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DEGREE ASSOCIATED RECONSTRUCTION PARAMETERS OF TOTAL GRAPHS

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Abstract. A *card* (*ecard*) of a graph G is a subgraph formed by deleting a vertex (an edge). A *dacard* (*da-ecard*) specifies the degree of the deleted vertex (edge) along with the card (ecard). The *degree associated reconstruction number* (*degree associated edge reconstruction number*) of a graph G, drn(G) (dern(G)), is the minimum number of dacards (da-ecards) that uniquely determines G. In this paper, we investigate these two parameters for the total graph of certain standard graphs.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. For graph theoretic terms which are not defined here, see [3]. For brevity, a vertex of degree m is called an *m-vertex* and a neighbour of degree n of a vertex v is called an *n-neighbour* of v. A vertex adjacent to all other vertices is called a *complete vertex*. A complete subgraph of a graph G is called a *clique* of G; a clique on n vertices is an *n-clique*. The *total graph* of a graph G, denoted by $T(G)$, is the graph whose vertex set is the union of $V(G)$, whose elements will be called *special vertices*, with the vertices of the line graph $L(G)$, whose elements will be called *nonspecial vertices*. The edge set of $T(G)$ contains the edges of G, the edges of $L(G)$, and the following edges connecting G and $L(G)$: a special vertex x and a nonspecial vertex y are joined by an edge if and only if in $L(G)$ y represents an edge of G that is incident with x in G. The *corona* $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. The *double star* $D_{m,n}$ is the tree with $m+n+2$ vertices whose central vertices have m and n leaf neighbours, respectively. By $S(G)$, we mean the graph obtained from G by subdividing every edge exactly once.

A *vertex deleted subgraph* or a *card* $G-v$ of a graph G is the unlabeled graph obtained from G by deleting a vertex v and all edges incident with v. The ordered pair $(d(v), G-v)$ is called a *degree associated card* (or *dacard*)

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of the graph G, where $d(v)$ is the degree of v in G. The *deck* (*dadeck*) of a graph G is the collection of all its cards (dacards). Following the formulation in [4], a graph G is *reconstructible* if it can be uniquely determined from its deck. For a reconstructible graph G, Harary and Plantholt [4] defined the *reconstruction number*, $rn(G)$, to be the minimum number of cards which can only belong to its deck and not to the deck of any other graph H, $H \ncong G$, thus uniquely identifying G. For a reconstructible graph G from its dadeck, Ramachandran [8, 9] has defined and studied the *degree associated reconstruction number* of a graph G, $\mathrm{drn}(G)$, which is the size of the smallest subcollection of the dadeck of G that is not contained in the dadeck of any other graph H, $H \ncong G$. The degree of an edge e, denoted by $d(e)$, is the number of edges adjacent to e. The *edge reconstruction number*, the *degree associated edge card* (or *da-ecard*), the *degree associated edge deck* (or *da-edeck*), and the *degree associated edge reconstruction number* (or dern) of a graph are defined analogously with edge deletions instead of vertex deletions. For further reading, see [2, 5, 6, 7, 8, 9] for some articles on this topic.

The following weakening of the reconstruction problem has also been considered by Harary and Plantholt [4]. A graph G in a given class of graphs $\mathscr{C}$ is called class-reconstructible if whenever $H \in \mathscr{C}$ has the same deck as G, then $G \cong H$. The *class reconstruction number*, $\mathrm{Crn}(G)$, of a graph G is the minimum number of cards which can only belong to its deck and not to the deck of any other graph $H \in \mathscr{C}$, $H \ncong G$, thus uniquely identifying G within the class $\mathscr{C}$. If a graph is degree associated reconstructible then it is class-reconstructible, and vice versa, where the class is the class of graphs with a given number m of edges. Bange et al. [1] have proved that if the class is the family of all total graphs G, then $\mathrm{Crn}(G)$ is one. Here, we shall prove that if the class is the family of graphs with a specified number of edges, then drn cannot be one in general. In this paper, we show that drn and dern is 1 or 2 for the total graphs of a path, complete bipartite, bidegreed double star, subdivision of a star, $C_n \circ K_1$ and $P_n \circ K_1$.

An *s-blocking set* of G is a family $\mathscr{F}$ of graphs not isomorphic to G such that every collection of s dacards (da-ecards) of G will appear in the dadeck (da-edeck) of some graph of $\mathscr{F}$ and every graph in $\mathscr{F}$ will have s dacards (da-ecards) in common with G. Table 1 shows drn for the total graphs of all connected graphs G on at most four vertices and the dark vertex of graphs denotes the vertex whose removal results in a dacard in common with $T(G)$. The total graph of a connected graph on at most four vertices except P_4 and $K_4 - e$ have dern one and $\mathrm{dern}(P_4) = \mathrm{dern}(K_4 - e) = 2$.

2. Total Graph of Standard Graphs

An *extension* of a dacard $(d(v), G - v)$ of G is a graph obtained from the dacard by adding a new vertex x and joining it to $d(v)$ vertices of the dacard and it is denoted by $H(d(v), G-v)$ (or simply by H). Throughout this paper, H and x are used in the sense of this definition.

Barrus and West [2] proved that $\mathrm{drn}(P_n) = 2$; Monikandan and Sundar Raj [6] proved that $\mathrm{dern}(P_n) = 1$. We shall determine drn and dern for the total graph of paths. Since $T(P_2)$ and $T(P_3)$ contain a complete vertex, $\mathrm{drn}(T(P_n)) = 1$ for $n = 2, 3$.

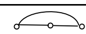
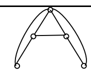
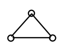
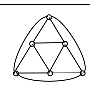
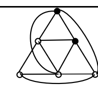
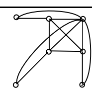
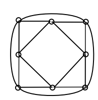
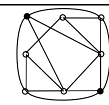
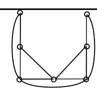
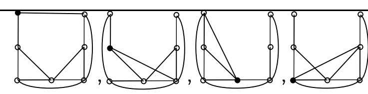
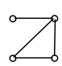
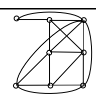
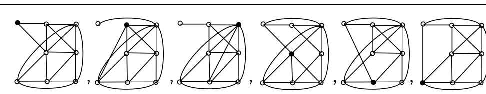
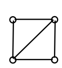
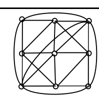
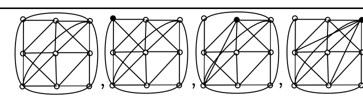
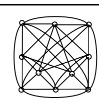
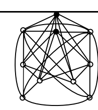
$\lvert V(G)\rvert$	G	$T(G)$	$(\mathrm{drn}(T(G)) - 1)$-blocking set	$\mathrm{drn}(T(G))$
2	K_2		-	1
3	P_3		-	1
3				3
4	$K_{1,3}$		-	1
4	C_4			3
4	P_4			2
4				2
4				2
4	K_4			3

TABLE 1. The drn of total graphs of all connected graphs on at most four vertices.

Theorem 2.1. *For $n > 3$, $\mathrm{drn}(T(P_n)) = 2$.*

Proof. Let $G = T(P_n)$, $n > 3$.
Upper bound. Consider the two dacards $(2, G - v_1)$ and $(4, G - v_2)$, where v_1 and v_2 are special vertices and all neighbours of v_2 are of degree 4 (when $n = 5$) or v_2 is a nonspecial vertex having a 3-neighbour (when $n \neq 5$). Let G' be a graph having these two dacards in its dadeck. Then G' can be constructed from $(2, G - v_1)$ by adding a new vertex x and joining it to

two vertices in $G - v_1$. In $(2, G - v_1)$, if we join x to two adjacent vertices of degree 2 and 3, then the resulting extension is isomorphic to G. For all other possibilities, the resulting extension has no 4-vertex (when $n = 4$) or the removal of any 4-vertex from the resulting extension results in a dacard with at least one of the following structures.

(i) A 1-vertex or a 5-vertex.
(ii) One or two vertices of degree two (when $n \neq 5$).
(iii) Three 2-vertices (when $n = 5$).
(iv) Four 2-vertices.
(v) Three independent 2-vertices.
(vi) Two adjacent 2-vertices with no common neighbour.
(vii) Two nonadjacent 2-vertices with a common neighbour.
(vii) Two nonadjacent vertices of degree 3 and 4 having a common 2-neighbour.

However, the dacard $(4, G - v_2)$ has none of the above structures. Hence the resulting extension does not contain the dacard $(4, G - v_2)$ in its dadeck. Hence $G' \cong G$ and $\text{drn}(G) \leq 2$.
Lower bound. Consider the extension $H_1(2, G - v_1)$ (obtained by joining the new vertex to two 2-vertices), $H_2(3, G - v_2)$ (obtained by joining the new vertex to the neighbours of a 3-vertex) and the extensions of the dacards with associated degree 4 (obtained by joining the new vertex to a 3-vertex and its neighbours). Then the set of all above extensions forms a 1-blocking set of G. Hence $\text{drn}(G) \geq 2$. □

For an edge $e = uv$ of a graph G, $(\deg_G\ u, \deg_G\ v)$ is called the *degree pair* of e and is denoted by $\text{dep}(e)$.

Theorem 2.2. *For* $n = 2, 3$, $\text{dern}(T(Pn)) = 1$, *while for* $n > 3$, $\text{dern}(T(Pn)) = 2$.

Proof. Let $G = T(P_n)$, $n > 1$. For $n = 2$, $\text{dern}(G) = 1$, since G is regular. Consider the two da-ecards $(3, G - e_1)$ and $(6, G - e_2)$, where $\text{dep}(e_1) = (2, 3)$ and $\text{dep}(e_2) = (4, 4)$ and these two 4-vertices (ends of e_2) have a common 3-neighbour. For $n = 3$, in $(3, G - e_1)$, the only possibility to get an extension $H(3, G - e_1)$ is to join the unique 1-vertex to a 2-vertex (since otherwise the degree sum of the two vertices to be joined would be greater than 3). Hence $H(3, G - e_1) \cong G$ and $\text{dern}(G) = 1$.

Now we shall consider the case when $n > 3$. Consider the two da-ecards $(3, G - e_1)$ and $(6, G - e_2)$.
Upper bound. Let G' be a graph having these two da-ecards in its dadeck. Then G' can be constructed from $(3, G - e_1)$ by adding a new edge joining the unique 1-vertex and a 2-vertex in $G - e_1$. In $(3, G - e_1)$, if we add the edge joining the unique 1-vertex and a 2-vertex having a common 4-neighbour, then $G' \cong G$. For the remaining possibility, the resulting extension does not have the dacard $(6, G - e_2)$ in its dadeck, since the removal of any edge of degree 6 results in a dacard having two nonadjacent 2-vertices with a

common neighbour. Hence G' is the unique extension and so it is isomorphic to G. Therefore $\mathrm{dern}(G) \leq 2$.
Lower bound. Consider the extensions $H_1(3, G-e_1)$ (obtained by adding a new edge joining the 1-vertex and a 2-vertex having no common neighbour), $H_2(4, G-e_2)$ (obtained by adding a new edge joining the 1-vertex and a 3-vertex having no common neighbour), the extensions of the da-ecards with associated edge degree 5 (obtained by adding a new edge joining a 2-vertex and a 3-vertex having no common neighbour) and the extensions of the da-ecards with associated edge degree 6 (obtained by adding a new edge joining two 3-vertices having no common neighbour). The set of all above extensions forms a 1-blocking set of G. Hence $\mathrm{dern}(G) \geq 2$. □

It is clear that $\mathrm{drn}(T(K_{1,n})) = \mathrm{dern}(T(K_{1,n})) = 1$. We shall determine drn and dern for the total graph of double stars $D_{n,n}$. For $n = 1$, $\mathrm{drn}(T(D_{n,n})) = 2$ by Theorem 2.1.

Theorem 2.3. *For $n > 1$, $\mathrm{drn}(T(D_{n,n})) = 2$.*

Proof. Let $G = T(D_{n,n})$, $n > 1$ (Figure 1). The dacards of G are $(2, G-v_1)$, $(n+2, G-v_2)$, $(2n+2, G-v_3)$, and $(2n+2, G-v_4)$, where v_1 and v_4 are special vertices, v_2 and v_3 are nonspecial vertices.

Consider the case when $n = 2$.

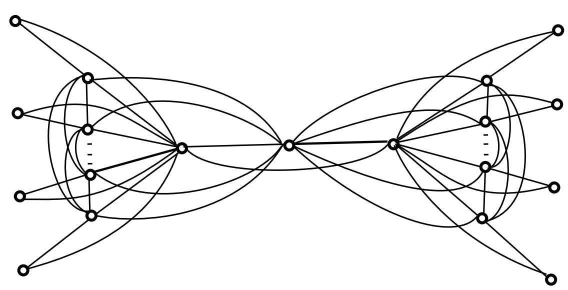


FIGURE 1. $T(D_{n,n})$

Upper bound. Consider the two dacards $(2, G-v_1)$ and $(4, G-v_2)$. Let G' be a graph having these two dacards in its dadeck. Then G' can be constructed from $(2, G-v_1)$ by adding a new vertex x and joining it to two vertices in $G-v_1$. In $(2, G-v_1)$, if we join x to the unique 3-vertex and the unique 5-vertex, then the resulting extension is isomorphic to G. For all other possibilities, the removal of any 4-vertex from the resulting extension results in a dacard having at least one of the following structures.

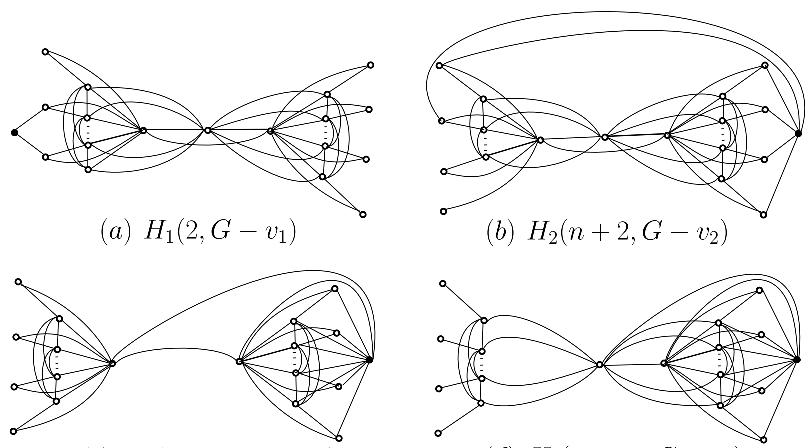
(i) No vertex of degree 1.
(ii) Exactly one or two vertices of degree 2.
(iii) Exactly four 2-vertices and two vertices of degree 6.
(iv) A 1-vertex adjacent to a 4-vertex.
(v) A 2-vertex adjacent to a 3-vertex and a 4-vertex.

(vi) Two 2-vertices with a common neighbour of degree 4 or 5.
(vii) No two 2-vertices having a common neighbour.

But it is clear that the dacard $(4, G - v_2)$ has none of the above structures. Hence the resulting extension does not contain the dacard $(4, G - v_2)$ in its dadeck. Therefore the only possibility is G' must be isomorphic to G and hence $\mathrm{drn}(G) \leq 2$.

Lower bound. Consider the extensions $H_1(2, G - v_1)$ (obtained by joining the new vertex to two 2-vertices), $H_2(4, G - v_2)$ (obtained by joining the new vertex to three 2-vertices and a 1-vertex), $H_3(6, G - v_3)$ where v_3 is a special vertex in G (obtained by joining the new vertex to two 2-vertices, two 4-vertices and two 5-vertices), and $H_4(6, G - v_4)$ where v_4 is a nonspecial vertex in G (obtained by joining the new vertex to two 2-vertices, two 3-vertices and two 5-vertices). Then the set $\{H_1, H_2, H_3, H_4\}$ clearly forms a 1-blocking set of G. Hence $\mathrm{drn}(G) \geq 2$.

Now we shall consider the case when $n > 2$.



(a) $H_1(2, G - v_1)$ (b) $H_2(n + 2, G - v_2)$

(c) $H_3(2n + 2, G - v_3)$ (d) $H_4(2n + 2, G - v_4)$

FIGURE 2

Upper bound. Consider the two dacards $(2, G - v_1)$ and $(2n + 2, G - v_3)$. Let G' be a graph having these two dacards in its dadeck. Then G' can be constructed from $(2, G - v_1)$ by adding a new vertex x and joining it to two vertices in $G - v_1$. In $(2, G - v_1)$, if we join x to the unique $(n + 1)$-vertex and the unique $(2n + 1)$-vertex, then the resulting extension is isomorphic to G. For all other possibilities, the resulting extension does not have the dacard $(2n+2, G-v_3)$ in its dadeck, since the removal of any $(2n+2)$-vertex results in a dacard having a 1-vertex, $2n - 2$ or $2n - 1$ vertices of degree 2, an $(n + 2)$-vertex or $(2n + 2)$-vertex. Hence $G' \cong G$ and $\mathrm{drn}(G) \leq 2$.

Lower bound. Consider the extensions $H_1(2, G-v_1)$ (obtained by joining the new vertex to two 2-vertices) (Figure 2(a)), $H_2(n + 2, G - v_2)$ (obtained by joining the new vertex to $n + 2$ vertices of degree 2) (Figure 2(b)), $H_3(2n + 2, G-v_3)$ where v_3 is a nonspecial vertex (obtained by joining the new vertex to n vertices of degree 2, n vertices of degree $n+1$ and two vertices of degree $2n+1$) (Figure 2(c)), and $H_4(2n+2, G-v_4)$ where v_4 is a special vertex (obtained by joining the new vertex to n vertices of degree 2, n vertices of degree $n+2$ and two vertices of degree $2n+1$) (Figure 2(d)). Then the set $\{H_1, H_2, H_3, H_4\}$ clearly forms a 1-blocking set of G. Hence $\mathrm{drn}(G) \geq 2$. $\square$

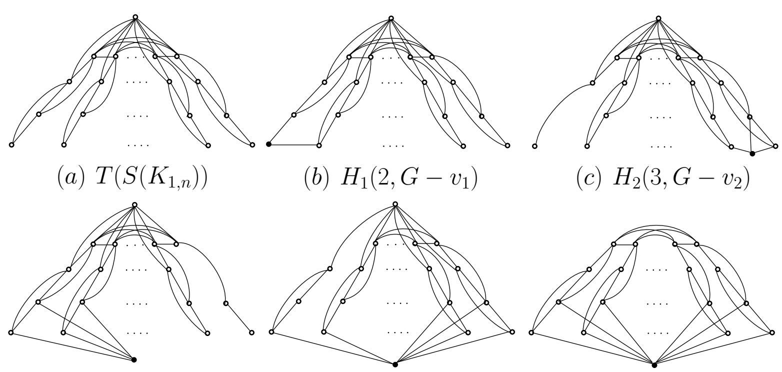
Theorem 2.4. *The degree associated edge reconstruction number of* $T(D_{1,1})$ *equals* 2 *while for* $n > 1$, $\mathrm{dern}(T(D_{n,n})) = 1$.

Proof. Let $G = T(D_{n,n})$, $n \geq 1$. For $n = 1$, $G = T(P_4)$ and hence $\mathrm{dern}(G) = 2$ by Theorem 2.2.

We now consider the case when $n > 1$. Consider the da-ecard $(4n+2, G-e)$, where $\mathrm{dep}(e) = (2n+2, 2n+2)$ and these two $2n+2$ vertices have a 2-neighbour. In $(4n+2, G-e)$, if we join add a new edge joining the two $(2n+1)$-vertices, then the resulting extension $H(4n+2, G-e)$ is isomorphic to G. To get an extension not isomorphic to G, at least one of the two vertices to be joined must be different from these two vertices. But then the degree sum of the two vertices to be joined would be 4, $n+4$, $2n+3$, $2n+4$ or $3n+3$. Hence $\mathrm{dern}(G) = 1$. $\square$

Theorem 2.5. *For* $n > 1$, $\mathrm{drn}(T(S(K_{1,n}))) = 2$.

Proof. Let $G = T(S(K_{1,n}))$, $n > 1$ (Figure 3(a)). The dacards of G are $(2, G-v_1)$, $(3, G-v_2)$, $(4, G-v_3)$, $(n+2, G-v_4)$, and $(2n, G-v_5)$, where v_1, v_3, and v_5 are special vertices, v_2 and v_4 are nonspecial vertices. For $n = 2$, $G = T(P_5)$ and hence $\mathrm{drn}(G) = 2$ by Theorem 2.1. Assume that $n > 2$ and consider the dacards $(2, G-v_1)$ and $(2n, G-v_5)$, where v_1 and v_5 are special vertices.



(a) $T(S(K_{1,n}))$ (b) $H_1(2, G-v_1)$ (c) $H_2(3, G-v_2)$

(d) $H_3(4, G-v_3)$ (e) $H_4(n+2, G-v_4)$ (f) $H_5(2n, G-v_5)$

FIGURE 3

Upper bound. Let G' be a graph having these two dacards in its dadeck. Then G' can be constructed from $(2, G-v_1)$ by adding a new vertex x and joining it to two vertices in $G-v_1$. The dacard $(2n, G-v_5)$ forces every extension to have exactly one vertex of maximum degree $2n$ and the next maximum degree to be either $n+1$ or $n+2$. Hence in $(2, G-v_1)$, the new vertex x cannot be adjacent to a vertex of degree $2n$ or $n+2$. In $(2, G-v_1)$, if we join the newly added vertex x to two adjacent vertices of degree 2 and 3 having a common neighbour of degree $n+2$, then the resulting extension is isomorphic to G. For all other possibilities, the resulting extension does not have the dacard $(2n, G-v_5)$ in its dadeck, since the removal of any $2n$-vertex results in a dacard having two 2-vertices with a common neighbour. Hence $G' \cong G$ and $\mathrm{drn}(G) \leq 2$.

Lower bound. Consider the extensions $H_1(2, G-v_1)$ (obtained by joining the new vertex to two 2-vertices), $H_2(3, G-v_2)$ (obtained by joining the new vertex to two 2-vertices and a 3-vertex), $H_3(4, G-v_3)$ (obtained by joining the new vertex to two 2-vertices and two 3-vertices), $H_4(n+2, G-v_4)$ (obtained by joining the new vertex to n vertices of degree 2 and two vertices of degree 3), and $H_5(2n, G-v_5)$ (obtained by joining the new vertex to n vertices of degree 2 and n vertices of degree 3). Then the set $\{H_1, H_2, H_3, H_4, H_5\}$ (Figure 3) clearly forms a 1-blocking set of G. Hence $\mathrm{drn}(G) \geq 2$. □

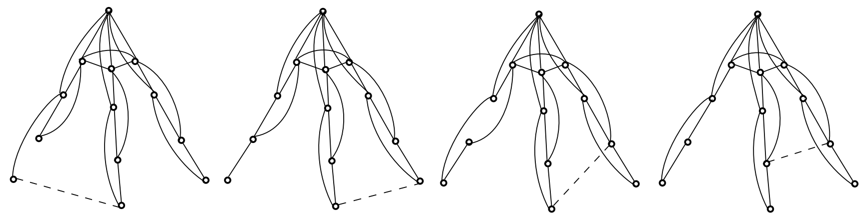
Theorem 2.6. *For* $n = 2, 3$, $\mathrm{dern}(T(S(K_{1,n}))) = 2$, *while for* $n > 3$, $\mathrm{dern}(T(S(K_{1,n}))) = 1$.

Proof. Let $G = T(S(K_{1,n}))$, $n > 1$. The da-ecards of G are $(3, G-e_1)$, $(4, G-e_2)$, $(5, G-e_3)$, $(n+3, G-e_4)$, $(n+4, G-e_5)$, $(3n, G-e_6)$, $(2n+2, G-e_7)$ and $(2n+2, G-e_8)$, where $dep(e_7) = (n+2, n+2)$, and $dep(e_8) = (4, 2n)$. For $n = 2$, $G = T(P_5)$ and hence $\mathrm{dern}(G) = 2$ by Theorem 2.2. For $n = 3$, consider the da-ecards $(3, G-e_1)$ and $(8, G-e_7)$.

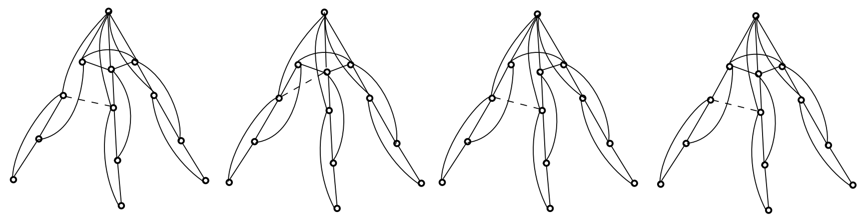
Upper bound. We use these two da-ecards to reconstruct G. The da-ecard $(8, G-e_7)$ forces every extension to have n vertices of degree 2 such that no two of them have a common neighbour. Hence G must be obtained from $(3, G-e_1)$, by adding an edge joining the unique 1-vertex and the 2-vertex having a common 4-vertex. Therefore $\mathrm{dern}(G) \leq 2$.

Lower bound. Consider the extensions $H_1(3, G-e_1)$ (obtained by adding a new edge joining a 1-vertex and a 2-vertex having no common neighbour) (Figure 4(a)), $H_2(4, G-e_2)$ (obtained by adding a new edge joining two 2-vertices) (Figure 4(b)), $H_3(5, G-e_3)$ (obtained by adding a new edge joining a 3-vertex and a 2-vertex having no common neighbour) (Figure 4(c)), $H_4(6, G-e_4)$ (obtained by adding a new edge joining two 3-vertices) (Figure 4(d)), $H_5(7, G-e_5)$ (obtained by adding a new edge joining a 3-vertex and a 4-vertex having no common neighbour) (Figure 4(e)), $H_6(9, G-e_6)$ (obtained by adding a new edge joining a 5-vertex and a 4-vertex adjacent to a vertex of degree 2) (Figure 4(f)), $H_7(8, G-v_7)$, and $H_8(8, G-v_8)$ (each of which is obtained by adding a new edge joining two 4-vertices each adjacent to a 2-vertex) (Figures 4(g) and 4(h)). The set $\{H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8\}$ forms a 1-blocking set of G. Hence $\mathrm{dern}(G) \geq 2$.

Now we shall consider the case when $n > 3$. Consider the da-ecard $(3n, G-$



(a) $H_1(3, G-e_1)$ (b) $H_2(4, G-e_2)$ (c) $H_3(5, G-e_3)$ (d) $H_4(6, G-e_4)$



(e) $H_5(7, G-e_5)$ (f) $H_6(9, G-e_6)$ (g) $H_7(8, G-e_7)$ (h) $H_8(8, G-e_8)$

FIGURE 4

e_6). In $(3n, G-e_6)$, the only possibility to get an extension $H(3n, G-e_6)$ is to join the $(2n-1)$-vertex to the $(n+1)$-vertex (since otherwise the degree sum of the two vertices to be joined will be less than $3n$). Clearly $H(3n, G-e_6) \cong G$ and hence $\text{dern}(G) = 1$. $\square$

Ramachandran [9] proved that $\text{drn}(K_{n,m}) = 2$ for $2 \leq n < m$ and in [6] it is proved that $\text{dern}(K_{n,m}) = 1, 2,$ or 3 for $1 \leq n \leq m$. Next we shall determine drn and dern of the total graph of complete bipartite graphs with partite sets of different size.

Theorem 2.7. *For $1 < n < m$, $\text{drn}(T(K_{n,m})) = 2$.*

Proof. Let $G = T(K_{n,m})$, $1 < n < m$ (Figure. 5). The non-isomorphic dacards of G are $(2n, G-v_1)$, $(n+m, G-v_2)$ and $(2m, G-v_3)$, where v_1 and v_3 are special vertices, and v_2 is a nonspecial vertex. Consider the two dacards $(2n, G-v_1)$ and $(2m, G-v_3)$.
Upper bound. Let G' be a graph having these two dacards in its dadeck. Then G' can be constructed from $(2n, G-v_1)$ by adding a new vertex x and joining it to some set of $2n$ vertices in $G-v_1$. These two dacards force every extension to have maximum degree $2m$, minimum degree $2n$ and hence $(2m, G-v_3)$ forces every extension to have m independent $2n$-vertices with n independent common $2m$-neighbours. Hence in $(2n, G-v_1)$, the vertex x must be adjacent to n independent $(2m-1)$-vertices having $m-1$ common $2n$-neighbours but it cannot be adjacent to these $2n$-vertices. If the remaining neighbours of x are $(n+m-1)$-vertices, then the resulting extension is isomorphic to G. For all other possibilities, the resulting extension does not have the dacard $(2m, G-v_3)$ in its dadeck, since the removal of any $2m$-vertex results in a dacard having an $(m+n-2)$-vertex (when $m > n+1$),

two vertices of degree $2n-1$ (when $m=n+1$), three vertices of degree $2n-1$ (when $m=n+1$ and $n>2$), $n+2$ vertices of degree $2n-1$ (when $m=n+1$), or no m-clique formed by vertices of degree $m+n$. Hence $G' \cong G$ and $\operatorname{drn}(G) \leq 2$.
Lower bound. Consider the extensions $H_1(2n, G-v_1)$ (obtained by joining the new vertex to n vertices of degree $2n$ and to n vertices of degree $2m-1$), $H_2(n+m, G-v_2)$ (obtained by joining the new vertex to $m-1$ vertices of degree $2n$, $n-1$ vertices of degree $2m$, two vertices of degree $n+m-1$), and $H_3(2m, G-v_3)$ (obtained by joining the new vertex to m vertices of degree $2n-1$ and m vertices of degree $n+m$). Then the set $\{H_1, H_2, H_3\}$ clearly forms a 1-blocking set of G. Hence $\operatorname{drn}(G) \geq 2$. □

Theorem 2.8. *For $1<n<m$, $\operatorname{dern}(T(K_{n,m})) = 1$ or 2.*

Proof. Let $G = T(K_{n,m})$, $1<n<m$. First we shall consider the case when $m=n+1$. Consider the da-ecards $(4n-1, G-e_1)$ and $(4n+1, G-e_2)$, where $\operatorname{dep}(e_1) = (2n, 2n+1)$ and $\operatorname{dep}(e_2) = (2n+1, 2n+2)$. The da-ecard $(4n+1, G-e_2)$ has $n+1$ independent $2n$-vertices having $n-1$ common $(2n+2)$-neighbours and a common $(2n+1)$-neighbour.
Upper bound. Let G' be a graph having these two da-ecard in its dadeck. Then G' can be constructed from $(4n-1, G-e_1)$ by adding a new edge joining the unique $(2n-1)$-vertex and a $2n$-vertex in $G-e_1$. In $(4n-1, G-e_1)$, if we add the new edge joining the unique $(2n-1)$-vertex and a $2n$-vertex adjacent to exactly one $(2n+2)$-vertex, then $G' \cong G$. For all other possibilities, the resulting extension does not have the second dacard, since the removal of any edge of degree $4n+1$, results in a da-ecard having no $n+1$ independent $2n$-vertices having a common $(2n+1)$-neighbour (when $n=2$) or $n-1$ common $(2n+2)$-neighbours (when $n>2$). Hence G' is the unique extension and is isomorphic to G. Therefore $\operatorname{dern}(G) \leq 2$.
Lower bound. Consider the extensions $H_1(4n-1, G-e_1)$ (obtained by adding a new edge joining the $(2n-1)$-vertex and a $2n$-vertex having n neighbours of degree $2n+2$), $H_2(4n+1, G-e_2)$ (obtained by adding a new edge joining a $(2n+1)$-vertex and a $2n$-vertex having $n-1$ neighbours of degree $2n+2$), and the extensions of the da-ecards with associated edge degree $4n$ (obtained by adding a new edge joining two vertices of degree $2n$ having n common $(2n+2)$-neighbours). The set of all above extensions forms a 1-blocking set of G. Hence $\operatorname{dern}(G) \geq 2$.

For $m=n+2$, consider the da-ecards $(4n, G-e_1)$ and $(4n+4, G-e_2)$, where $\operatorname{dep}(e_1) = (2n, 2n+2)$ and $\operatorname{dep}(e_2) = (2n+2, 2n+4)$.
Upper bound. We use these two da-ecards to reconstruct G. The da-ecard $(4n+4, G-e_2)$ forces every extension to have minimum degree $2n$. Hence in $(4n, G-e_1)$, the unique $(2n-1)$-vertex must be joined to the unique $(2n+1)$-vertex and so G is the only extension. Therefore $\operatorname{dern}(G) \leq 2$.
Lower bound. Consider the extensions $H_1(4n, G-e_1)$ (obtained by adding a new edge joining two $2n$-vertex), $H_2(4n+4, G-e_2)$ (obtained by adding a new edge joining two $(2n+2)$-vertices), and the extensions of the da-ecards

with associated edge degree $4n$ (obtained by adding a new edge joining a $2n$-vertex and a $(2n+2)$-vertex). The set of all above extensions forms a 1-blocking set of G. Hence $\mathrm{dern}(G) \geq 2$.

For $m > n+2$, consider the da-ecard $(3n+m-2, G-e)$, where $\mathrm{dep}(e) = (2n, n+m)$. In $(3n+m-2, G-e)$, the only possibility to get an extension $H(3n+m-2, G-e)$ is to join unique $(2n-1)$-vertex to the unique $(n+m-1)$-vertex, since otherwise the degree sum of the two vertices to be joined will be $4n-1,\ 4n,\ 3n+m-1,\ 3n+m,\ 2n+2m-1,\ 2n+2m,\ n+3m-1$, or $n+3m$. Clearly $H(3n+m-2, G-e) \cong G$ and hence $\mathrm{dern}(G) = 1$. □

3. Corona Product

In [5], it is proved that $\mathrm{dern}(C_n \circ K_1) = 2$ and $\mathrm{dern}(P_n \circ K_1) = 1$ or 2. In this section, we shall determine the drn and dern for their total graphs.

Theorem 3.1. *For $n > 2$, $\mathrm{drn}(T(C_n \circ K_1)) = 2$.*

Proof. Let $G = T(C_n \circ K_1),\ n > 2$.
Upper bound. Consider the two dacards $(2, G-v_1)$ and $(6, G-v_2)$, where v_1 is a special vertex and v_2 is a nonspecial vertex. Let G' be a graph having these two dacards in its dadeck. Then G' can be constructed from $(2, G-v_1)$ by adding a new vertex x and joining it to two vertices in $G-v_1$. In $(2, G-v_1)$, if we join x to the unique 3-vertex and to the unique 5-vertex, then the resulting extension is isomorphic to G. For all other possibilities, the removal of any 6-vertex from the resulting extension results in a dacard with at least one of the following structures.

(i) A 1-vertex or a 7-vertex.
(ii) $(n-1)$ or $(n-2)$ vertices of degree 2.
(iii) Two adjacent 2-vertices.
(iv) A 2-vertex adjacent to two vertices of degree 6.
(v) A 6-vertex adjacent to a 3-vertex.
(vi) Two 2-vertices having a common neighbour.
(vii) A 2-vertex adjacent to vertices of degree 3 and 6, or 4 and 5.

However, the dacard $(6, G-v_2)$ has none of the above structures. Hence the dadeck of the resulting extension does not contain the dacard $(6, G-v_2)$ in its dadeck. Hence the only possibility is that $G' \cong G$ and therefore $\mathrm{drn}(G) \leq 2$.
Lower bound. Consider the extensions $H_1(2, G-v_1)$ (obtained by joining the new vertex to two 2-vertices), $H_2(4, G-v_2)$ (obtained by joining the new vertex to two 2-vertices and two 4-vertices), $H_3(6, G-v_3)$ where v_3 is a special vertex (obtained by joining the new vertex to two 2-vertices, two 4-vertices and two 5-vertices), and $H_4(6, G-v_4)$ where v_4 is a nonspecial vertex (obtained by joining the new vertex to two 2-vertices, two 3-vertices and two 5-vertices). Then the set $\{H_1, H_2, H_3, H_4\}$ clearly forms a 1-blocking set of G. Hence $\mathrm{drn}(G) \geq 2$. □

Theorem 3.2. *For $n > 2$, $\mathrm{dern}(T(C_n \circ K_1)) = 2$.*

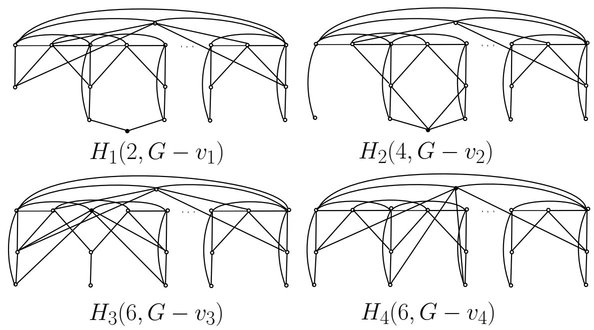


FIGURE 5

Proof. Let $G = T(C_n \circ K_1),\ n > 2$.
Upper bound. Consider the two da-ecards $(4, G - e_1)$ and $(10, G - e_2)$, where $\text{dep}(e_1) = (2, 4)$ and $\text{dep}(e_2) = (6, 6)$ and these two 6-vertices are not adjacent to any 2-vertex. We use these two da-ecards to reconstruct G. The da-ecard $(10, G - e_2)$ forces every extension to have n vertices of degree 2. Hence in $(4, G - e_1)$, the unique 1-vertex must be joined to the unique 3-vertex and therefore G is the only extension and $\text{dern}(G) \leq 2$.
Lower bound. Consider the extensions $H_1(4, G - e_1)$ (obtained by adding a new edge joining the two 2-vertices), $H_2(6, G - e_2)$ (obtained by adding a new edge joining a 2-vertex and a 4-vertex), the extensions of the da-ecards with associated edge degree 8 (obtained by adding a new edge joining a 2-vertex and a 6-vertex), and the extensions of the da-ecards with associated edge degree 10 (obtained by joining a 4-vertex and a 6-vertex). The set of all above extensions forms a 1-blocking set of G. Hence $\text{dern}(G) \geq 2$. □

Theorem 3.3. *For* $n > 1,\ \text{drn}(T(P_n \circ K_1)) = 2$.

Proof. Let $G = T(P_n \circ K_1),\ n > 1$. For $n = 2,\ G = T(P_4)$ and hence $\text{drn}(G) = 2$ by Theorem 2.1. Consider the case when $n > 2$.
Upper bound. Consider the dacards $(2, G - v_1)$ and $(6, G - v_2)$, where v_1 and v_2 are special vertices, v_1 has no 6-neighbour and v_2 has a 5-neighbour. Let G' be a graph having these two dacards in its dadeck. Then G' can be constructed from $(2, G - v_1)$ by adding a new vertex x and joining it to two vertices in $G - v_1$. In $(2, G - v_1)$, if we join x to a 2-vertex and its 3-neighbour adjacent to a 6-vertex, then the resulting extension is isomorphic to G. For $n = 3$, in $(2, G - v_1)$, if we join x to a 6-vertex and a vertex of degree $2,\ 3$, or 4, then the resulting extension has no 6-vertex and for all other possibilities, the resulting extension has the dacard $(2, G - v_1)$ but does not have the dacard $(4, G - v_2)$ in its dadeck, since the removal of any 4-vertex from that extension results in a dacard having exactly one 2-vertex, a 5-vertex, a 6-vertex, a 2-vertex adjacent to a 4-vertex, or no 1-vertex. Hence $G' \cong G$. When $n > 3$, for the remaining possibilities, the removal of any 6-vertex

from the resulting extension results in a dacard having at least one of the following structures.

(i) No 1-vertex.
(ii) A 7-vertex.
(iii) $(2n-7)$ or $(2n-6)$ vertices of degree 6.
(iv) At least four vertices of degree 5.
(v) A 1-vertex adjacent to a 4-vertex.
(vi) Two 2-vertices with a common neighbour.
(vii) $n-2$ vertices of degree 2.

However, the dacard $(6, G-v_2)$ has none of the above structures. Hence the resulting extension contains the dacard $(2, G-v_1)$ but does not contain the dacard $(6, G-v_2)$ in its dadeck. Hence $G' \cong G$ and $\mathrm{drn}(G) \leq 2$.
Lower bound. Consider the extension of the dacards with associated degree 2 (obtained joining the new vertex to two 2-vertices), the extensions of the dacards with associated degree 4 (obtained by joining the new vertex to two 2-vertices and two 4-vertices), the extensions of the dacards with associated degree 3 (obtained by joining the new vertex to two 2-vertices and a 6-vertex), and the extensions of the dacards with associated degree 6 (obtained by joining the new vertex to two 5-vertices, two 3-vertices and two 2-vertices). Then set of all above extensions clearly forms a 1-blocking set of G. Hence $\mathrm{drn}(G) \geq 2$. □

Theorem 3.4. *For $n > 1$, $\mathrm{dern}(T(P_n \circ K_1)) = 2$.*

Proof. Let $G = T(P_n \circ K_1)$, $n > 1$. For $n = 2$, $G = T(P_4)$ and hence $\mathrm{dern}(G) = 2$ by Theorem 2.2.
Upper bound. Consider the two da-ecards $(3, G-e_1)$ and $(8, G-e_2)$, where $\mathrm{dep}(e_1) = (2,3)$ and $\mathrm{dep}(e_2) = (4,6)$, and the 4-vertex is adjacent to a 3-vertex and the 6-vertex is adjacent to a 4-vertex. Let G' be a graph having these two da-ecard in its dadeck. Then G' can be constructed from $(3, G-e_1)$ by adding a new edge joining the 1-vertex and a 2-vertex in $G-e_1$. In $(3, G-e_1)$, if we add the new edge joining the unique 1-vertex and a 2-vertex having a common neighbour with the 1-vertex, then $G' \cong G$. For all other possibilities, the resulting extension does not have the dacard $(8, G-e_2)$ in its dadeck, since the removal of any edge of degree 8 results in a dacard having two 2-vertices with a common neighbour. Hence G' is the unique extension and is isomorphic to G. Therefore $\mathrm{dern}(G) \leq 2$.
Lower bound. Consider the extension $H_1(3, G-e_1)$ (obtained by adding a new edge joining the 1-vertex and a 2-vertex having no common neighbours), $H_2(4, G-e_2)$ (obtained by adding a new edge joining a 1-vertex and a 3-vertex having no common neighbour), $H_3(5, G-e_3)$ (obtained by adding a new edge joining a 3-vertex and a 2-vertex adjacent to a 4-vertex), the extensions of the da-ecards with associated edge degree 7 (obtained by joining a 2-vertex and a 5-vertex), the extensions of the da-ecards with associated edge degree 8 (obtained by adding a new edge joining a two 4-vertices each adjacent to a 2-vertex), the extensions of the da-ecards with associated edge

degree 9 (obtained by adding a new edge joining a 4-vertex adjacent to a 2-vertex and a 5-vertex), and the extensions of the da-ecards with associated edge degree 10 (when $n > 3$) (obtained by adding a new edge joining a 4-vertex and a 6-vertex). Clearly the set of all above extensions forms a 1-blocking set of G. Hence $\mathrm{dern}(G) \geq 2$. $\square$

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