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CLOSING THE GAP: ETERNAL DOMINATION ON $3 \times n$ GRIDS

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ABSTRACT. The domination number for grid graphs has been a long studied problem. Grid graphs are a natural class of graphs to consider for the eternal dominating set problem as the domination number forms a lower bound for the eternal domination number. The $3 \times n$ grid has been considered in several papers, and the difference between the upper and lower bounds for the eternal domination number in the all-guards move model has been reduced to a linear function of n. In this short paper, we provide an upper bound for the eternal domination number which exceeds the lower bound by at most 3.

1. INTRODUCTION

A methodological approach to optimizing security strategies using graph theory was introduced in [1] and retrospectively applied to the situation faced by Emperor Constantine as the Roman Empire confronted potential decline in the 4th century. Follow up papers appeared in *John Hopkins Magazine* [11], *Scientific American* [13], and *American Mathematical Monthly* [12] and then several graph protection models were born. In *graph protection*, mobile agents are placed on vertices of a graph to form a dominating set and then a vertex is attacked. Subject to restrictions on their movements, the collective goal of the agents is generally to form a new dominating set containing the attacked vertex. See the survey [10] for more background and the state of the art of graph protection.

We consider the *all guards move* model of the eternal dominating set problem: a set of guards occupy the vertices of a dominating set in a graph and then a vertex of the graph is attacked. In response, each guard may remain where he is or move to a neighbouring vertex with the common goal of occupying a dominating set that contains the attacked vertex. In achieving this goal, the guards have *defended the attack*. The *eternal domination number* of a graph G, denoted $\gamma_{\text{all}}^{\infty}(G)$, is the minimum number of guards necessary to defend any sequence of attacks.

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Grid graphs are a natural class of graphs to consider for the eternal dominating set problem as the domination number for all grid graphs was recently concluded (see [7, 8]) and forms a lower bound for the eternal domination number. In this paper, we substantially close the gap between the upper and lower bounds for the eternal domination number on $3 \times n$ grids, i.e. $P_3 \,\square\, P_n$. In [5] and [6] the authors found that for $n \geq 9$

$$\left\lfloor \frac{3n+4}{4} \right\rfloor = \gamma(P_3 \,\square\, P_n) \leq \gamma^{\infty}_{\text{all}}(P_3 \,\square\, P_n) \leq \left\lceil \frac{8n}{9} \right\rceil \tag{1.1}$$

and in [6], it was conjectured that $\gamma^{\infty}_{\text{all}}(P_3 \,\square\, P_n) = \lceil 4n/5 \rceil + 1$ for $n > 9$. The rightmost inequality of (1.1) is interesting because although $\gamma^{\infty}_{\text{all}}(P_3 \,\square\, P_n) = n$ for $2 \leq n \leq 8$, [6] found the surprising result that $\gamma^{\infty}_{\text{all}}(P_3 \,\square\, P_9) = 8$. In [4], both the upper and lower bounds were substantially improved for $n \geq 11$ to

$$1 + \left\lceil \frac{4n+1}{5} \right\rceil \leq \gamma^{\infty}_{\text{all}}(P_3 \,\square\, P_n) \leq \left\lceil \frac{6n+2}{7} \right\rceil. \tag{1.2}$$

Though the lower bound of (1.2) shows the conjecture of [6] technically fails, in this paper we show the conjectured value of $\gamma^{\infty}_{\text{all}}(P_3 \,\square\, P_n)$ is actually extremely close to the exact value (within 3). The main contribution of this paper is the upper bound appearing in Theorem 2.6 where we use the Iverson bracket,

$$[A] = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.6. *For $n \geq 12$,*

$$\left\lceil \frac{4n}{5} \right\rceil + 1 + [n \equiv 0 \pmod 5] \leq \gamma^{\infty}_{all}(P_3 \,\square\, P_n) \leq \left\lceil \frac{4n}{5} \right\rceil + 2 + [n \equiv 0, 1, 3 \pmod 5].$$

We conclude this section with formal definitions and a simple observation. A *dominating set* of graph G is a subset of $V(G)$ whose closed neighbourhood is $V(G)$. The smallest cardinality of a dominating set is denoted $\gamma(G)$ and is called the *domination number* of G. Let $\mathbb{D}_q(G)$ be the set of all dominating sets of G which have cardinality q. In each dominating set $D \in \mathbb{D}_q(G)$, there is one guard located at each vertex of D. Let $D, D' \in \mathbb{D}_q(G)$. We will say D can be *transformed* to D' if $D = \{v_1, v_2, \ldots, v_q\}$, $D' = \{u_1, u_2, \ldots, u_q\}$ and $u_i \in N[v_i]$ for $i = 1, 2, \ldots, q$.

In the *eternal dominating set problem*, a defender is given q guards to protect the graph from a series of attacks on vertices made by an attacker. An *eternal dominating family of* G is a subset $\mathcal{E} \subseteq \mathbb{D}_q(G)$ for some q so that for every $D \in \mathcal{E}$ and every possible attack $v \in V(G)$, there is a dominating set $D' \in \mathcal{E}$ so that $v \in D'$ and D transforms to D'. When the value of q in the above definition is known we will refer to this family as an eternal dominating family with q guards. A set $D \in \mathbb{D}_q(G)$ is an *eternal dominating set* if it is a member of some eternal dominating family. The eternal domination number of G, denoted $\gamma^{\infty}_{all}(G)$, is the smallest integer q for which $\mathbb{D}_q(G)$ is non-empty and an eternal dominating family $\mathcal{E} \subseteq \mathbb{D}_q(G)$ exists. Unfortunately, a variety of notation and terminology appears in the literature for the all guards move model, and the eternal domination number has been denoted $\sigma_m(G)$ [5], $\gamma_m(G)$ [9], and $\gamma_m^\infty(G)$ [10]. The $\gamma_{\text{all}}^\infty(G)$ notation, which we make use of in this paper, has been used more recently in [2, 3, 4].

The *Cartesian product* of G and H, $G \,\square\, H$, has vertex set $V(G \,\square\, H) = \{(u,v) | u \in V(G), v \in V(H)\}$ and two vertices (u,v), (u',v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. In this paper, we consider only $P_3 \,\square\, P_n$. Label the vertices of P_3 as a, b, c and label the vertices of P_n as $1, 2, 3, \ldots, n$. A vertex in $P_3 \,\square\, P_n$ containing first coordinate a, b, or c is referred to as a *top* vertex, a *middle* vertex, or a *bottom* vertex, respectively. The three vertices in $P_3 \,\square\, P_n$ containing second coordinate i ($1 \leq i \leq n$) will be referred to as "column i" and the "first i columns" counts from the left to the right: column 1 through i, inclusive. In the figures that appear throughout the paper, squares have been used to represent vertices in order to indicate arrangements of guards more simply.

To conclude this section, we make a simple observation:

Observation 1.1. $\gamma_{all}^\infty(P_3 \,\square\, P_5) = 5$.

In [6], it was shown that $\gamma_{\text{all}}^\infty(P_3 \,\square\, P_5) \leq 5$, so it remains to show that 4 guards are insufficient. Certainly, there can be at most one guard in column 1. Otherwise, there are at most 2 guards to dominate the vertices of columns 3, 4, and 5, which is impossible as $\gamma(P_3 \,\square\, P_3) = 3$ (see [8]). Suppose, w.l.o.g., no guard occupies the top vertex of column 1. Then if the top vertex of column 1 is attacked, a guard must move to that vertex. However, this leaves three guards to dominate one vertex in column 1, two vertices in column 2, and three vertices in each of columns 3, 4, and 5. We leave to the reader the easy exercise of showing that the remaining vertices cannot be dominated by three vertices (by three guards occupying three vertices).

2. The Main Result

In this section, we determine the bound

$$\gamma_{\text{all}}^\infty(P_3 \,\square\, P_n) \leq \left\lceil \frac{4n}{5} \right\rceil + 2 + [n \equiv 0, 1, 3 \pmod 5].$$

Section 2.1 introduces notation, while Section 2.2 provides the 16 different guard configurations used to defend all possible attacks. Section 2.3 provides the desired upper bound.

2.1. **Notation.** Let $n = 5k + 2$ for some integer $k \geq 2$ and decompose $P_3 \,\square\, P_n$ into k vertex-disjoint copies of $P_3 \,\square\, P_5$ and two vertex disjoint copies of $P_3 \,\square\, P_1$ such that the degree 2 vertices of $P_3 \,\square\, P_n$ are found in the two copies of $P_3 \,\square\, P_1$. That is, from left-to-right in $P_3 \,\square\, P_n$, we find a copy of $P_3 \,\square\, P_1$ followed by k copies of $P_3 \,\square\, P_5$, followed by one copy of $P_3 \,\square\, P_1$. For ease, we will refer to such a copy of $P_3 \,\square\, P_5$ in $P_3 \,\square\, P_n$ as a *block* and each copy of $P_3 \,\square\, P_1$ as a *sub-block*. Throughout Sections 2.1 and 2.2, we will consider only the "blocks" of $P_3 \,\square\, P_5$ and consider the sub-blocks in Section 2.3.

Consider the guard arrangement given in Figure 1, where a red "X" indicates the position of a guard. Because we will consider a number of different arrangements of guards in blocks, colours are used as a visualization tool. That is, if the guards in a block are arranged such that a guard is located at the middle vertex of the second column, the middle vertex of the third column, and the top and bottom vertices of the fifth column, we refer to that arrangement of guards as the "red" configuration.

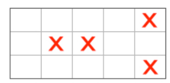

FIGURE 1. A red block.

Consider now, a sequence of red blocks; that is, a sequence of blocks in which the guards in each block are located at the positions given in Figure 1. A sequence of red blocks is shown in Figure 3 (a) and we denote this guard arrangement as block-guard configuration R.

Given an attack at a vertex unoccupied by a guard, the guards must move to form a new dominating set containing the attacked vertex. If, for example, the middle vertex of the leftmost column of a red block is attacked, the guards can transition to the arrangement given in Figure 2 (a): by each guard moving to the neighbouring vertex to the left. The new arrangement of guards in the block is called the "yellow" configuration and the "X"'s indicating guard positions are recoloured yellow to distinguish the new guard arrangement. Thus, one can easily see that the guards of block-guard configuration R can transition to the configuration given in Figure 3 (b).

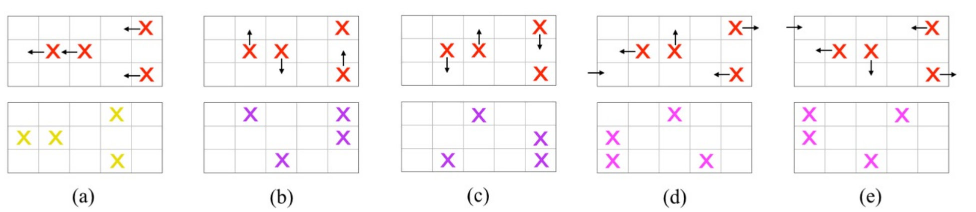

FIGURE 2. The transition from a red block to a yellow, purple, or magenta block.

If the middle vertex of the rightmost column in a red block is attacked, one can easily see that the guards can transition to either of the guard arrangements given in Figure 2 (b) and (c). The new arrangement of guards in these blocks are called the "purple" configuration and the "X"'s indicating guard positions are recoloured purple. Note that the two purple guard configurations given in Figure 2 (b) and (c) are simply vertical reflections of one another (and in general will not be distinguished from one another). Thus, given an attack at a vertex of R, the guards can move to any of the guard arrangements given in Figure 3 (b), (c), or (d). A sequence of yellow blocks is called block-guard configuration Y and a sequence of purple blocks is called block-guard configuration P. Note that in (c) and (d) of Figure 3, each purple block is the vertical reflection of its neighbours. This ensures the vertices of the interior blocks (all blocks, but not necessarily sub-blocks) are all dominated by vertices with guards.

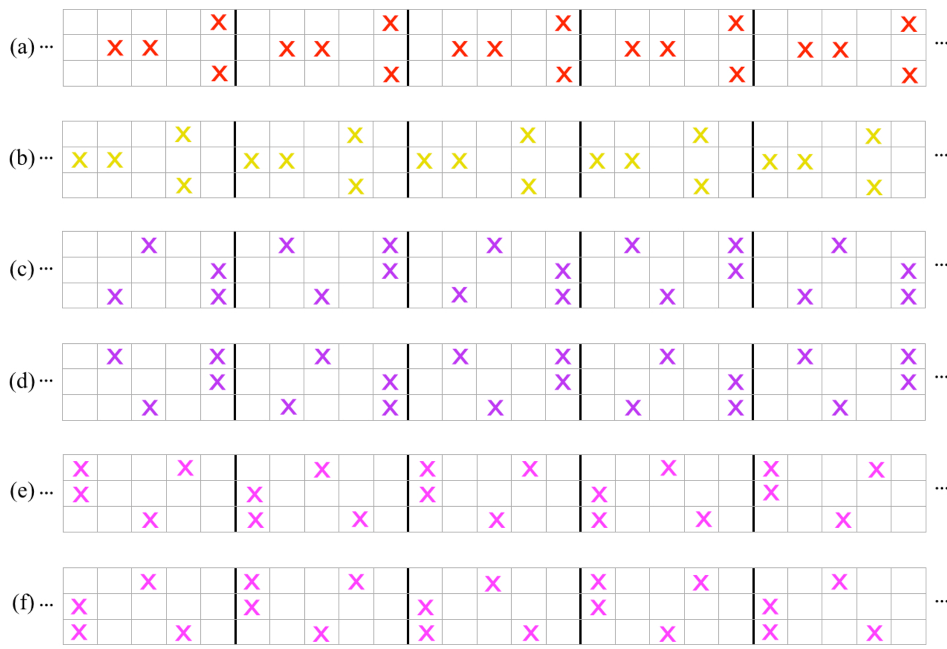

FIGURE 3. Block-guard configurations R, Y, P, and M.

Although $\gamma(P_3 \square P_5) = 4$, recall that $\gamma^{\infty}_{\text{all}}(P_3 \square P_5) = 5$ from Observation 1.1. Consequently, given a particular attack, it may not be possible for the 4 guards in a block to dominate all vertices in that block. As a result, in response to some attacks, a guard may have to move to a neighbouring block temporarily. For example, consider an attack at the top or bottom vertex of the leftmost column of a red block. Since the four guards in a red block do not dominate the top or bottom vertex in the leftmost column, if either vertex is attacked, a guard from the neighbouring red block must temporarily move to the attacked vertex. Once all the block-guard configurations have been introduced, it will be clear that such a guard in a neighbouring block will always exist; the only potential problem occurs if the neighbouring block is a sub-block, rather than a block. In Section 2.3, we will see that a sub-block will always have a guard that can move to the neighbouring block.

Consider block-configuration R and suppose the bottom (or top) vertex in the leftmost column of any block is attacked. As there exists a guard in the bottom (or top) vertex of the rightmost column of the block to the left, that guard moves to the attacked vertex. The remaining guards (in that block along with all other blocks) move as shown in Figure 2 (d) (or (e)). The resulting arrangement of guards is called the "magenta" configuration and the "X"'s indicating guard positions are recoloured magenta. In a magenta block, a guard is temporarily *borrowed* from the left neighbouring block. A sequence of magenta blocks (where each block is the vertical reflection of neighbouring blocks) is called block-guard configuration M and is shown in Figure 3 (e) and (f). Following the movements of the guards given in Figure 2 (d) and (e), it is easy to see that block-guard configuration R (shown in Figure 3 (a)) can transition to either of the configurations shown in Figure 3 (e) and (f). In other words, regardless of which block of R contains the attacked vertex, the guards in *every* block move as shown in Figure 2 (d) or (e): the movement of guards between blocks propagates through all blocks.

Thus, block-guard configuration R can transition to block-guard configuration Y, P, or M. We express this information in Figure 4 where a vertex labeled "Y" indicates that the guards of R can transition to Y if such a vertex is attacked. Similarly, a vertex labeled "P" (or "M") indicates that the guards of R can transition to P in either the form of Figure 3 (c) or (d) (or M in either the form of Figure 3 (e) or (f)) if such a vertex is attacked. For simplicity, at each vertex of Figure 4, we list only *one* block-guard configuration to which R can transition, although more than one may exist; for example, if the middle vertex of the leftmost column of a red block is attacked, the guards can transition to either Y or M, though only Y is given in Figure 4. Practically speaking, it is unimportant to which of the 16 block-guard configurations the guards move, it is simply important that there *exists* a block-guard configuration among the 16 to which the guards can move in response to an attack.

	M	P	P	Y	X	M	P	P	Y	X	M	P	P	Y	X	M	P	P	Y	X	M	P	P	Y	X	
R ...	Y	X	X		P	Y	X	X		P	Y	X	X		P	Y	X	X		P	Y	X	X		P	...
	M	P	P	Y	X	M	P	P	Y	X	M	P	P	Y	X	M	P	P	Y	X	M	P	P	Y	X	

FIGURE 4. Possible configurations to which R can transition.

In Section 2.2, we consider the movements of guards in response to an attack at the middle vertex of the fourth column (from the left) in any red block and introduce the remaining block-guard configurations. To conclude this section, we introduce some additional terminology and notation.

Some of the 16 block-guard configurations will contain blocks of only one colour, such as R, Y, P, or M. However, others block-guard configurations will contain some blocks of one colour and some blocks of a different colour. In each block, the arrangement of guards will be one of 11 possibilities given in Figure 5.

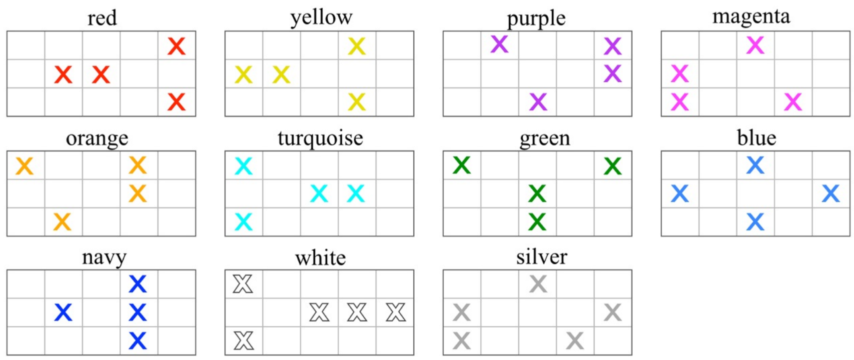

FIGURE 5. The 11 possible arrangements of guards in each block.

As in the situation described earlier for red blocks, there can be vertices in a particular block that are not dominated by guards in that block; an attack at such a vertex will force a guard from the neighbouring block to temporarily move to the block containing the attacked vertex. In a magenta or turquoise block, one guard is temporarily borrowed from the left neighbouring block and one guard is temporarily *loaned* to the right neighbouring block. This distinguishes magenta and turquoise from simply being horizontal reflections of purple and yellow blocks, respectively. In a white or silver block, one guard is temporarily borrowed from the left neighbouring block, but no guard is borrowed from or loaned to the right neighbouring block (thus, white and silver blocks each contain 5 guards, rather than 4). This idea of loaning and borrowing guards is described further in Section 2.2 after all block-guard configurations have been introduced.

2.2. **The 16 Guard Configurations.** Four block-guard configurations R, Y, P, and M were presented in the previous section. Next, we present the remaining 12 block-guard configurations and illustrate that in response to an attack at any vertex in the 16 configurations, the guards can move to another of the 16 configurations. This will yield the desired family $\mathcal{E}$ of eternal dominating sets.

Figure 6 presents R, Y, P, and M along with additional block-guard configurations B, O, G, and T where all blocks are blue, orange, green, and turquoise (respectively) and each block is the vertical reflection of its neighbouring blocks. The "overline" notation in Figure 6 is used to indicate a horizontal reflection; that is, $\overline{Y}$ indicates a horizontal reflection of the sequence of yellow blocks. The bullets are used to indicate the vertical reflection of the current guard configuration. For example, in G, the bullets indicate the vertical reflection of the sequence of green blocks. In Y, each yellow block can move to an orange block or the vertical reflection of an orange block. Although the guards of Y can move to O in two ways (one is the vertical reflection of the other), these are not distinguished from one another in Figure 6.

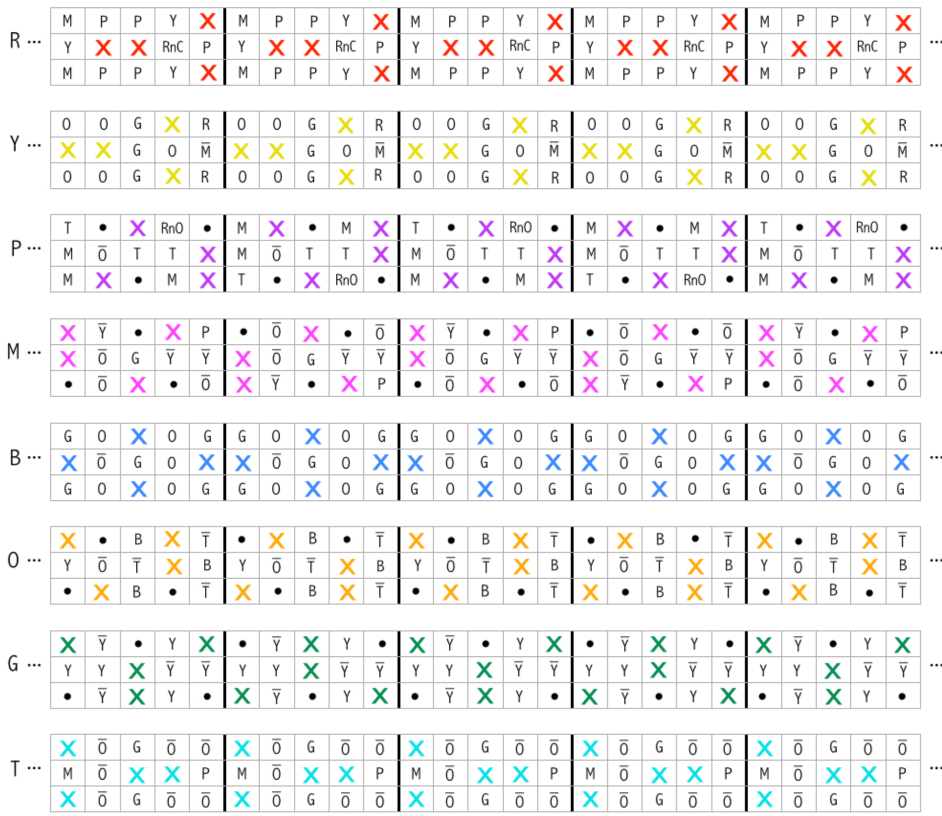

FIGURE 6. Eight monochromatic guard configurations.

In Section 2.1, we considered several possible attacks at vertices in block-guard configuration R and showed the guards could transition to Y, P, or M in response (see Figure 4). Consider an attack at the middle vertex in the fourth column of a red block in R. In response to such an attack, the guards in the block containing the attacked vertex transition to a navy block, any block to the left will remain a red block, and any block to the right will transition to a yellow block. Note that such transitions are possible: we observed earlier that a red block can transition to a yellow block, but one can easily see that a red block can also transition to a navy block by shifting the guards in the last 3 columns all to the fourth column. Label such a block-guard configuration, consisting (from left-to-right) of a sequence of 0 or more red blocks, followed by exactly one 1 navy block, followed by a sequence of 0 or more blocks coloured yellow or blue or both (the yellow and blue blocks are intermixed) as RnC. This will be the naming convention when the blocks in a guard configuration are not all the same colour: the capital letter R indicates a sequence of 0 or more red blocks, lowercase n indicates exactly one navy block, and C indicates a sequence of intermixed yellow and blue blocks (0 or more of each). In Figure 7, there simply happens to be 0 blue blocks.

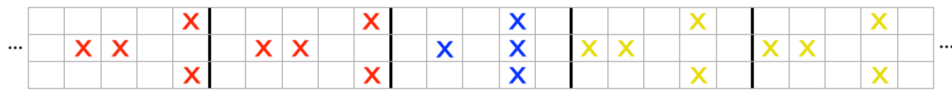

Figure 7. An example of configuration RnC.

The notation $R \to P$ will be used to indicate that in response to an attack at a vertex in any block, the block-guard configuration R can transition to the configuration of P. The notation $R \to \{P, M, Y, RnC\}$ will be used to indicate that in response to an attack at *any* vertex in *any* block, the guards in configuration R can transition to one of P, M, Y, RnC and consequently say that $\{P, M, Y, RnC\}$ *cover* R. Observe that all vertices in the k blocks will be dominated by guards – with the possible exception of the leftmost column of the leftmost block and the rightmost column of the rightmost block. The leftmost column of the leftmost block and rightmost column of the rightmost block will be considered in the next section along with the left sub-block and the right sub-block.

Determining the sets of block-guard configurations that cover configurations G, Y, O, M, T, B, P is very similar to the above situation for R. We omit an explanation of the sets of block-guard configurations that cover configurations G, Y, O, M, T, B, P and instead refer the reader to the list below and Figure 6, which illustrates some configurations to which the guards can move, in response to an attack at any vertex.

$G \to \{Y, \overline{Y}, G\}$ $\quad Y \to \{O, G, \overline{M}\}$ $\quad O \to \{P, \overline{O}, \overline{M}, \overline{T}\}$

$M \to \{M, T, \overline{O}\}$ $\quad T \to \{P, M, \overline{O}\}$ $\quad B \to \{G, O, \overline{O}\}$

$P \to \{R, M, T, RnO\}$ $\quad R \to \{P, M, Y, RnC\}$.

Again, note that $\overline{O}$, $\overline{M}$, $\overline{T}$ denote a *horizontal reflection* of an orange, magenta, or turquoise block, respectively. The naming conventions (from left-to-right) for the remaining eight guard configurations are stated below.

- PG: A sequence of at least 1 purple block, followed by a sequence of at least 1 green block.
- PO: A sequence of at least 1 purple block, followed by a sequence of at least 1 orange block.
- $R\overline{O}$: A sequence of at least 1 red block, followed by a sequence of at least 1 horizontal reflection of orange blocks
- RnO: A sequence of 0 or more red blocks, followed by 1 navy block, followed by a sequence of 0 or more orange blocks.

MsO: A sequence of 0 or more magenta blocks, followed by 1 silver block, followed by a sequence of 0 or more orange blocks.

TwO: A sequence of 0 or more turquoise blocks, followed by 1 white block, followed by a sequence of 0 or more orange blocks.

$R\overline{s}\overline{M}$: A sequence of 0 or more red blocks, followed by 1 horizontal reflection of a silver block, followed by a sequence of 0 or more horizontal reflections of magenta blocks.

RnC: A sequence of 0 or more red blocks, followed by 1 navy block, followed by a sequence of 0 or more yellow and blue blocks intermixed.

In Section 2.1, the concept of guards moving between blocks was introduced in the situation where a top (or bottom) vertex in the leftmost column of a block in R was attacked. It was shown that the block containing the attacked vertex borrowed a guard from the left neighbouring block and that this movement of guards between blocks propagated: as a result, each block borrowed a guard from the left neighbouring block. In this case, $R \to M$. We now make several observations with respect to guards moving between blocks. The first observation is that a guard may move from a block of one colour to a neighbouring block of another colour. For example, a navy block could borrow a guard from a left neighbouring red block. The second observation is that the borrowing of guards may not propagate, or may only propagate over a subset of sequential blocks. Both of these situations occur when block-configuration RnC transitions to MsO as explained below in Lemmas 2.1 and 2.2. In such a case, the red blocks transition to magenta blocks (as described in Section 2.1), the navy block transitions to a silver block (by borrowing a guard from the block to the left), and C, the sequence of intermixed yellow and blue blocks transitions to O (where neither the blocks of C nor O contain any borrowed guards).

The next two lemmas demonstrate configurations that cover the guard configuration RnC.

Lemma 2.1. *Let C be a sequence of intermixed yellow and blue blocks. Then $C \to \{G, O, \overline{O}, \overline{s}\overline{M}\}$ where $\overline{s}\overline{M}$ is a horizontal reflection of one silver block, followed by a sequence of the horizontal reflection of magenta blocks.*

Proof. We begin by noting that a yellow or blue block can move to a green, orange, or a horizontal reflection of an orange block. Since yellow and blue blocks are vertically symmetric, they can also move to the vertical reflection of one of these blocks. Figure 8 (a)–(d) illustrates how a yellow or blue block can move to the horizontal reflection of magenta or to the horizontal reflection of silver. (Again, as yellow and blue blocks are vertically symmetric, they can also move to the vertical reflections of these.) To do this, each block borrows a guard from the middle row of the neighbouring block to the right. The horizontal reflection of a magenta block also loans a guard from its middle row to the neighbouring block to the left. As a consequence, $C \to \overline{s}\overline{M}$ and upon inspection, we conclude that $\{G, O, \overline{O}, \overline{s}\overline{M}\}$ covers C. □

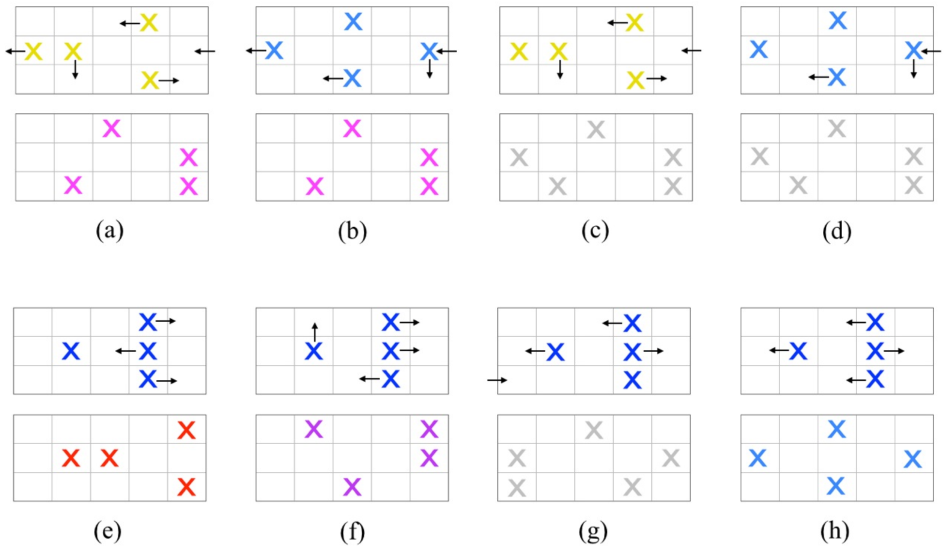

FIGURE 8. Transitions of guards.

Lemma 2.2. $RnC \to \{Rs\overline{M}, R\overline{O}, PO, PG, MsO, RnC\}$.

Proof. First, $Rn \to R$, since a navy block can move to a red block (see Figure 8 (e)). Combining this with Lemma 2.1, we find that $RnC \to Rs\overline{M}, R\overline{O}$. Second, $Rn \to P$, since both red and navy can move to a purple block or the vertical reflection of a purple block (see Figures 2 (b),(c), and 8 (f)). Combining this with Lemma 2.1, we find that $RnC \to PO, PG$.

From Figure 2 (d) and (e), a red block can move to a magenta block (or its vertical reflection), where each magenta block borrows a guard from the left neighbouring block (and loans a guard to the right neighbouring block). From Figure 8 (g), a navy block can move to a silver block (or a vertical reflection of the silver block as navy is vertically symmetric), whereupon it borrows a guard from the left neighbouring block (it does not loan to, or borrow from, the right neighbouring block). Combining this with Lemma 2.1, we find that $RnC \to MsO$.

Finally, RnC can move to a *different* block-configuration of the form RnC. Suppose that the vertex with the symbol $\star$ in Figure 9 (a) is attacked. The red block containing the attacked vertex $\star$ moves to navy (see Figure 8 (e)); the red blocks to the left remain red blocks; and obviously any red blocks to the right can move to yellow. The navy block to the right of the block containing the attacked vertex $\star$ can move to blue (as shown in Figure 8 (h)). Thus we recover a different block-guard configuration of the form RnC.

Consequently, we know that $RnC \to Rs\overline{M}, R\overline{O}, PO, PG, MsO, RnC$. Next, to show that $\{Rs\overline{M}, R\overline{O}, PO, PG, MsO, RnC\}$ *covers* RnC, we use Lemma 2.1 with Figure 9 (b) to observe that all possible attacks on RnC are covered by these block-guard configurations (and their vertical reflections as

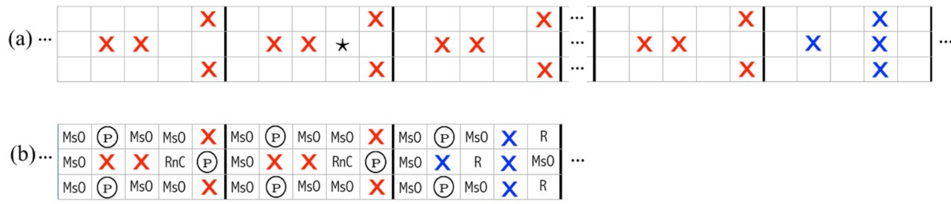

FIGURE 9. (a) An attack that leads from RnC to another configuration of the form RnC. (b) Possible configurations to which RnC can transition.

RnC is vertically symmetric). Certainly a red or a navy block can move to a purple block (or its vertical reflection). However, as C moves to O or G, depending on what vertex is attacked, RnC moves to PG or PO. In Figure 9 (b), since only a portion of RnC is given, we use the notation Ⓟ to indicate that the configuration RnC could move to PG or PO where the red and navy blocks move to purple blocks. □

The remaining 7 block-guard configurations are simpler than RnC and consequently, we omit the explanation, but refer the reader to Figure 10, which illustrates a possible response to an attack at any vertex in the remaining 7 block-guard configurations (note there is often more than one possible response to an attack). We also note that a configuration may be shifted left or right from how the illustration is given in Figure 10; for example, in RnO, suppose there is an attack in the fourth column of a red block. That particular attacked red block moves to a navy configuration, the blocks to the left remain red blocks, and the blocks to the right move to yellow or blue (noting that red blocks can move yellow, a navy block can move to blue, and orange blocks can move to yellow or blue); in other words, RnO moves to RnC in a number of different forms, depending on which red block in RnO is attacked.

The results of this section, along with Figure 10 illustrate that given an attack at a vertex in any block (excluding the leftmost column of the leftmost block and the rightmost column or the rightmost block) guards can successfully defend attacks when each block has 4 associated guards. With respect to Figure 10, we comment on the notation used: $\overline{TwO}$ is used to indicate the horizontal reflection of TwO, that is (from left-to-right): 0 or more blocks that are horizontal reflections of orange; then exactly one block that is the horizontal reflection of white; then 0 or more blocks that are horizontal reflections of turquoise. For example, in Figure 10, there is one vertex in each orange block of PO that is labeled $\overline{TwO}$. Given an attack at such a vertex, the leftmost orange block of PO moves to $\overline{w}$, the purple blocks of PO move to a horizontal reflection of orange, and the remaining orange blocks of PO move to the horizontal reflection of turquoise. Finally, note that $\overline{R\overline{s}\overline{M}}$ is simply 0 or more magenta blocks, followed by exactly one

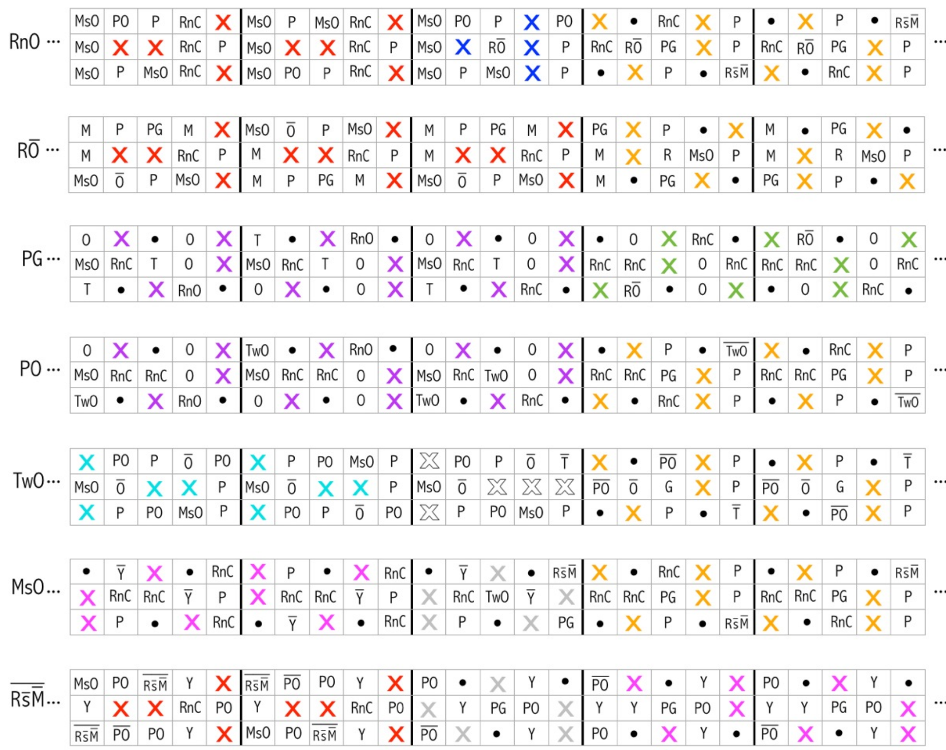

FIGURE 10. The remaining 7 guard configurations.

silver block, followed by 0 or more blocks that are horizontal reflections of red. However, to make it clear that this is simply the horizontal reflection of $R\bar{s}\overline{M}$ and not a new configuration, we use the notation $\overline{R\bar{s}\overline{M}}$.

In Section 2.3, we consider the leftmost column of the leftmost block and the rightmost column of the rightmost block along with the sub-blocks.

2.3. **The Upper Bound.**

Theorem 2.3. *For $n \geq 12$ and $n \equiv 2 \pmod 5$, $\gamma^{\infty}_{all}(P_3 \Box P_n) \leq \lceil 4n/5 \rceil + 2$.*

Proof. Let $n = 5k + 2 \geq 12$ and decompose $P_3 \Box P_n$ into k vertex-disjoint copies of $P_3 \Box P_5$ (blocks) along with two vertex-disjoint copies of $P_3 \Box P_1$ such that the degree 2 vertices of $P_3 \Box P_n$ are found in the two copies of $P_3 \Box P_1$.

We use the 16 block-guard configurations given in Section 2.2 to defend any sequence of attacks in the k blocks. Recall, however, that vertices in the leftmost column of the leftmost block and the rightmost column of the rightmost block, may not be dominated by the $4k$ guards in k blocks. Consequently, we place two additional guards in each of the two copies of $P_3 \Box P_1$. This yields a total of $4k+4$ guards. Label the two guards in the left copy of $P_3 \square P_1$ as g, g'. Given the 16 block-guard configurations, we now consider the arrangement of guards in the leftmost block.

(a) If two vertices in the leftmost column of the leftmost block need to be dominated by guards outside the block, then the two guards in the 1×3 sub-block move to the required vertices in the 1×3 sub-block and dominate the two vertices in the neighbouring block.

(b) If one vertex in the leftmost column of the leftmost block needs to be dominated by guards outside the block, one of the guards of the 1×3 sub-block moves to the required vertex in the 1×3 sub-block (to dominate the vertex in the neighbouring block) and the other guard moves to a vertex in the 1×3 sub-block that ensures the 1×3 sub-block is dominated. In other words, the second guard moves to the middle vertex of the 1×3 sub-block if it is not occupied.

(c) If a guard (other than g, g') moves from the leftmost column of leftmost block to the 1×3 sub-block, then g, g' move within the 1×3 sub-block so that one guard occupies each of the three vertices. An example of such a situation would be if guard configuration B transitioned to guard configuration $\overline{M}$.

(d) If a guard is required to move from the 1×3 sub-block to the leftmost column of the leftmost block, then w.l.o.g., g moves to the neighbouring block and g' moves to the middle vertex of the 1×3 sub-block (to dominate the vertices of the sub-block).

Note that in all situations, the vertices of the 1×3 sub-grid remain dominated by guard(s) in the 1×3 sub-grid. An identical argument can be made for two guards in the right 1×3 sub-block and the rightmost column of the rightmost block. Thus, $4k+4$ guards can defend any sequence of attacks on a $3 \times 5k+2$ grid. □

Lemma 2.4 ([6]). *If* $\gamma^{\infty}_{all}(P_3 \square P_n) \leq t$ *and* $\gamma^{\infty}_{all}(P_3 \square P_s) \leq r$ *then*

$$\gamma^{\infty}_{all}(P_3 \square P_{n+s}) \leq t + r.$$

Theorem 2.5. *For* $n \geq 12$,

$$\gamma^{\infty}_{all}(P_3 \square P_n) \leq \left\lceil \frac{4n}{5} \right\rceil + 2 + \delta_{n \equiv 0,1,3 \pmod 5}.$$

Proof. Suppose $n \geq 12$ and $n = 5k+2+i$ where $i \in \{1, 2, 3, 4\}$. From [4, 6], $\gamma^{\infty}_{\text{all}}(P_3 \square P_1) = 2$, $\gamma^{\infty}_{\text{all}}(P_3 \square P_2) = 2$, $\gamma^{\infty}_{\text{all}}(P_3 \square P_3) = 3$, and $\gamma^{\infty}_{\text{all}}(P_3 \square P_4) = 4$. Combining this with Theorem 2.3 and Lemma 2.4, we find:

$$\begin{aligned}
\gamma^{\infty}_{\text{all}}(P_3 \square P_{5k+3}) &\leq 4k + 4 + 2 = 4k + 6,\\
\gamma^{\infty}_{\text{all}}(P_3 \square P_{5k+4}) &\leq 4k + 4 + 2 = 4k + 6,\\
\gamma^{\infty}_{\text{all}}(P_3 \square P_{5k+5}) &\leq 4k + 4 + 3 = 4k + 7,\\
\gamma^{\infty}_{\text{all}}(P_3 \square P_{5k+6}) &\leq 4k + 4 + 4 = 4k + 8.
\end{aligned}$$

□

Combining the lower bound of [4] with Theorem 2.5 yields the following result where the values of the upper and lower bound differ by at most 3.

Theorem 2.6. *For $n \geq 12$,*

$$\left\lceil \frac{4n}{5} \right\rceil + 1 + \delta_{n \equiv 0 \pmod 5} \leq \gamma_{all}^{\infty}(P_3 \,\square\, P_n) \leq \left\lceil \frac{4n}{5} \right\rceil + 2 + \delta_{n \equiv 0,1,3 \pmod 5}.$$

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