11;3,8,0)\}$.

□

Lemma 3.4. *There exists a* $(C_4, K_{1,3})$-URGDD$(2;r,s)$ *of type* 12^2 *with* $(r,s)\in\{(12,0),(6,8),(0,16)\}$.

Proof. Take the groups to be $\{a_1,a_2\dots,a_{12}\}$ and $\{b_1,b_2,\dots,b_{12}\}$.

- The case $(12,0)$ is given in [8].
- $(0,16)$
  Sixteen resolution classes of 3-stars are obtained by considering $i=1,4,7,10$ in the blocks listed below:
  $\{(a_i;b_i,b_{i+1},b_{i+2}),\ (a_{i+1};b_{i+3},b_{i+4},b_{i+5}),$
  $(a_{i+2};b_{i+6},b_{i+7},b_{i+8}),(b_{i+9};a_{i+3},a_{i+4},a_{i+5}),$
  $(b_{i+10};a_{i+6},a_{i+7},a_{i+8}),(b_{i+11};a_{i+9},a_{i+10},a_{i+11})\},$
  $\{(a_i;b_{i+3},b_{i+4},b_{i+5}),(a_{i+1};b_{i+6},b_{i+7},b_{i+8}),$
  $(a_{i+2};b_{i+9},b_{i+10},b_{i+11}),(b_i;a_{i+3},a_{i+4},a_{i+5}),$
  $(b_{i+1};a_{i+6},a_{i+7},a_{i+8}),(b_{i+2};a_{i+9},a_{i+10},a_{i+11})\},$
  $\{(a_i;b_{i+6},b_{i+7},b_{i+8}),(a_{i+1};b_{i+9},b_{i+10},b_{i+11}),$
  $(a_{i+2};b_i,b_{i+1},b_{i+2}),(b_{i+3};a_{i+3},a_{i+4},a_{i+5}),$
  $(b_{i+4};a_{i+6},a_{i+7},a_{i+8}),(b_{i+5};a_{i+9},a_{i+10},a_{i+11})\},$
  $\{(a_i;b_{i+9},b_{i+10},b_{i+11}),(a_{i+1};b_i,b_{i+1},b_{i+2}),$
  $(a_{i+2};b_{i+3},b_{i+4},b_{i+5}),(b_{i+6};a_{i+3},a_{i+4},a_{i+5}),$
  $(b_{i+7};a_{i+6},a_{i+7},a_{i+8}),(b_{i+8};a_{i+9},a_{i+10},a_{i+11})\}.$

- $(6,8)$
  Six resolution classes of 4-cycles are:
  $\{(a_{12},b_{12},a_5,b_1),(a_1,b_2,a_7,b_3),(a_2,b_4,a_8,b_5),(a_3,b_7,a_4,b_8),$
  $(a_6,b_9,a_9,b_{10}),(a_{10},b_6,a_{11},b_{11})\}$
  $\{(a_{12},b_2,a_8,b_3),(a_1,b_4,a_3,b_5),(a_2,b_6,a_{11},b_7),(a_4,b_8,a_6,b_{11}),$
  $(a_5,b_9,a_7,b_{10}),(a_9,b_{12},a_{10},b_1)\}$
  $\{(a_{12},b_2,a_2,b_7),(a_1,b_4,a_9,b_6),(a_3,b_5,a_4,b_9),(a_5,b_{12},a_7,b_{10}),$

$(a_6, b_8, a_8, b_{11}), (a_{10}, b_1, a_{11}, b_3)\}$
$\{(a_{12}, b_4, a_{10}, b_5), (a_1, b_6, a_6, b_7), (a_2, b_9, a_4, b_{10}), (a_3, b_3, a_{11}, b_8),$
$(a_5, b_2, a_9, b_{11}), (a_7, b_{12}, a_8, b_1)\}$
$\{(a_{12}, b_4, a_{11}, b_5), (a_1, b_7, a_5, b_9), (a_2, b_8, a_7, b_{11}), (a_3, b_{12}, a_4, b_6),$
$(a_6, b_1, a_8, b_{10}), (a_9, b_2, a_{10}, b_3)\}$
$\{(a_{12}, b_6, a_2, b_9), (a_1, b_5, a_5, b_8), (a_3, b_7, a_4, b_{10}), (a_6, b_{12}, a_8, b_3),$
$(a_7, b_1, a_9, b_{11}), (a_{10}, b_2, a_{11}, b_4)\}$.
Eight resolution classes of 3-stars are the last eight resolution classes of the case $(0, 16)$.

□

Lemma 3.5. $J(24, 2; C_4, K_{1,3}) = I(24)$.

Proof. Take a $(C_4, K_{1,3})$-URGDD$(2; r, s)$ of type 12^2 and index 2 with $(r, s) \in \{(12, 0), (0, 16)\}$ which exists by Lemma 3.4. Replace each group of size 12 with the same $(C_4, K_{1,3})$-URD$(2K_{12}; r, s)$, where $(r, s) \in \{(11, 0), (8, 4), (5, 8), (2, 12)\}$ which exists by Lemma 3.3. □

Lemma 3.6. $J(36, 2; C_4, K_{1,3}) = I(36)$.

Proof. Take a $(C_4, K_{1,3})$-URGDD$(12, 0)$ of type 12^3 and index 1 which is given in [8] and a $(C_4, K_{1,3})$-URGDD$(0, 16)$ of type 12^3 and index 1 with is given in [19]. Combine the two above designs to obtain a $(C_4, K_{1,3})$-URGDD$(2; r, s)$ of type 12^3 and index 2 with $(r, s) \in \{(24, 0), (12, 16), (0, 32)\}$. Replace each group of size 12 with the same $(C_4, K_{1,3})$-URD$(2K_{12}; r, s)$, where $(r, s) \in \{(11, 0), (8, 4), (5, 8), (2, 12)\}$ which exists by Lemma 3.3. □

Lemma 3.7. *There exists a* $(C_4, K_{1,3})$-IURD$(2K_{20} - 2K_8; [r, s], [\bar{r}, \bar{s}])$ *with* $(r, s) \in \{(7, 0), (4, 4), (1, 8)\}$ *and* $(\bar{r}, \bar{s}) \in \{(12, 0), (9, 4), (6, 8), (3, 12), (0, 16)\}$.

Proof. Let the point set of K_{20} be $\mathbb{Z}_{20}$ and the point set $\{0, 1, \ldots, 7\}$ be the hole. The following partial resolution classes $(7, 0)$, $(4, 4)$, $(1, 8)$ cover the same edges on the point set $K_{20} - K_8$.

- $(7, 0)$
  Seven partial resolution classes of 4-cycles are:
  $\{(8, 9, 11, 10), (12, 13, 15, 14), (16, 17, 19, 18)\}$,
  $\{(8, 11, 15, 12), (9, 10, 19, 16), (13, 14, 18, 17)\}$,
  $\{(8, 13, 10, 15), (9, 18, 11, 19), (12, 16, 14, 17)\}$,
  $\{(8, 14, 16, 10), (12, 18, 15, 9), (11, 17, 19, 13)\}$,
  $\{(8, 16, 15, 12), (14, 10, 19, 11), (18, 9, 13, 17)\}$,
  $\{(8, 18, 10, 15), (14, 13, 16, 19), (12, 11, 9, 17)\}$,
  $\{(8, 9, 15, 14), (10, 11, 17, 16), (12, 13, 19, 18)\}$.
- $(4, 4)$
  Four partial resolution classes of 4-cycles are the last four resolution classes of the case $(7, 0)$ above. Four partial resolution classes of 3-stars are:
  $\{(8; 9, 10, 11), (14; 12, 13, 15), (19; 16, 17, 18)\}$,
  $\{(9; 10, 11, 19), (15; 8, 12, 13), (17; 14, 16, 18)\}$,

$\{(10; 11, 15, 19), (12; 8, 13, 17), (18; 9, 14, 16)\}$,
$\{(11; 15, 18, 19), (13; 8, 10, 17), (16; 9, 12, 14)\}$.

- $(1, 8)$
  One partial resolution class of 4-cycles is the last resolution class of the case $(7, 0)$ above. Eight partial resolution classes of 3-stars are:
  $\{(8; 9, 10, 11), (12; 13, 14, 15), (16; 17, 18, 19)\}$,
  $\{(8; 10, 12, 13), (9; 11, 15, 16), (14; 17, 18, 19)\}$,
  $\{(9; 10, 11, 12), (13; 14, 17, 19), (15; 8, 16, 18)\}$,
  $\{(10; 11, 13, 19), (14; 8, 15, 16), (17; 9, 12, 18)\}$,
  $\{(10; 14, 15, 16), (17; 11, 12, 13), (18; 8, 9, 19)\}$,
  $\{(11; 12, 14, 18), (15; 8, 10, 13), (19; 9, 16, 17)\}$,
  $\{(11; 13, 15, 19), (16; 8, 12, 14), (18; 9, 10, 17)\}$,
  $\{(12; 8, 15, 18), (13; 9, 14, 16), (19; 10, 11, 17)\}$.

Now all the edges that are not covered in the above partial resolution classes on the point set $K_{20} - K_8$ will be covered by the following full resolution classes on the point set K_{20}.

- $(12, 0)$
  Twelve full resolution classes of 4-cycles are:
  $\{(0, 8, 1, 9), (2, 10, 3, 11), (4, 12, 5, 13), (6, 14, 18, 16), (7, 17, 15, 19)\}$,
  $\{(0, 8, 1, 9), (2, 10, 3, 11), (4, 12, 5, 13), (6, 16, 7, 18), (14, 17, 15, 19)\}$,
  $\{(0, 10, 1, 11), (2, 8, 3, 9), (4, 14, 12, 16), (5, 15, 13, 18), (6, 17, 7, 19)\}$,
  $\{(0, 10, 1, 11), (2, 8, 3, 9), (4, 17, 5, 19), (6, 12, 7, 14), (13, 16, 15, 18)\}$,
  $\{(0, 12, 1, 13), (2, 14, 3, 15), (4, 8, 16, 9), (5, 17, 6, 19), (7, 10, 18, 11)\}$,
  $\{(0, 12, 1, 13), (2, 14, 11, 16), (3, 15, 4, 17), (5, 8, 19, 9), (6, 10, 7, 18)\}$,
  $\{(0, 14, 1, 15), (2, 18, 3, 19), (4, 9, 7, 16), (5, 8, 17, 10), (6, 12, 11, 13)\}$,
  $\{(0, 14, 1, 15), (2, 13, 8, 17), (3, 16, 5, 18), (4, 10, 12, 19), (6, 9, 7, 11)\}$,
  $\{(0, 16, 1, 17), (2, 15, 4, 18), (3, 13, 10, 14), (5, 9, 6, 11), (7, 8, 19, 12)\}$,
  $\{(0, 16, 1, 17), (2, 12, 3, 19), (4, 11, 8, 18), (5, 10, 9, 14), (6, 13, 7, 15)\}$,
  $\{(0, 18, 1, 19), (2, 13, 3, 16), (4, 11, 5, 14), (6, 8, 7, 15), (9, 12, 10, 17)\}$,
  $\{(0, 18, 1, 19), (2, 12, 3, 17), (4, 8, 6, 10), (5, 15, 11, 16), (7, 13, 9, 14)\}$.
- $(9, 4)$
  Nine full resolution classes of 4-cycles are:
  $C_1 = \{(0, 10, 1, 11), (2, 8, 3, 9), (4, 17, 5, 19), (6, 12, 7, 14), (13, 16, 15, 18)\}$,
  $C_2 = \{(0, 12, 1, 13), (2, 14, 3, 15), (4, 8, 16, 9), (5, 17, 6, 19), (7, 10, 18, 11)\}$,
  $C_3 = \{(0, 14, 1, 15), (2, 12, 11, 16), (3, 13, 6, 18), (4, 17, 8, 19), (5, 9, 7, 10)\}$,
  $C_4 = \{(0, 10, 1, 11), (2, 8, 3, 9), (4, 14, 12, 16), (5, 15, 13, 18), (6, 17, 7, 19)\}$,
  $C_5 = \{(0, 18, 1, 19), (2, 13, 3, 17), (4, 8, 11, 16), (5, 10, 6, 15), (7, 12, 9, 14)\}$
  $C_6 = \{(0, 8, 1, 9), (2, 10, 3, 11), (4, 12, 5, 13), (6, 14, 18, 16), (7, 17, 15, 19)\}$,
  $C_7 = \{(0, 8, 1, 9), (2, 10, 3, 11), (4, 12, 5, 13), (6, 16, 7, 18), (14, 17, 15, 19)\}$,
  $C_8 = \{(0, 12, 1, 13), (2, 14, 3, 15), (4, 10, 6, 11), (5, 16, 7, 18), (8, 17, 9, 19)\}$,
  $C_9 = \{(0, 14, 4, 18), (1, 16, 2, 19), (3, 12, 10, 17), (5, 8, 13, 11), (6, 9, 7, 15)\}$.
  Four full resolution classes of 3-stars are:
  $S_1 = \{(0; 15, 16, 17), (7; 8, 11, 13), (12; 6, 10, 19), (14; 1, 5, 9), (18; 2, 3, 4)\}$,
  $S_2 = \{(1; 16, 17, 18), (4; 9, 10, 15), (5; 8, 11, 14), (13; 2, 6, 7), (19; 0, 3, 12)\}$,

$S_3 = \{(2; 12, 18, 19), (6; 8, 9, 11), (10; 13, 14, 17), (15; 1, 4, 7), (16; 0, 3, 5)\}$,
$S_4 = \{(3; 12, 16, 19), (8; 6, 7, 18), (9; 5, 10, 13), (11; 4, 14, 15), (17; 0, 1, 2)\}$.

- $(6, 8)$
  Six full resolution classes of 4-cycles are $C_1, C_2, C_3, C_4,\ C_5$ of the case $(9, 4)$ above together with the following full resolution class of 4-cycles:
  $\{(0, 8, 17, 9), (1, 12, 4, 13), (2, 10, 6, 11), (3, 14, 19, 15), (5, 16, 7, 18)\}$.
  Eight full resolution classes of 3-stars are S_1, S_2, S_3, S_4 of the case $(9, 4)$ above together with the following four resolution classes of 3-stars:
  $S_5 = \{(0; 9, 14, 18), (5; 11, 12, 13), (10; 3, 4, 17), (16; 1, 2, 6), (19; 7, 8, 15)\}$,
  $S_6 = \{(1; 8, 9, 19), (7; 15, 16, 18), (11; 2, 3, 13), (12; 0, 5, 10), (14; 4, 6, 17)\}$,
  $S_7 = \{(2; 10, 14, 19), (4; 11, 12, 13), (6; 9, 16, 18), (8; 0, 1, 5), (17; 3, 7, 15)\}$,
  $S_8 = \{(3; 10, 11, 12), (9; 1, 7, 19), (13; 0, 5, 8), (15; 2, 6, 17), (18; 4, 14, 16)\}$.
- $(3, 12)$
  Three full resolution classes of 4-cycles are C_1, C_2, C_3 of the case $(9, 4)$.
  Twelve full resolution classes of 3-stars are S_1, S_2, S_3, S_4 of the case $(9, 4)$, S_5, S_6, S_7, S_8 of the case $(6, 8)$ above together with the following four resolution classes of 3-stars:
  $S_9 = \{(0; 9, 10, 18), (1; 11, 12, 13), (4; 8, 14, 16), (15; 3, 5, 19), (17; 2, 6, 7)\}$,
  $S_{10} = \{(2; 9, 10, 11), (3; 8, 13, 17), (5; 15, 16, 18), (12; 4, 7, 14), (19; 0, 1, 6)\}$,
  $S_{11} = \{(6; 10, 11, 15), (8; 0, 2, 17), (14; 3, 9, 19), (16; 4, 7, 12), (18; 1, 5, 13)\}$,
  $S_{12} = \{(7; 14, 18, 19), (9; 3, 12, 17), (10; 1, 5, 6), (11; 0, 8, 16), (13; 2, 4, 15)\}$.
- $(0, 16)$
  Sixteen full resolution classes of 3-stars are $S_i,\ i = 2, 3, ..., 12$ above together with the following five resolution classes of 3-stars:
  $\{(0; 10, 11, 17), (2; 8, 12, 15), (13; 7, 16, 18), (14; 1, 3, 9), (19; 4, 5, 6)\}$,
  $\{(0; 13, 15, 16), (5; 14, 17, 19), (7; 8, 9, 10), (12; 1, 6, 11), (18; 2, 3, 4)\}$,
  $\{(1; 10, 13, 14), (12; 0, 7, 19), (16; 2, 8, 9), (17; 4, 5, 6), (18; 3, 11, 15)\}$,
  $\{(3; 8, 13, 15), (4; 9, 17, 19), (10; 5, 12, 18), (11; 1, 7, 16), (14; 0, 2, 6)\}$,
  $\{(6; 12, 13, 18), (7; 10, 11, 14), (8; 4, 17, 19), (9; 2, 3, 5), (15; 0, 1, 16)\}$.

□

Lemma 3.8. $J(2K_{20}; C_4, K_{1,3}) = I(20)$.

Proof. Replace the hole of size 8 in Lemma 3.7 by a $(C_4, K_{1,3})$-URD$(2K_8; r, s)$, with $(r, s) \in \{(7, 0), (4, 4), (1, 8)\}$ which exists by Lemma 3.2. □

4. Main results

Lemma 4.1. *For every* $v \equiv 0 \pmod{12}$, $I(v) \subseteq J(2K_v; C_4, K_{1,3})$.

Proof. For $v = 12,\ 24,\ 36$ the conclusion follows from Lemmas 3.3, 3.5 and 3.6, respectively. For $v \geq 48$ start with a 4-RGDD of type $12^{\frac{v}{12}}$ [7] and apply Lemma 1.2 with $g = 12, u = v/12$ and $t = (v - 12)/3$ (The input designs are a $(C_4, K_{1,3})$-URD$(2K_4; r, s)$, with $(r, s) \in \{(3, 0), (0, 4)\}$, which exists by Lemma 3.1 and a $(C_4, K_{1,3})$-URD$(2K_{12}; r, s)$, with $(r, s) \in \{(11, 0), (8, 4), (5, 8), (2, 12)\}$, which exists by Lemma 3.3). This implies

$$J(2K_v; C_4, K_{1,3}) \supseteq \{\{(11, 0), (8, 4), (5, 8), (2, 12)\} + \frac{(v-12)}{3} * \{(3, 0), (0, 4)\}\}.$$

Since

$$\frac{v-12}{3} * \{(3, 0), (0, 4)\} = \left\{(v - 12 - 3x, 4x), x = 0, \dots, \frac{v-12}{3}\right\},$$

it is easy to see that

$$\{\{(11, 0), (8, 4), (5, 8), (2, 12)\} + \frac{(v-12)}{3} * \{(3, 0), (0, 4)\}\} = I(v).$$

This completes the proof. □

Lemma 4.2. *For every* $v \equiv 8 \pmod{24}$, $I(v) \subseteq J(2K_v; C_4, K_{1,3})$.

Proof. For $v = 8$ the result follows by Lemma 3.2. For $v \geq 32$, start with a 4-RGDD of type $8^{\frac{v}{8}}$ [7] and apply Lemma 1.2 with $g = 8, u = v/8$ and $t = (v - 8)/3$ (The input designs are a $(C_4, K_{1,3})$-URD$(2K_4; r, s)$, with $(r, s) \in \{(3, 0), (0, 4)\}$, which exists by Lemma 3.1 and a $(C_4, K_{1,3})$-URD$(2K_8; r, s)$, with $(r, s) \in \{(7, 0), (4, 4), (1, 8)\}$, which exists by Lemma 3.2). Proceeding as in Lemma 4.1 the result follows. □

Lemma 4.3. *For every* $v \equiv 4 \pmod{12}$, $I(v) \subseteq J(2K_v; C_4, K_{1,3})$.

Proof. For $v = 4$ the result follows by Lemma 3.1. For $v \geq 16$, start with a 4-RGDD of type $4^{\frac{v}{4}}$ [7] and apply Lemma 1.2 with $g = 4, u = v/4$ and $t = (v - 4)/3$ (The input design is a $(C_4, K_{1,3})$-URD$(2K_4; r, s)$, with $(r, s) \in \{(3, 0), (0, 4)\}$, which exists by Lemma 3.1). Proceeding as in Lemma 4.1 the result follows. □

Lemma 4.4. *For every* $v \equiv 20 \pmod{24}$, $I(v) \subseteq J(2K_v; C_4, K_{1,3})$.

Proof. The case $v = 20$ follows by Lemma 3.8. For $v \geq 44$ start from a 2-frame $\mathcal{F}$ of type $1^{\frac{v-8}{12}}$ with groups G_i, $i = 1, 2, \dots, (v - 8)/12$ [7]. Then expand each point by 12 points and add a set $H = \{a_1, a_2, \dots, a_8\}$. For $i = 1, 2, \dots, (v - 8)/12$, let P_i be the partial factor which misses the group G_i.

Replace each block $b \in P_i$ by a $(C_4, K_{1,3})$-URGDD$(2; r_1, s_1)$ of type 12^2 and index 2, say D_i^b on the vertex set of $b \times \{1, 2, ..., 12\}$ with $(r_1, s_1) \in \{(12, 0), (6, 8), (0, 16)\}$, which exists by Lemma 3.4.

For $i = 1, 2, \dots, (v - 8)/12$ place on $H \cup (G_i \times \{1, 2, \dots, 12\})$ a copy of a $(C_4, K_{1,3})$-IURD$(2K_{20} - 2K_8; [x_1, y_1], [x, y])$, say D_i with $(x_1, y_1) \in \{(7, 0), (4, 4), (1, 8)\}$ and $(x, y) \in \{(12, 0), (6, 8), (0, 16)\}$, which exists by Lemma 3.7. Combine the resolution classes of D_i^b with the full resolution classes of D_i so to obtain r_2 resolution classes of C_4 and s_2 resolution classes of $K_{1,3}$ with $(r_2, s_2) \in \{((v - 8)/12) * \{(12, 0), (6, 8), (0, 16)\}\}$.

Fill the hole H with a copy of a $(C_4, K_{1,3})$-URD$(2K_8; r, s)$ say D with $(r_4, s_4) \in \{(7, 0), (4, 4), (1, 8)\}$, which exists by Lemma 3.2. Combine the resolution classes of D with the partial of D_i to obtain r_4 resolution classes of C_4 and s_4 resolution classes of $K_{1,3}$ with $(r_4, s_4) \in \{(7,0), (4,4), (1,8)\}$.

This gives a $(C_4, K_{1,3})$-URD$(2K_v; r, s)$, with $(r, s) \in \{\{(7,0), (4,4), (1,8)\} +((v-8)/12) * \{(12,0), (6,8), (0,16)\}\}$. Proceeding as in Lemma 4.1 we obtain the result. □

Combining Lemmas 4.1, 4.2, 4.3, and 4.4 we obtain the main theorem of this article.

Theorem 4.5. *For each $v \equiv 0 \pmod 4$, we have $J(2K_v; C_4, K_{1,3}) = I(v)$.*

References

1. P. Adams, E. Billington, D. Bryant, and S. El-Zanati, *On the Hamilton–Waterloo problem*, Graphs Combin. **18** (2002), 31–51.
2. B. Alspach, P. J. Shellenberg, D. R. Stinson, and D. Wagner, *The Oberwolfach problem and factors of uniform length*, J. Combin. Theory Ser. A **52** (1989), 20–43.
3. L. Berardi, M. Gionfriddo, and R. Rota, *Balanced and strongly balanced P_k-designs*, Discrete Math. **312** (2012), 633–636.
4. D. Bryant and P. Danziger, *On bipartite 2-factorizations of $K_n - I$ and the Oberwolfach problem*, J. Graph Theory **68** (2011), 22–37.
5. D. Bryant, P. Danziger, and W. Pettersson, *Bipartite 2-factorizations of complete multipartite graphs*, J. Graph Theory **78** (2015), 287–294.
6. D. Bryant and V. Scharaschkin, *Complete solutions to the Oberwolfach problem for an infinite set of orders*, J. Combin. Theory Ser. B **78** (2009), 904–918.
7. C. J. Colbourn and J. H. Dinitz, *The CRC Handbook of Combinatorial Designs*, Chapman and Hall/CRC, Boca Raton, FL, 2007, Online updates `www.emba.uvm.edu/~dinitz/newresults.html`.
8. P. Danziger, G. Quattrocchi, and B. Stevens, *The Hamilton–Waterloo problem for cycle sizes 3 and 4*, J. Combin. Des. **12** (2004), 221–232.
9. J. H. Dinitz and A. C. H. Ling, *The Hamilton–Waterloo problem: The case of triangle-factors and one Hamilton cycle*, J. Combin. Des. **17** (2009), no. 2, 160–176.
10. J. H. Dinitz, A. C. H. Ling, and P. Danziger, *Maximum uniformly resolvable designs with block sizes 2 and 4*, Discrete Math. **309** (2009), 4716–4721.
11. M. Gionfriddo, S. S. Küçükçifçi, and L. Milazzo, *Balanced and strongly balanced 4-kite designs*, Util. Math. **91** (2013), 121–129.
12. M. Gionfriddo, L. Milazzo, and V. Voloshin, *Hypergraphs and Designs*, Mathematics Research Developments, Nova Science Publishers Inc, 2014.
13. M. Gionfriddo and S. Milici, *On the existence of uniformly resolvable decompositions of k_v and $k_v - i$ into paths and kites*, Discrete Math. **313** (2013), 2830–2834.
14. ———, *Uniformly resolvable $\mathcal{H}$-designs with $\mathcal{H} = \{P_3, P_4\}$*, Australas. J. Combin. **60** (2014), no. 3, 325–332.
15. ———, *Uniformly resolvable $\{K_2, P_k\}$-designs with $k = \{3, 4\}$*, Contr. Discrete Math. **10** (2015), no. 1, 126–133.
16. P. Horak, R. Nedela, and A. Rosa, *The Hamilton-Waterloo problem: the case of Hamilton factors and triangle-factors*, Discrete Math. **284** (2004), 181–188.
17. E. Kohler, *Das Oberwolfacher problem*, Mitt. Math. Gesellsch. Hamburg **10** (1973), 124–129.
18. S. Küçükçifçi, G. Lo Faro, S. Milici, and A. Tripodi, *Resolvable 3-star designs*, Discrete Math. **338** (2015), 608–614.
19. S. Küçükçifçi, S. Milici, and Zs. Tuza, *Maximum uniformly resolvable decompositions of k_v into 3-stars and 3-cycles*, Discrete Math. **338** (2015), 1667–1673.

20. J. Liu, *The equipartite Oberwolfach problem with uniform tables*, J. Combin. Theory Ser. A **101** (2003), 20–34.
21. J. Liu and D. R. Lick, *On λ-fold equipartite Oberwolfach problem with uniform table sizes*, Ann. Combin. **7** (2003), 315–323.
22. S. Milici, *A note on uniformly resolvable decompositions of K_v and $K_v - I$ into 2-stars and 4-cycles*, Australas. J. Combin. **56** (2013), 195–200.
23. S. Milici and Zs. Tuza, *Uniformly resolvable decompositions of K_v into P_3 and K_3 graphs*, Discrete Math. **331** (2014), 137–141.
24. W. L. Piotrowski, *The solution of the bipartite analogue of the Oberwolfach problem*, Discrete Math. **97** (1991), 339–356.
25. R. Rees, *Uniformly resolvable pairwise balanced designs with block sizes two and three*, J. Combin. Theory Ser. A **45** (1987), 207–225.
26. E. Schuster, *Uniformly resolvable designs with index one and block sizes three and five and up to five with blocks of size five*, Discrete Math. **309** (2009), 4435–4442.
27. ———, *Uniformly resolvable designs with index one and block sizes three and four-with three or five parallel classes of block size four*, Discrete Math. **309** (2009), 2452–2465.
28. ———, *Small uniformly resolvable designs for block sizes 3 and 4*, J. Combin. Des. **21** (2013), 481–523.
29. E. Schuster and G. Ge, *On uniformly resolvable designs with block sizes 3 and 4*, Des. Codes Cryptogr. **57** (2010), 47–69.
30. T. Traetta, *A complete solution to the two-table Oberwolfach problems*, J. Combin. Theory Ser. A. **120** (2013), 984–997.

Dipartimento di Matematica e Informatica, Università di Catania, Catania, Italy

E-mail address: `gionfriddo@dmi.unict.it`

Department of Mathematics, Koç University, Istanbul, Turkey

E-mail address: `skucukcifci@ku.edu.tr`

Dipartimento di Matematica e Informatica, Università di Catania, Catania, Italy

E-mail address: `milici@dmi.unict.it`

Department of Mathematics, Koç University, Istanbul, Turkey

E-mail address: `eyazici@ku.edu.tr`