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ON THE TRANSMISSION OF UNIFORM UNICYCLIC HYPERGRAPHS

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Abstract. The transmission of a connected hypergraph is defined as the sum of distances between all unordered pairs of distinct vertices. We determine the unique uniform unicyclic hypergraphs of fixed size with minimum and maximum transmissions, respectively.

1. Introduction

A hypergraph G is a pair (V, E), where $V = V(G)$ is a nonempty finite set called the vertex set of G and $E = E(G)$ is a family of subsets of $V(G)$ called the edge set of G [2]. The size of G is the cardinality of $E(G)$. For an integer $k \geq 2$, a hypergraph is k-uniform if all its edges have cardinality k. A hypergraph G is linear if $|e \cap f| \leq 1$ for any $\{e, f\} \subseteq E(G)$. An ordinary graph is a 2-uniform (linear) hypergraph.

A path in a hypergraph G is an alternating sequence of distinct vertices and edges of the form $(v_1, e_1, v_2, e_2, \ldots, v_k)$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leq i \leq k-1$. The number of edges of a path is its length. If there is a path from u to v for any $u, v \in V(G)$, then we say that G is connected.

A cycle in a hypergraph G is an alternating sequence of distinct vertices and edges of the form $(v_1, e_1, v_2, e_2, \ldots, v_{\ell+1})$ except $v_{\ell+1} = v_1$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leq i \leq \ell$, see [2, p. 391]. The number of edges of a cycle is its length. Note that the length of a cycle of a graph is at least 3, while the length of a cycle of a k-uniform hypergraph with $k \geq 3$ is at least and can be 2.

A hypertree is a connected acyclic hypergraph. A unicyclic hypergraph is a connected hypergraph with exactly one cycle. Evidently, a hypertree

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is linear, while a k-uniform unicyclic hypergraph is linear if and only if the length of its unique cycle is at least 3.

Note that a k-uniform unicyclic hypergraph of size $m \geq 2$ always has $m(k-1)$ vertices, where $m \geq 3$ if $k = 2$, see [2, p. 393].

Let G be a connected hypergraph. For $u, v \in V(G)$, the distance between u and v is the length of a shortest path from u to v in G, denoted by $d_G(u, v)$. In particular, $d_G(u, u) = 0$. The transmission $\sigma(G)$ of G is defined as the sum of distances between all unordered pairs of distinct vertices in G, i.e., $\sigma(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v)$.

Any hypergraph G corresponds naturally to a graph O_G with the same vertex set such that two vertices u and v are adjacent in O_G if and only if $u, v \in e$ for some $e \in E(G)$. So an edge of G with size k corresponds to a clique of size k in O_G. If G is connected, then $d_G(u, v) = d_{O_G}(u, v)$ for $u, v \in V(G)$, so $\sigma(G) = \sigma(O_G)$. In this sense, the study of transmission of a hypergraph G is the study of the transmission of its corresponding 'clique graph' O_G.

For ordinary graphs, the transmission (see [11]) is also known as the sum of the distances [10] and the Wiener index [1, 3, 4, 7, 8].

Tang and Deng [13] determined the unique unicyclic graphs with maximum and minimum transmissions, respectively. The transmissions of a connected hypergraph were discussed in [9]. Sun et al. [12] computed the transmissions of some special k-uniform hypergraphs, and provided a lower bound for the transmission of a k-uniform hypergraph with given circumference. Guo et al. [6] determined the unique k-uniform hypertrees with maximum, second maximum and third maximum transmissions, as well as the unique k-uniform hypertrees with minimum, second minimum and third minimum transmissions, respectively. A relation between the transmission and the degree distance (Gutman index, respectively) of uniform hypertrees was established in [5].

In this article, we characterize the unique uniform unicyclic hypergraphs of fixed size with minimum and maximum transmissions, respectively.

2. Preliminaries

Let G be a connected hypergraph. For $A \subseteq V(G)$, let

$$\sigma_G(A) = \sum_{\{u,v\} \subseteq A} d_G(u, v).$$

For $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, let

$$\sigma_G(A, B) = \sum_{a \in A, b \in B} d_G(a, b).$$

For $X \subseteq V(G)$ with $X \neq \emptyset$, let $G[X]$ be the subhypergraph of G induced by X, i.e., $G[X]$ has vertex set X and edge set $\{e \subseteq X : e \in E(G)\}$. For $u \in V(G)$, let $G - u$ be the subhypergraph of G obtained by deleting u

and all edges containing u. A component of a hypergraph G is a maximal connected subhypergraph of G.

For a hypergraph G with $v \in V(G)$, let $E_G(v)$ be the set of edges of G containing v. The degree of a vertex v in G, denoted by $d_G(v)$, is defined as $d_G(v) = |E_G(v)|$.

A path $P = (v_0, e_1, v_1, \ldots, v_{p-1}, e_p, v_p)$ in a k-uniform hypergraph G is called a pendant path at v_0, if $d_G(v_0) \geq 2$, $d_G(v_i) = 2$ for $1 \leq i \leq p-1$, $d_G(v) = 1$ for $v \in e_i \setminus \{v_{i-1}, v_i\}$ with $1 \leq i \leq p$, and $d_G(v_p) = 1$. A pendant edge is a pendant path of length one.

If P is a pendant path of a hypergraph G at u, we say G is obtained from H by attaching a pendant path P at u with $H = G[V(G) \setminus (V(P) \setminus \{u\})]$. If P is a pendant edge at u in G, then we also say that G is obtained from H by attaching a pendant edge at u.

Let G be a connected k-uniform hypergraph with $|E(G)| \geq 1$. For $u \in V(G)$ and positive integers p and q, let $G_u(p,q)$ be the k-uniform hypergraph obtained from G by attaching two pendant paths of lengths p and q at u, respectively, and $G_u(p,0)$ the k-uniform hypergraph obtained from G by attaching a pendant path of length p at u.

Lemma 2.1 ([6]). *Let G be a connected k-uniform hypergraph with $|E(G)| \geq 1$ and $u \in V(G)$. For integers $p \geq q \geq 1$, $\sigma(G_u(p,q)) < \sigma(G_u(p+1, q-1))$.*

Let G be a connected k-uniform hypergraph with $u, v \in e \in E(G)$. For positive integers p and q, let $G_{u,v}(p,q)$ be the k-uniform hypergraph obtained from G by attaching a pendant path of length p at u and a pendant path of length q at v, and $G_{u,v}(p,0)$ the k-uniform hypergraph obtained from G by attaching a pendant path of length p at u.

Lemma 2.2 ([6]). *Let G be a connected k-uniform hypergraph with $|E(G)| \geq 2$, $u, v \in e \in E(G)$ and $d_G(u) = 1$. For integers $p \geq q \geq 1$, $\sigma(G_{u,v}(p,q)) < \sigma(G_{u,v}(p+1, q-1))$.*

Let G be a k-uniform hypergraph with $u, v \in V(G)$ and $e_1, \ldots, e_r \in E(G)$ such that $u \in e_i$, $v \notin e_i$ and $e_i' \notin E(G)$ for $1 \leq i \leq r$, where $e_i' = (e_i \setminus \{u\}) \cup \{v\}$. Let G' be the hypergraph with $V(G') = V(G)$ and $E(G') = (E(G) \setminus \{e_1, \ldots, e_r\}) \cup \{e_1', \ldots, e_r'\}$. Then we say that G' is obtained from G by moving edges $e_1, \ldots, e_r$ from u to v.

Denote by $C_{m,k}$ the k-uniform unicyclic hypergraph of m edges that is a cycle. Particularly, $C_m := C_{m,2}$ is just an ordinary graph that is a cycle of length m when $m \geq 3$.

A hyperstar is a hypertree such that each edge has a common vertex, which is its center. A trivial hyperstar consists of only one vertex (with no edges).

Let G be a k-uniform unicyclic hypergraph, where $k \geq 3$. Let C be the unique cycle and g its length, where $g \geq 2$. Label the vertices of $C_{g,k}$ as $v_1, \ldots, v_{g(k-1)}$ with $v_{(g-1)(k-1)+k} = v_1$ such that its edges are

$$e_i = \{v_{(i-1)(k-1)+1}, \ldots, v_{(i-1)(k-1)+k}\}$$

with $i = 1, \ldots, g$. The hypergraph obtained from G by deleting all edges on C consists of $g(k-1)$ components, denoted by $H_1, \ldots, H_{g(k-1)}$, where $v_i \in V(H_i)$ for $i = 1, \ldots, g(k-1)$. In this case, we denote by $C_g^k\left(H_1, \ldots, H_{g(k-1)}\right)$ for G, see Figure 1. If $H_{(i-1)(k-1)+1}$ is a hyperstar of size t_i with center $v_{(i-1)(k-1)+1}$ and H_j is trivial when $j = (i-1)(k-1)+2, \ldots, i(k-1)$ for each for $i = 1, \ldots, g$, then we denote $C_g^k\left(H_1, \ldots, H_{g(k-1)}\right)$ by $C_g^k(t_1, \ldots, t_g)$.

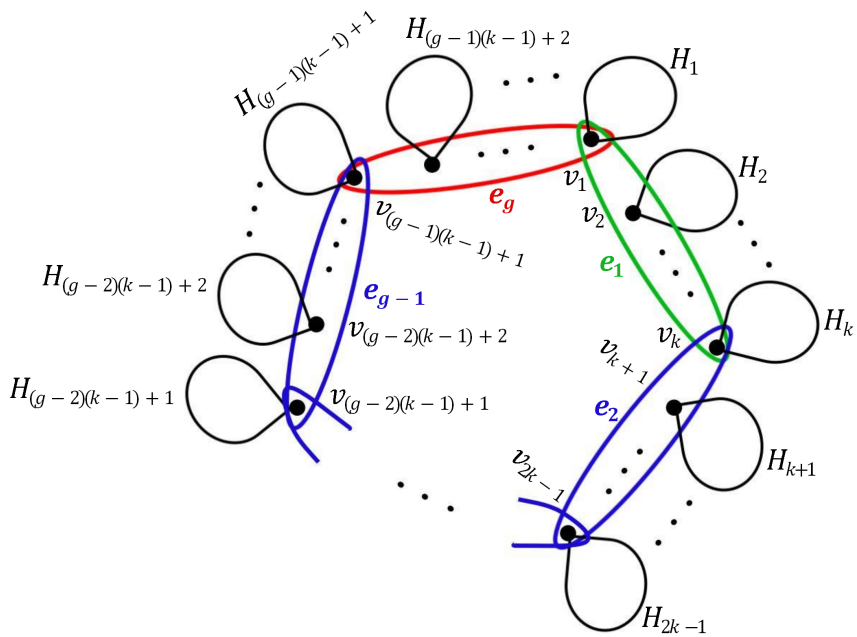

FIGURE 1. Hypergraph $C_g^k\left(H_1, \ldots, H_{g(k-1)}\right)$.

The diameter of a connected hypergraph G is the maximum distance between all vertex pairs of G.

3. Uniform unicyclic hypergraphs with minimum and maximum transmissions

For an integer $m \geq 3$, let U_m be the unicyclic graph of m edges obtained from a triangle by attaching $m-3$ pendant edges at a common vertex. Let G be a unicyclic graph of size m, where $m \geq 3$. If $m = 3$, then G is a triangle, and if $m = 4$, then $G \cong C_4$ or U_4, and it is easy to see that these two graphs have equal transmissions. If $m \geq 5$, then $\sigma(G) \geq m + 2\left(\binom{m}{2} - m\right) = m(m-2)$ with equality if and only if the diameter of G is at most 2, i.e., $G \cong C_5$ or U_5 if $m = 5$, and U_m if $m \geq 6$.

Theorem 3.1. *For an integer $k \geq 3$, let G be a k-uniform unicyclic hypergraph of size $m \geq 2$. Then*

$$\sigma(G) \geq \begin{cases} (k-1)m\left((k-1)m-1-\frac{k}{2}\right) & \text{if } m=3 \\ (k-1)m\left((k-1)m-1-\frac{k}{2}\right)+1 & \text{if } m \neq 3 \end{cases}$$

with equality if and only if $G \cong C_{3,k}$ if $m = 3$ and $G \cong C_2^k(m-2,0)$ if $m = 2$ or $m \geq 4$.

Proof. The order of G is $n = (k-1)m$. If the cycle length of G is 2, then there are exactly 2 edges with exactly 2 vertices in common, and thus

$$\begin{aligned} \sigma(G) &\geq \left(m\binom{k}{2}-1\right)\cdot 1+\left(\binom{n}{2}-m\binom{k}{2}+1\right)\cdot 2 \\ &= (k-1)m\left((k-1)m-1-\frac{k}{2}\right)+1 \end{aligned}$$

with equality if and only if the diameter of G is at most 2, i.e., $G \cong C_2^k(m-2,0)$.

If the cycle length of G is at least 3, then any 2 edges have at most one vertex in common, and thus

$$\begin{aligned} \sigma(G) &\geq m\binom{k}{2}\cdot 1+\left(\binom{n}{2}-m\binom{k}{2}\right)\cdot 2 \\ &= (k-1)m\left((k-1)m-1-\frac{k}{2}\right) \end{aligned}$$

with equality if and only if the diameter of G is at most 2, i.e., $G \cong C_{3,k}$ and $m = 3$.

Suppose that the cycle length of G is at least 3 and $G \ncong C_{3,k}$. Then the diameter of G is at least 3. If the diameter of G is at least 4, then it is obvious that $\sigma(G) > (k-1)m\left((k-1)m-1-\frac{k}{2}\right)+1$. If the diameter of G is 3, then it is impossible that there is exactly one pair of vertices with distance 3, and thus $\sigma(G) > (k-1)m\left((k-1)m-1-\frac{k}{2}\right)+1$. □

Note that, for $m \geq 5$, the graph obtained from a triangle by attaching a pendant path of length $m-3$ is the unique unicyclic graph of size m with maximum transmission, see [13]. In the following, we determine the unique hypergraphs with maximum transmission in the set of k-uniform unicyclic hypergraphs of fixed size for $k \geq 3$.

Lemma 3.2. *For $k \geq 3$ and $g \geq 3$, let $G = C_g^k(H_1, \ldots, H_{g(k-1)})$. Let G^* be the k-uniform hypergraph obtained from G by moving e_1 from v_k to $v_{(g-1)(k-1)+1}$, and G^{**} the k-uniform hypergraph obtained from G by moving e_g from $v_{(g-1)(k-1)+1}$ to v_k, see Figures 2 and 3. Then $\sigma(G^*) > \sigma(G)$ or $\sigma(G^{**}) > \sigma(G)$.*

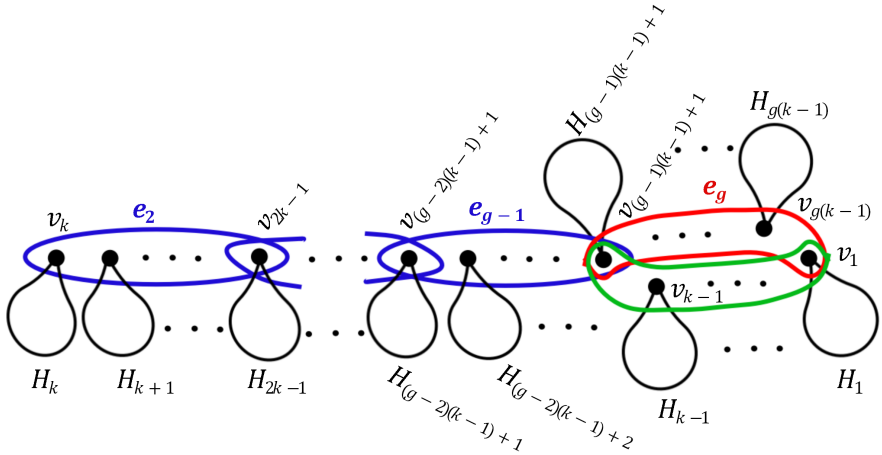

FIGURE 2. Hypergraph G^* in Lemma 3.2.

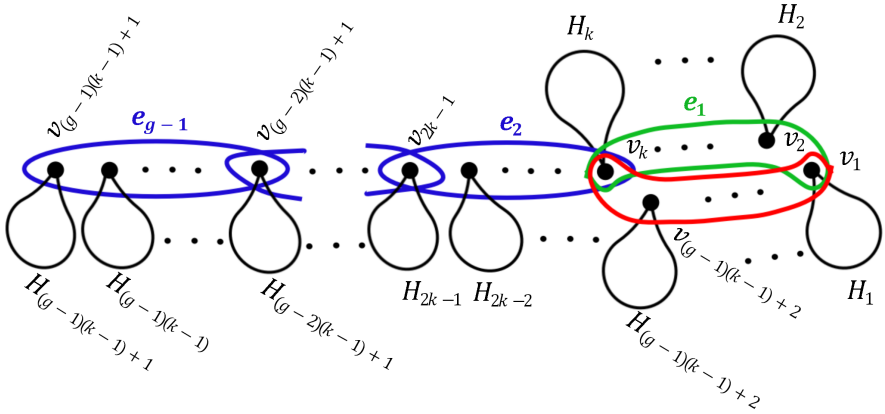

FIGURE 3. Hypergraph G^{**} in Lemma 3.2.

Proof. For $1 \le i \le g(k-1)$, let $V_i = V(H_i)$. Let $A_1 = \bigcup_{i=k}^{g(k-1)/2} V_i$ and $A_2 = \bigcup_{i=g(k-1)/2+2}^{(g-1)(k-1)+1} V_i$ if g is even, and $A_1 = \bigcup_{i=k}^{(g-1)(k-1)/2+1} V_i$ and $A_2 = \bigcup_{i=(g+1)(k-1)/2+1}^{(g-1)(k-1)+1} V_i$ if g is odd.

Suppose that $|A_1| \ge |A_2|$. As we pass from G to G^*, the distance between a vertex of $U_1 = \bigcup_{i=2}^{k-1} V_i$ and a vertex of A_1 is increased by at least one, the distance between a vertex of U_1 and a vertex of A_2 is decreased by 1, the distance between a vertex of V_1 and V_k is increased by $g-2$, and the distance between any other vertex pair is increased or remains unchanged.

Thus

$$\begin{aligned}\sigma_{G^*}(U_1, A_1) - \sigma_G(U_1, A_1) &\geq |U_1||A_1|,\\ \sigma_{G^*}(U_1, A_2) - \sigma_G(U_1, A_2) &= -|U_1||A_2|,\\ \sigma_{G^*}(V_1, V_k) - \sigma_G(V_1, V_k) &= (g-2)|V_1||V_k|,\end{aligned}$$

and so

$$\begin{aligned}\sigma(G^*) - \sigma(G) &\geq \sigma_{G^*}(U_1, A_1) + \sigma_{G^*}(U_1, A_2) + \sigma_{G^*}(V_1, V_k)\\ &\quad - (\sigma_G(U_1, A_1) + \sigma_G(U_1, A_2) + \sigma_G(V_1, V_k))\\ &\geq |U_1||A_1| - |U_1||A_2| + (g-2)|V_1||V_k|\\ &= |U_1|(|A_1| - |A_2|) + (g-2)|V_1||V_k|\\ &> 0.\end{aligned}$$

Therefore, it follows that $\sigma(G^*) > \sigma(G)$.

Now suppose that $|A_1| < |A_2|$. As we pass from G to G^{**}, the distance between a vertex of $U_2 = \bigcup_{i=(g-1)(k-1)+2}^{g(k-1)} V_i$ and a vertex of A_1 is decreased by 1, the distance between a vertex of U_2 and a vertex of A_2 is increased by at least 1, the distance between a vertex of V_1 and a vertex of $V_{(g-1)(k-1)+1}$ is increased by $g-2$, and the distance between any other vertex pair is increased or remains unchanged. Thus

$$\begin{aligned}\sigma_{G^{**}}(U_2, A_1) - \sigma_G(U_2, A_1) &= -|U_2||A_1|,\\ \sigma_{G^{**}}(U_2, A_2) - \sigma_G(U_2, A_2) &\geq |U_2||A_2|,\\ \sigma_{G^{**}}(V_1, V_{(g-1)(k-1)+1}) - \sigma_G(V_1, V_{(g-1)(k-1)+1}) &= (g-2)|V_1||V_{(g-1)(k-1)+1}|,\end{aligned}$$

and so

$$\begin{aligned}\sigma(G^{**}) - \sigma(G) &\geq \sigma_{G^{**}}(U_2, A_1) + \sigma_{G^{**}}(U_2, A_2) + \sigma_{G^{**}}(V_1, V_{(g-1)(k-1)+1})\\ &\quad - \big(\sigma_G(U_2, A_1) + \sigma_G(U_2, A_2) + \sigma_G(V_1, V_{(g-1)(k-1)+1})\big)\\ &\geq \quad -|U_2||A_1| + |U_2||A_2| + (g-2)|V_1||V_{(g-1)(k-1)+1}|\\ &= |U_2|(|A_2| - |A_1|) + (g-2)|V_1||V_{(g-1)(k-1)+1}|\\ &> 0.\end{aligned}$$

Therefore, we have $\sigma(G^{**}) > \sigma(G)$. □

Lemma 3.3. *Let $G = C_2^k\left(H_1, \ldots, H_{2(k-1)}\right)$ with $k \geq 3$. Suppose that $\sum_{i=k+1}^{2(k-1)} |V(H_i)| \geq |V(H_2)|$ and $|E(H_k)| \geq 1$. Let G^* be the k-uniform hypergraph obtained from G by moving all edges in $E_{H_k}(v_k)$ from v_k to v_2. Then $\sigma(G^*) > \sigma(G)$.*

Proof. As we pass from G to G^*, the distance between a vertex of $U_1 = V(H_k) \setminus \{v_k\}$ and a vertex of $U_2 = \bigcup_{i=k+1}^{2(k-1)} V(H_i)$ is increased by 1, the distance between a vertex of U_1 and a vertex of $V(H_2)$ is decreased by 1, the distance between a vertex of U_1 and v_k is increased by 1, and the distance

between any other vertex pair remains unchanged. Then

$$\begin{aligned}\sigma_{G^*}(U_1,U_2)-\sigma_G(U_1,U_2)&=|U_1||U_2|\geq|U_1||V(H_2)|,\\ \sigma_{G^*}(U_1,V(H_2))-\sigma_G(U_1,V(H_2))&=-|U_1||V(H_2)|,\\ \sigma_{G^*}(U_1,\{v_k\})-\sigma_G(U_1,\{v_k\})&=|U_1|,\end{aligned}$$

and thus

$$\begin{aligned}\sigma(G^*)-\sigma(G)&=\sigma_{G^*}(U_1,U_2)+\sigma_{G^*}(U_1,V(H_2))+\sigma_{G^*}(U_1,\{v_k\})\\ &\quad-(\sigma_G(U_1,U_2)+\sigma_G(U_1,V(H_2))+\sigma_G(U_1,\{v_k\}))\\ &=|U_1||U_2|-|U_1||V(H_2)|+|U_1|\\ &>0.\end{aligned}$$

Therefore, $\sigma(G^*)>\sigma(G)$. □

For $k\geq 3$ and $p\geq q\geq 1$, let $\widetilde{C}_2^k(p,q)=G_{w_1,w_1'}(p,q)$, where $G=C_{2,k}$ with edges $e=\{u,v,w_1,\ldots,w_{k-2}\}$ and $f=\{u,v,w_1',\ldots,w_{k-2}'\}$. Let $\widetilde{C}_2^k(0,0)=C_{2,k}$.

Lemma 3.4. *For $k\geq 3$ and $m\geq 4$, if $1\leq p\leq q-2$ and $p+q=m-2$, then $\sigma\left(\widetilde{C}_2^k(p+1,q-1)\right)>\sigma\left(\widetilde{C}_2^k(p,q)\right)$.*

Proof. Let $G=\widetilde{C}_2^k(p,q)$. Let e,f be the edges of the cycle in G with $e\cap f=\{u,v\}$. Let $(u_0,e_1,u_1,\ldots,u_{p-1},e_p,u_p)$ and $(v_0,f_1,v_1,\ldots,v_{q-1},f_q,v_q)$ be the pendant paths at u_0 and v_0, respectively, where $u_0\in e\setminus\{u,v\}$ and $v_0\in f\setminus\{u,v\}$, see Figure 4.

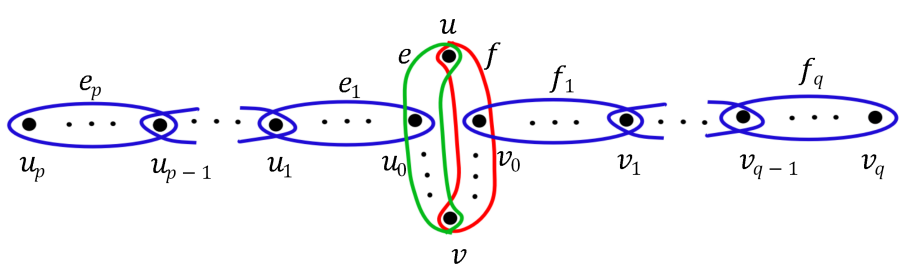

FIGURE 4. Hypergraph G in the proof of Lemma 3.4.

Let G' be the k-uniform hypergraph obtained from G by moving f_q from v_{q-1} to u_p. Obviously, $G'\cong\widetilde{C}_2^k(p+1,q-1)$. Let $U=V(G)\setminus(f_q\setminus\{v_{q-1}\})$. By direct calculation, we have

$$\begin{aligned}&\sigma_G(f_q\setminus\{v_{q-1}\},U)\\ &=\sum_{w\in f_q\setminus\{v_{q-1}\}}\Bigg(\sum_{i=0}^{q-1}d_G(w,v_i)+\sum_{i=1}^{p}d_G(w,u_i)+\sum_{i=1}^{q-1}\sum_{z\in f_i\setminus\{v_{i-1},v_i\}}d_G(w,z)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{p}\sum_{z\in e_i\setminus\{u_{i-1},u_i\}} d_G(w,z)+\sum_{z\in f\setminus\{v_0\}} d_G(w,z)+\sum_{z\in e\setminus\{u,v\}} d_G(w,z)\Bigg)\\
&=\sum_{w\in f_q\setminus\{v_{q-1}\}}\Bigg(\sum_{i=0}^{q-1}(q-i)+\sum_{i=1}^{p}(q+2+i)+\sum_{i=1}^{q-1}\sum_{z\in f_i\setminus\{v_{i-1},v_i\}}(q+1-i)\\
&+\sum_{i=1}^{p}\sum_{z\in e_i\setminus\{u_{i-1},u_i\}}(q+2+i)+\sum_{z\in f\setminus\{v_0\}}(q+1)+\sum_{z\in e\setminus\{u,v\}}(q+2)\Bigg)\\
&=(k-1)\Bigg(\frac{(q+1)q}{2}+\frac{(p+2q+5)p}{2}+\sum_{i=1}^{q-1}(k-2)(q+1-i)\\
&+\sum_{i=1}^{p}(k-2)(q+2+i)+(k-1)(q+1)+(k-2)(q+2)\Bigg)\\
&=(k-1)\Bigg(\frac{(q+1)q}{2}+\frac{(p+2q+5)p}{2}+\frac{(k-2)(q+2)(q-1)}{2}\\
&+\frac{(k-2)(p+2q+5)p}{2}+(k-1)(q+1)+(k-2)(q+2)\Bigg)\\
&=(k-1)\Bigg(\frac{(q+1)q}{2}+\frac{(p+2q+5)p}{2}\\
&+\frac{(k-2)(p^2+q^2+2pq+5p+q-2)}{2}+(2k-3)(q+1)+k-2\Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sigma_{G'}(f_q\setminus\{v_{q-1}\},U)\\
&=(k-1)\Bigg(\frac{(p+2)(p+1)}{2}+\frac{(q+2p+6)(q-1)}{2}\\
&+\frac{(k-2)(p^2+q^2+2pq+5q+p-6)}{2}+(2k-3)(p+2)+k-2\Bigg).
\end{aligned}
$$

Note that $\sigma_G(U)=\sigma_{G'}(U)$ and $\sigma_G(f_q\setminus\{v_{q-1}\})=\sigma_{G'}(f_q\setminus\{v_{q-1}\})$. Thus

$$
\begin{aligned}
&\sigma(G')-\sigma(G)\\
&=\sigma_{G'}(f_q\setminus\{v_{q-1}\})+\sigma_{G'}(U)+\sigma_{G'}(f_q\setminus\{v_{q-1}\},U)\\
&\quad-(\sigma_G(f_q\setminus\{v_{q-1}\})+\sigma_G(U)+\sigma_G(f_q\setminus\{v_{q-1}\},U))\\
&=\sigma_{G'}(f_q\setminus\{v_{q-1}\},U)-\sigma_G(f_q\setminus\{v_{q-1}\},U)\\
&=(k-1)\Bigg(\frac{(p+2)(p+1)}{2}+\frac{(q+2p+6)(q-1)}{2}\\
&+\frac{(k-2)(p^2+q^2+2pq+5q+p-6)}{2}+(2k-3)(p+2)+k-2\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&- (k-1)\left(\frac{(q+1)q}{2}+\frac{(p+2q+5)p}{2}\right.\\
&\left.+\frac{(k-2)(p^2+q^2+2pq+5p+q-2)}{2}+(2k-3)(q+1)+k-2\right)\\
&= (k-1)(q-p-1)\\
&> 0.
\end{aligned}
$$

So, it follows that $\sigma(G') > \sigma(G)$. □

Theorem 3.5. *For $k \geq 3$ and $m \geq 2$, let G be a k-uniform unicyclic hypergraph of size m. Then*

$$
\sigma(G) \leq \begin{cases} \frac{m^3(k-1)^2}{6}+\frac{m^2(k-1)(2k-3)}{4}-\frac{m(k-1)(k+5)}{6}+1 & \text{if } m \text{ is even} \\ \frac{m^3(k-1)^2}{6}+\frac{m^2(k-1)(2k-3)}{4}-\frac{m(k-1)(k+5)}{6}+1-\frac{k-1}{4} & \text{if } m \text{ is odd} \end{cases}
$$

with equality if and only if $G \cong \widetilde{C}_2^k\left(\lfloor\frac{m-2}{2}\rfloor, \lceil\frac{m-2}{2}\rceil\right)$.

Proof. Let $H = \widetilde{C}_2^k\left(\lfloor\frac{m-2}{2}\rfloor, \lceil\frac{m-2}{2}\rceil\right)$. Suppose first that m is even. Label the vertices of H as $1, \ldots, m(k-1)$ such that the edges of H are $e_i = \{(i-1)(k-1)+1, \ldots, i(k-1)+1\}$ for $i = 1, \ldots, \frac{m}{2}$, $e_{m/2+1} = \left\{\frac{m(k-1)}{2}, \ldots, \frac{(m+2)(k-1)}{2}\right\}$, and $e_i = \{(i-1)(k-1), \ldots, i(k-1)\}$ for $i = \frac{m}{2}+2, \ldots, m$. Here $\left(v_{m(k-1)/2}, e_{m/2}, v_{m(k-1)/2+1}, e_{m/2+1}, v_{m(k-1)/2}\right)$ is the cycle of H. Then

$$
\begin{aligned}
&\sigma(H)\\
&= \sum_{i=1}^{m(k-1)}\sum_{j=i+1}^{m(k-1)} d_H(i,j)\\
&= 1\cdot\left(m\binom{k}{2}-1\right)+\sum_{i=1}^{m/2-1}(k-1)\left(\sum_{j=2}^{m/2+1-i} j(k-1)\right.\\
&\quad\left.+\left(\frac{m}{2}+2-i\right)(k-2)+\sum_{j=1}^{m/2-1}\left(\frac{m}{2}+2-i+j\right)(k-1)\right)\\
&\quad+(k-2)\left(2(k-2)+\sum_{j=3}^{m/2+1} j(k-1)\right)+\sum_{i=1}^{m/2-1}(k-1)\sum_{j=2}^{m/2+1-i} j(k-1)\\
&= m\binom{k}{2}-1+2(k-1)^2\sum_{i=1}^{m/2-1}\sum_{j=2}^{m/2+1-i} j\\
&\quad+(k-2)(k-1)\left(\sum_{i=1}^{m/2-1}\left(\frac{m}{2}+2-i\right)+\sum_{j=3}^{m/2+1} j\right)+2(k-2)^2
\end{aligned}
$$

$$
\begin{aligned}
&+(k-1)^2\sum_{i=1}^{m/2-1}\sum_{j=1}^{m/2-1}\left(\frac{m}{2}+2-i+j\right)\\
&=m\binom{k}{2}-1+2(k-1)^2\left(\frac{m}{2}-1\right)\left(\frac{m}{2}+1\right)\frac{m}{4}\\
&\quad+(k-2)(k-1)\left(\frac{m}{2}+4\right)\left(\frac{m}{2}-1\right)+2(k-2)^2\\
&\quad+(k-1)^2\left(\frac{m}{2}-1\right)\left(\left(\frac{m}{2}-1\right)\left(\frac{m}{2}+2\right)-\frac{m(m-1)}{6}\right)\\
&=\frac{m^3(k-1)^2}{6}+\frac{m^2(k-1)(2k-3)}{4}-\frac{m(k-1)(k+5)}{6}+1.
\end{aligned}
$$

Suppose that m is odd. Let f be the pendant edge in the longer pendant path of H, and w be the vertex of degree 2 in f. Then

$$
\begin{aligned}
&\sigma(H)\\
&=\sigma_H(f)+\sigma_H(V(H)\setminus(f\setminus\{w\}))+\sigma_H(f\setminus\{w\},V(H)\setminus(f\setminus\{w\}))\\
&=\binom{k}{2}+\left(\frac{(m-1)^3(k-1)^2}{6}+\frac{(m-1)^2(k-1)(2k-3)}{4}\right.\\
&\quad\left.-\frac{(m-1)(k-1)(k+5)}{6}+1\right)\\
&\quad+(k-1)\left(\sum_{i=2}^{\frac{m+1}{2}}i(k-1)+\frac{m+3}{2}(k-2)+\sum_{i=\frac{m+5}{2}}^{m}i(k-1)\right)\\
&=\frac{m^3(k-1)^2}{6}+\frac{m^2(k-1)(2k-3)}{4}-\frac{m(k-1)(k+5)}{6}+1-\frac{k-1}{4}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\sigma\left(\widetilde{C}_2^k\left(\left\lfloor\frac{m-2}{2}\right\rfloor,\left\lceil\frac{m-2}{2}\right\rceil\right)\right)\\
&=\begin{cases}\frac{m^3(k-1)^2}{6}+\frac{m^2(k-1)(2k-3)}{4}-\frac{m(k-1)(k+5)}{6}+1 & \text{if } m \text{ is even,}\\ \frac{m^3(k-1)^2}{6}+\frac{m^2(k-1)(2k-3)}{4}-\frac{m(k-1)(k+5)}{6}+1-\frac{k-1}{4} & \text{if } m \text{ is odd.}\end{cases}
\end{aligned}
$$

Now suppose that G is a k-uniform unicyclic hypergraph of size m with maximum transmission. It suffices to show that $G\cong\widetilde{C}_2^k\left(\lfloor\frac{m-2}{2}\rfloor,\lceil\frac{m-2}{2}\rceil\right)$. This is trivial if $m=2$. Suppose that $m\geq 3$. Let C be the unique cycle of G. By Lemma 3.2, the length of C is 2. Let $e_1=\{v_1,\ldots,v_k\}$ and $e_2=\{v_k,\ldots,v_{2k-1}\}$ be the edges of C, where $v_{2k-1}=v_1$. Let H_i be the component of $G-E(C)$ containing v_i, where $1\leq i\leq 2k-2$. Then $G\cong C_2^k(H_1,\ldots,H_{2k-2})$.

Claim 1. $|E(H_1)| = |E(H_k)| = 0$.

Otherwise, say $|E(H_k)| \geq 1$. Assume that $|V(H_2)| = \min\{|V(H_i)| : 2 \leq i \leq 2(k-1), i \neq k\}$. Then $\sum_{i=k+1}^{2(k-1)} |V(H_i)| \geq |V(H_2)|$. Let G^* be the k-uniform hypergraph obtained from G by moving all edges at $E_{H_k}(v_k)$ from v_k to v_2. By Lemma 3.3, $\sigma(G^*) > \sigma(G)$, a contradiction. Claim 1 follows.

Claim 2. For each i with $1 \leq i \leq 2(k-1)$ and $i \neq 1, k$, if $|V(H_i)| > 1$, then H_i is a pendant path at v_i.

Suppose that $d_{H_i}(v_i) = 1$. Let Δ be the maximum degree of H_i. Suppose that $\Delta \geq 3$. Evidently, $d_{H_i}(w) = d_G(w)$ and $d_{H_i}(w, v_i) = d_G(w, v_i)$ for $w \in V(H_i) \setminus \{v_i\}$. Choose a vertex $v \in V(H_i)$ of degree at least 3 such that $d_G(v, v_i)$ is as large as possible. Let $F_1, \ldots, F_{d_G(v)}$ be the vertex–disjoint subhypergraphs of $G - v$ with $\bigcup_{j=1}^{d_G(v)} V(F_j) = V(G) \setminus \{v\}$ such that $G[V(F_j) \cup \{v\}]$ is a k-uniform hypertree for $2 \leq j \leq d_G(v)$ and $G[V(F_1) \cup \{v\}]$ is a k-uniform unicyclic hypergraph containing v_i. Suppose that $G[V(F_j) \cup \{v\}]$ is not a pendant path at v for some j with $2 \leq j \leq d_G(v)$. Then there is at least one edge in $E(G[V(F_j) \cup \{v\}])$ with at least 3 vertices of degree 2 in G. We choose such an edge $e = \{w_1, \ldots, w_k\}$ by requiring that $d_G(v, w_1)$ is as large as possible, where $d_G(v, w_1) = d_G(v, w_\ell) - 1$ for $2 \leq \ell \leq k$. Then there are 2 pendant paths at different vertices of e, say P at w_s and Q at w_t, where $2 \leq s < t \leq k$. Let $p \geq q \geq 1$ be the lengths of P and Q, respectively. Then $G \cong N_{w_s,w_t}(p, q)$, where $N = G[V(G) \setminus (V(P \cup Q) \setminus \{w_s, w_t\})]$. Obviously, $d_N(w_s) = d_N(w_t) = 1$ and $G' = N_{w_s,w_t}(p+1, q-1)$ is a k-uniform unicyclic hypergraph of size m. By Lemma 2.2, we have $\sigma(G) < \sigma(G')$, a contradiction. Thus $G[V(F_j) \cup \{v\}]$ is a pendant path at v for $2 \leq j \leq d_G(v)$. Let ℓ_j be the length of the pendant path $G[V(F_j) \cup \{v\}]$ at v, where $2 \leq j \leq d_G(v)$. Then $G \cong F_v(\ell_2, \ell_3)$, where $F = G[V(G) \setminus V(F_2 \cup F_3)]$. We may assume that $\ell_2 \geq \ell_3$. Obviously, $G'' = F_v(\ell_2 + 1, \ell_3 - 1)$ is a k-uniform unicyclic hypergraph of size m. By Lemma 2.1, $\sigma(G) < \sigma(G'')$, a contradiction. It thus follows that $\Delta \leq 2$.

If $\Delta = 1$, then H_i is a pendant edge at v_i in G, and the claim follows. Suppose that $\Delta = 2$. Suppose that H_i is not a pendant path at v_i. Then there is an edge in H_i with at least 3 vertices of degree 2 in G. Choose such an edge $e = \{w_1, \ldots, w_k\}$ in $E(H_i)$ such that $d_G(v_i, w_1)$ is as large as possible, where $d_G(v_i, w_1) = d_G(v_i, w_j) - 1$ for $2 \leq j \leq k$. Then there are 2 pendant paths at different vertices of e, say P' at w_s and Q' at w_t, where $2 \leq s < t \leq k$. Let p' and q' be the lengths of P' and Q', respectively, where $p' \geq q' \geq 1$. Then $G \cong \widetilde{N}_{w_s,w_t}(p', q')$, where $\widetilde{N} = G[V(G) \setminus (V(P' \cup Q') \setminus \{w_s, w_t\})]$. Obviously, $d_{\widetilde{N}}(w_s) = d_{\widetilde{N}}(w_t) = 1$ and $G^* = \widetilde{N}_{w_s,w_t}(p' + 1, q' - 1)$ is a k-uniform unicyclic hypergraph of size m. By Lemma 2.2, we have $\sigma(G) < \sigma(G^*)$, a contradiction. Thus H_i is a pendant path at v_i in G.

Next we show that $d_{H_i}(v_i) \geq 2$ is impossible. Otherwise, let $U_1, \ldots, U_{d_{H_i}(v_i)}$ be the subhypergraphs of $H_i - v_i$ with $\bigcup_{j=1}^{d_{H_i}(v_i)} V(U_j) = V(H_i) \setminus \{v_i\}$ such that $H_i[V(U_j) \cup \{v_i\}]$ is a k-uniform hypertree for $1 \leq j \leq$

$d_{H_i}(v_i)$, where $d_{H_i}(v_i) \geq 2$. Similarly as above, for each $1 \leq j \leq d_{H_i}(v_i)$, $H_i[V(U_j) \cup \{v_i\}]$ is a pendant path at v_i. Let $s_j \geq 1$ be the length of $H_i[V(U_j) \cup \{v_i\}]$ with $1 \leq j \leq d_{H_i}(v_i)$. We may assume that $s_1 \geq s_2$. Then $G \cong \widetilde{F}_{v_i}(s_1, s_2)$, where $\widetilde{F} = G[V(G) \setminus (V(U_1) \cup V(U_2))]$. Let $G^{**} \cong \widetilde{F}_{v_i}(s_1+1, s_2-1)$. Obviously, G^{**} is a k-uniform unicyclic hypergraph of size m. By Lemma 2.1, we have $\sigma(G) < \sigma(G^{**})$, a contradiction. Now Claim 2 follows.

Let $r_i = |E(H_i)|$ for $1 \leq i \leq 2k-2$. By Claim 1, $r_1 = r_k = 0$. Suppose that there are integers i and j with $2 \leq i < j \leq k-1$ such that $r_i \geq r_j \geq 1$. By Claim 2, we have $G = H_{v_i,v_j}(r_i, r_j)$, where $H = G[V(G) \setminus (V(H_i \cup H_j) \setminus \{v_i, v_j\})]$. Let $G' = H_{v_i,v_j}(r_i+1, r_j-1)$. Obviously, G' is a k-uniform unicyclic hypergraph of size m. By Lemma 2.2, $\sigma(G) < \sigma(G')$, a contradiction. Thus there is at most one integer i with $2 \leq i \leq k-1$ such that $r_i \geq 1$. Similarly, there is at most one integer i with $k+1 \leq i \leq 2k-2$ such that $r_i \geq 1$. Assume that $r_2 = \max\{r_i : 2 \leq i \leq k-1\}$, $r_{k+1} = \max\{r_i : k+1 \leq i \leq 2k-2\}$ and $r_2 \geq r_{k+1}$. Therefore $G \cong \widetilde{C}_2^k(r_2, r_{k+1})$, where $r_2 \geq r_{k+1} \geq 0$ and $r_2 + r_{k+1} = m-2$. By Lemma 3.4, we have $G \cong \widetilde{C}_2^k\left(\lfloor \frac{m-2}{2} \rfloor, \lceil \frac{m-2}{2} \rceil\right)$. $\square$

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References

1. A. Alochukwu and P. Dankelmann, *Wiener index in graphs with given minimum degree and maximum degree*, Discrete Math. Theor. Comput. Sci. **23** (2021), Paper 11.
2. C. Berge, *Graphs and Hypergraphs*, North-Holland Pub. Co., Amsterdam-London, 1973.
3. A. A. Dobrynin, R. Entringer, and I. Gutman, *Wiener index of trees: theory and applications*, Acta Appl. Math. **66** (2001), 211–249.
4. A. Georgakopoulos and S. Wagner, *Hitting times, cover cost, and the Wiener index of a tree*, J. Graph Theory **84** (2017), 311–326.
5. H. Guo and B. Zhou, *Properties of degree distance and Gutman index of uniform hypergraphs*, MATCH Commun. Math. Comput. Chem **78** (2017), 213–220.
6. H. Guo, B. Zhou, and H. Lin, *The Wiener index of uniform hypergraphs*, MATCH Commun. Math. Comput. Chem. **78** (2017), 133–152.
7. M. Knor, S. Majstorović, and R. Škrekovski, *Graphs whose Wiener index does not change when a specific vertex is removed*, Discrete Appl. Math. **238** (2018), 126–132.
8. M. Knor, R. Škrekovski, and A. Tepeh, *Mathematical aspects of Wiener index*, Ars Math. Contemp. **11** (2016), 327–352.
9. E. V. Konstantinova, *Chemical Hypergraph Theory*, Com^{2}MaC Lecture Note Series, Vol. 3, Combinatorial and Computational Mathematics Center, Pohang University of Science and Technology, 2001.
10. J. Plesník, *On the sum of all distances in a graph or digraph*, J. Graph Theory **8** (1984), 1–21.
11. L. Šoltés, *Transmission in graphs: a bound and vertex removing*, Math. Slovaca **41** (1991), 11–16.

12. L. Sun, J. Wu, H. Cai, and Z. Luo, *The Wiener index of r-uniform hypergraphs*, Bull. Malays. Math. Sci. Soc. **40** (2017), 1093–1113.
13. Z. Tang and H. Deng, *The (n, n)-graphs with the first three extremal Wiener indices*, J. Math. Chem. **43** (2008), 60–74.

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