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ON THE NUMBER OF PARTITIONS INTO ODD PARTS OR CONGRUENT TO $\pm 2 \bmod 10$

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Abstract. Let $R_2(n)$ denote the number of partitions of n into parts that are odd or congruent to $\pm 2 \pmod{10}$. In 2007, Andrews considered partitions with some negative parts and provided a second combinatorial interpretation for $R_2(n)$. In this paper, we give a collection of linear recurrence relations for the partition function $R_2(n)$. As a corollary, we obtain a simple criterion for deciding whether $R_2(n)$ is odd or even. Some identities involving overpartitions and partitions into distinct parts are derived in this context.

1. Introduction

We adopt the standard notations on partitions and q-series as used in [1, 7]. The q-shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}),$$

with $(a;q)_0 = 1$. For $|q|<1$, we also define

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$$

Because this infinite product diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q|<1$. Since products of q-shifted factorials occur so often, to simplify them we shall use the more compact notation

$$(a_1,a_2\ldots,a_n;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty.$$

Let $R_2(n)$ be the function which enumerates the number of ordinary partitions of n into parts that are either odd or congruent to $\pm 2 \pmod{10}$. The generating function for $R_2(n)$,

$$\sum_{n=0}^{\infty} R_2(n)q^n = \frac{1}{(q;q^2)_\infty(q^2,q^8;q^{10})_\infty},$$

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appears in the following q-identity of Rogers [2]

$$\sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(3n+1)/2}}{(q;q)_{2n+1}} = \frac{G(q^2)}{(q;q^2)_\infty},$$

where

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}$$

is the Rogers–Ramanujan function. Recall that $G(q)$ is the generating function for the number of partition of n into parts of the form $5k+1$ or $5k+4$.

In 2007, Andrews [2] considered partitions with some negative parts and provided a second combinatorial interpretation for the numbers $R_2(n)$. We remark that the sequence $\{R_2(n)\}_{n\geqslant 0}$ is known and can be seen in [15, A133153].

In this paper, motivated by these results, we give a collection of recurrence relations for the partition function $R_2(n)$. The first result states that the partition function $R_2(n)$ satisfies Euler's recurrence relation for the partition function $p(n)$ unless n is the form $k(5k+1)$ with $k \in \mathbb{Z}$.

Theorem 1.1. *For $n \geqslant 0$,*

$$\sum_{j=-\infty}^{\infty} (-1)^j R_2(n-j(3j+1)/2) = \begin{cases} (-1)^k, & \text{if } n = k(5k+1),\ k \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

The second result provides a linear recurrence relation for $R_2(n)$ that involves the triangular numbers, i.e., $k(k+1)/2,\ k \in \mathbb{N}_0$.

Theorem 1.2. *For $n \geqslant 0$,*

$$\sum_{j=0}^{\infty} (-1)^{\lceil j/2 \rceil} R_2(n-j(j+1)/2) = \begin{cases} (-1)^k, & \text{if } n = 30k^2-4k \text{ or} \\ & \quad n = 30k^2+16k+2,\ k \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

In the following recurrence relation, the partition function $R_2(n)$ is combined with the perfect square numbers.

Theorem 1.3. *For $n \geqslant 0$,*

$$\sum_{j=-\infty}^{\infty} (-1)^j R_2(n-j^2) = \begin{cases} 1, & \text{if } n = k(15k+1)/2,\ k \in \mathbb{Z}, \\ -1, & \text{if } n = (3k-1)(5k-2)/2,\ k \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

A more efficient recurrence relation for $R_2(n)$ is given by the following result.

Theorem 1.4. *For $n \geqslant 0$,*

$$\sum_{j=-\infty}^{\infty} (-1)^j R_2(n-2j^2) = \begin{cases} (-1)^{k(k-1)/2}, & \text{if } n = k(15k+1)/2 \text{ or} \\ & \quad n = (3k-1)(5k-2)/2,\ k \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

If n is small, then it is relatively easy to evaluate $R_2(n)$, namely

$$\begin{aligned}
R_2(0) &= 1,\\
R_2(1) &= 1,\\
R_2(2) &= 2R_2(0) = 2,\\
R_2(3) &= 2R_2(1) - 1 = 3,\\
R_2(4) &= 2R_2(2) = 4,\\
R_2(5) &= 2R_2(3) = 6,\\
R_2(6) &= 2R_2(4) = 8,\\
R_2(7) &= 2R_2(5) - 1 = 11,\\
R_2(8) &= 2R_2(6) - 2R_2(0) + 1 = 15,\\
R_2(9) &= 2R_2(7) - 2R_2(1) = 20.
\end{aligned}$$

As a corollary of Theorem 1.3 or 1.4, we derive a simple criterion for deciding whether the number $R_2(n)$ is odd or even.

Corollary 1.5. *The number of ordinary partitions of n into parts that are either odd or congruent to $\pm 2 \pmod{10}$ is odd if and only if $n = k(15k+1)/2$ or $n = (3k-1)(5k-2)/2,\ k \in \mathbb{Z}$.*

As far as we know, this parity result is new. A collection of identities involving overpartitions and partitions into distinct parts are derived in this context.

2. Proof of Theorem 1.1

By the Jacobi triple product identity [4, Theorem 11],

$$\sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n(n-1)/2} = (q, z, q/z; q)_\infty, \tag{2.1}$$

with q replaced by q^{10} and z replaced by q^6, we obtain

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+1)} = (q^4, q^6, q^{10}; q^{10})_\infty. \tag{2.2}$$

On the other hand, considering Euler's pentagonal number theorem

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q; q)_\infty, \tag{2.3}$$

we can write

$$\left(\sum_{n=0}^{\infty} R_2(n) q^n\right)\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}\right) = \frac{(q;q)_\infty}{(q;q^2)_\infty (q^2, q^8; q^{10})_\infty}. \tag{2.4}$$

Taking into account that

$$\begin{aligned}\frac{(q;q)_\infty}{(q;q^2)_\infty(q^2,q^8;q^{10})_\infty} &= \frac{(q;q)_\infty}{(q,q^3,q^5,q^7,q^9;q^{10})_\infty(q^2,q^8;q^{10})_\infty}\\ &= (q^4,q^6,q^{10};q^{10})_\infty,\end{aligned}$$

by (2.2) and (2.4), we give the identity

$$\left(\sum_{n=0}^{\infty} R_2(n)q^n\right)\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}\right) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(5n+1)}.$$

The proof follows easily applying the well known Cauchy product of two power series on this identity. □

3. Proof of Theorem 1.2

By the Jacobi triple product identity (2.1), with q replaced by q^4 and z replaced by q^3, we obtain the relation

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n} = (q,q^3,q^4;q^4)_\infty,$$

that can be written as

$$\sum_{n=0}^{\infty}(-1)^{\lceil n/2\rceil} q^{n(n+1)/2} = (q,q^3,q^4;q^4)_\infty.$$

It is clear that

$$\left(\sum_{n=0}^{\infty} R_2(n)q^n\right)\left(\sum_{n=0}^{\infty}(-1)^{\lceil n/2\rceil} q^{n(n+1)/2}\right) = \frac{(q,q^3,q^4;q^4)_\infty}{(q;q^2)_\infty(q^2,q^8;q^{10})_\infty}. \tag{3.1}$$

Watson's quintuple product identity [5, 16] states that

$$\sum_{n=-\infty}^{\infty} z^{3n}q^{n(3n-1)/2}(1-zq^n) = (q,z,q/z;q)_\infty(qz^2,q/z^2;q^2)_\infty. \tag{3.2}$$

By this identity, with q replaced by q^{20} and z replaced by $-q^2$, we get

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}(-1)^n\left(q^{30n^2-4n}+q^{30n^2+16n+2}\right)\\ &\qquad = (-q^2,-q^{18},q^{20};q^{20})_\infty(q^{16},q^{24};q^{40})_\infty.\end{aligned} \tag{3.3}$$

Considering that

$$\begin{aligned}(-q^2,-q^{18},q^{20};q^{20})_\infty&(q^{16},q^{24};q^{40})_\infty\\ &=\frac{(q^4,q^{20},q^{36},q^{40};q^{40})_\infty}{(q^2,q^{18};q^{20})_\infty}\cdot(q^{16},q^{24};q^{40})_\infty\\ &=\frac{(q^4,q^{16},q^{20};q^{20})_\infty}{(q^2,q^{18};q^{20})_\infty}\\ &=\frac{(q^4,q^8,q^{12},q^{16},q^{20};q^{20})_\infty}{(q^8,q^{12};q^{20})_\infty(q^2,q^{18};q^{20})_\infty}\\ &=\frac{(q^4;q^4)_\infty}{(q^2,q^8;q^{10})_\infty}\\ &=\frac{(q,q^3,q^4;q^4)_\infty}{(q;q^2)_\infty(q^2,q^8;q^{10})_\infty}\end{aligned}$$

by (3.1) and (3.3), we obtain the relation

$$\begin{aligned}\left(\sum_{n=0}^{\infty}R_2(n)q^n\right)&\left(\sum_{n=0}^{\infty}(-1)^{\lceil n/2\rceil}q^{n(n+1)/2}\right)\\ &=\sum_{n=-\infty}^{\infty}(-1)^n\left(q^{30n^2-4n}+q^{30n^2+16n+2}\right).\end{aligned}$$

Applying the well known Cauchy product of two power series on this identity, we arrive at our conclusion. □

4. Proof of Theorem 1.3

By the Jacobi triple product identity (2.1) with q replaced by q^2 and z replaced by q, we obtain

$$\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}=(q,q,q^2;q^2)_\infty=(q;q^2)_\infty(q;q)_\infty. \tag{4.1}$$

So we can write

$$\left(\sum_{n=0}^{\infty}R_2(n)q^n\right)\left(\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}\right)=\frac{(q;q)_\infty}{(q^2,q^8;q^{10})_\infty}. \tag{4.2}$$

By Watson's quintuple product identity (3.2), with q replaced by q^5 and z replaced by q, we get

$$\sum_{n=-\infty}^{\infty}\left(q^{n(15n+1)/2}-q^{(3n-1)(5n-2)/2}\right)=(q,q^4,q^5;q^5)_\infty(q^3,q^7;q^{10})_\infty. \tag{4.3}$$

It is an easy exercise to show that

$$\frac{(q;q)_\infty}{(q^2,q^8;q^{10})_\infty}=(q,q^4,q^5;q^5)_\infty(q^3,q^7;q^{10})_\infty. \tag{4.4}$$

Thus, by the identities (4.2) and (4.3), we deduce the relation

$$\left(\sum_{n=0}^{\infty} R_2(n)q^n\right)\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}\right) = \sum_{n=-\infty}^{\infty}\left(q^{n(15n+1)/2} - q^{(3n-1)(5n-2)/2}\right).$$

The proof follows easily considering the Cauchy product of two power series. □

5. Proof of Theorem 1.4

By the Jacobi triple product identity (2.1), with q replaced by q^4 and z replaced by q^2, we obtain

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2} = (q^2, q^2, q^4; q^4)_\infty. \tag{5.1}$$

So we can write

$$\left(\sum_{n=0}^{\infty} R_2(n)q^n\right)\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}\right) = \frac{(q^2, q^2, q^4; q^4)_\infty}{(q; q^2)_\infty (q^2, q^8; q^{10})_\infty}. \tag{5.2}$$

By Watson's quintuple product identity (3.2), with q replaced by $-q^5$ and z replaced by $-q$, we get

$$\sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2}\left(q^{n(15n+1)/2} + q^{(3n-1)(5n-2)/2}\right) = (-q, q^4, -q^5; -q^5)_\infty(-q^3, -q^7; q^{10})_\infty. \tag{5.3}$$

Taking into account that

$$\begin{aligned}
&(-q, q^4, -q^5; -q^5)_\infty(-q^3, -q^7; q^{10})_\infty \\
&= (-q, -q^5, -q^9; q^{10})_\infty (q^4, q^6, q^{10}; q^{10})_\infty \frac{(q^6, q^{14}; q^{20})_\infty}{(q^3, q^7; q^{10})_\infty} \\
&= \frac{(q^2, q^{10}, q^{18}; q^{20})_\infty}{(q, q^5, q^9; q^{10})_\infty} \cdot \frac{(q^2, q^4, q^6, q^8, q^{10}; q^{10})_\infty}{(q^2, q^8; q^{10})_\infty} \cdot \frac{(q^6, q^{14}; q^{20})_\infty}{(q^3, q^7; q^{10})_\infty} \\
&= \frac{(q^2, q^6, q^{10}, q^{14}, q^{18}; q^{20})_\infty}{(q, q^3, q^5, q^7, q^9; q^{10})_\infty} \cdot \frac{(q^2; q^2)_\infty}{(q^2, q^8; q^{10})_\infty} \\
&= \frac{(q^2; q^4)_\infty}{(q; q^2)_\infty} \cdot \frac{(q^2; q^2)_\infty}{(q^2, q^8; q^{10})_\infty} \\
&= \frac{(q^2, q^2, q^4; q^4)_\infty}{(q; q^2)_\infty (q^2, q^8; q^{10})_\infty} \qquad (5.4)
\end{aligned}$$

by (5.2) and (5.3), we deduce that

$$\left(\sum_{n=0}^{\infty} R_2(n)q^n\right)\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}\right)$$
$$= \sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2}\left(q^{n(15n+1)/2} + q^{(3n-1)(5n-2)/2}\right).$$

Applying again the Cauchy product of two power series, we arrive at our result. □

6. Connections with overpartitions and partitions into distinct parts

We start with two q-identities that involves the Rogers–Ramanujan function $G(q)$. These follow easily from the previous sections. In order to simplify the writing, for any integer n we define

$$a_n = \frac{n(15n+1)}{2} \qquad \text{and} \qquad b_n = \frac{(3n-1)(5n-2)}{2}.$$

Theorem 6.1. *For* $|q| < 1$,

$$\frac{G(q^2)}{(q;q^2)_\infty} = \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\left(q^{a_n} - q^{b_n}\right).$$

Proof. To derive this identity, we consider the relations (4.1), (4.3), and (4.4). □

Theorem 6.2. *For* $|q| < 1$,

$$\frac{G(q^2)}{(q;q^2)_\infty} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}(-1)^{n(n-1)/2}\left(q^{a_n} + q^{b_n}\right).$$

Proof. This identity follows considering the relations (5.1), (5.3), and (5.4). □

Recall that an overpartition of the nonnegative integer n is a partition of n where the first occurrence of parts of each size may be overlined. Let $\bar{p}(n)$ denote the number of overpartitions of n. For example, the overpartitions of the integer 3 are:

$$3,\ \bar{3},\ 2+1,\ \bar{2}+1,\ 2+\bar{1},\ \bar{2}+\bar{1},\ 1+1+1 \text{ and } \bar{1}+1+1.$$

We see that $\bar{p}(3) = 8$. The overpartition function $\bar{p}(n)$ has the generating function

$$\sum_{n=0}^{\infty}\bar{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}$$

and its properties have been the subject of many recent studies [3, 6, 8, 9, 10, 11, 12, 13]. It is clear that, the partition function $R_2(n)$ can be expressed in terms of the overpartition function $\bar{p}(n)$ as combinatorial interpretations of Theorems 6.1 and 6.2.

Corollary 6.3. *For any nonnegative integer n,*

$$R_2(n) = \sum_{k=-\infty}^{\infty} \left[\bar{p}\left(n-a_k\right) - \bar{p}\left(n-b_k\right)\right].$$

Proof. By Theorem 6.1, we obtain

$$\sum_{n=0}^{\infty} R_2(n)q^n = \left(\sum_{n=0}^{\infty} \bar{p}(n)q^n\right)\left(\sum_{n=-\infty}^{\infty} \left(q^{a_n} - q^{b_n}\right)\right).$$

The proof follows easily applying the Cauchy product on this identity. □

Corollary 6.4. *For any nonnegative integer n,*

$$R_2(n) = \sum_{k=-\infty}^{\infty} (-1)^{k(k-1)/2}\left[\bar{p}\left(\frac{n}{2}-\frac{a_k}{2}\right) + \bar{p}\left(\frac{n}{2}-\frac{b_k}{2}\right)\right],$$

with $\bar{p}(x) = 0$ if x is not an integer.

Proof. By Theorem 6.2, we can write

$$\sum_{n=0}^{\infty} R_2(n)q^n = \left(\sum_{n=0}^{\infty} \bar{p}(n)q^{2n}\right)\left(\sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2}(q^{a_n} + q^{b_n})\right).$$

The proof follows equating the coefficient of q^n on each side of this identity. □

It is well known that

$$\bar{p}(n) \equiv 0 \pmod 2 \quad \text{if and only if} \quad n \neq 0.$$

This parity result can be deduced from the recurrence relation

$$\sum_{j=-\infty}^{\infty} (-1)^j \bar{p}(n-j^2) = \delta_{0,n}$$

and can be used together with Corollary 6.3 or 6.4 to derive another proof of Corollary 1.5.

Recall that the number of partitions of n into distinct parts is usually denoted by $q(n)$ and the generating function of $q(n)$ is given by

$$\sum_{n=0}^{\infty} q(n)q^n = (-q;q)_\infty = \frac{1}{(q;q^2)_\infty}.$$

We have the following results.

Corollary 6.5. *For any nonnegative integer n, the partition functions $q(n)$ and $R_2(n)$ are related by:*

(1) $\displaystyle\sum_{k=0}^{n}(-1)^k R_2(n-k(3k-1)/2) = \sum_{k=-\infty}^{\infty} \left[q\left(n-a_k\right) - q\left(n-b_k\right)\right];$

(2) $\displaystyle\sum_{k=0}^{n}(-1)^k R_2(n-k(3k-1))$

$$= \sum_{k=-\infty}^{\infty} (-1)^{k(k-1)/2} \left[q\left(\frac{n}{2}-\frac{a_k}{2}\right) + q\left(\frac{n}{2}-\frac{b_k}{2}\right) \right],$$

where $q(x)=0$ if x is not an integer.

Proof. Considering Theorems 6.1 and 6.2, we obtain

$$(q;q)_\infty \sum_{n=0}^{\infty} R_2(n) q^n = (-q;q)_\infty \sum_{n=-\infty}^{\infty} \left(q^{a_n} - q^{b_n} \right)$$

and

$$(q^2;q^2)_\infty \sum_{n=0}^{\infty} R_2(n) q^n = (-q^2;q^2)_\infty \sum_{n=-\infty}^{\infty} (-1)^{n(n+1)/2} \left(q^{a_n} + q^{b_n} \right),$$

respectively. The proof follows equating the coefficients of q^n in these relations. □

As a consequence of Theorem 1.1 and Corollary 6.5, we derive the following recurrence relation for the partition function $q(n)$.

Corollary 6.6. *For $n \geqslant 0$,*

$$\sum_{j=-\infty}^{\infty} \left[q\left(n-a_j\right) - q\left(n-b_j\right) \right] = \begin{cases} (-1)^k, & \text{if } n = k(5k+1),\ k \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

The number of partitions of n into distinct odd parts is denoted in this paper by $q_{odd}(n)$ and the generating function of $q_{odd}(n)$ is given by

$$\sum_{n=0}^{\infty} q_{odd}(n) q^n = (-q;q^2)_\infty = \frac{1}{(q;-q)_\infty}.$$

Corollary 6.7. *For any nonnegative integer n, the partition functions $p(n)$, $q_{odd}(n)$ and $R_2(n)$ are related by:*

(1) $$\sum_{k=0}^{n} (-1)^k q_{odd}(k) R_2(n-k) = \sum_{k=-\infty}^{\infty} \left[p\left(n-a_k\right) - p\left(n-b_k\right) \right];$$

(2) $$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q_{odd}(k) R_2(n-2k)$$
$$= \sum_{k=-\infty}^{\infty} (-1)^{k(k-1)/2} \left[p\left(\frac{n}{2}-\frac{a_k}{2}\right) + p\left(\frac{n}{2}-\frac{b_k}{2}\right) \right]$$

where $p(x)=0$ if x is not an integer;

(3) $$\sum_{k=0}^{n} q_{odd}(k) R_2(n-k) = \sum_{k=-\infty}^{\infty} (-1)^{k(k-1)/2} \left[p\left(n-a_k\right) + p\left(n-b_k\right) \right].$$

Proof. Rewriting Theorem 6.1 as

$$\frac{1}{(-q;q)_\infty} \sum_{n=0}^{\infty} R_2(n) q^n = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \left(q^{a_n} - q^{b_n} \right),$$

we derive the first identity.

By Theorem 6.2, we give

$$\frac{1}{(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}R_2(n)q^n=\frac{1}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}(-1)^{n(n-1)/2}\left(q^{a_n}+q^{b_n}\right)$$

and

$$(-q;q^2)\sum_{n=0}^{\infty}R_2(n)q^n=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^{n(n-1)/2}\left(q^{a_n}+q^{b_n}\right).$$

The proof of the next two identities follows easily. □

On the other hand, we denote by $q_{e-o}(n)$ the number of partitions of n into distinct parts with an even number of odd parts minus partitions of n into distinct parts with an odd number of odd parts. Because

$$\sum_{n=0}^{\infty}q_{e-o}(n)q^n=\frac{1}{(-q;q^2)_\infty}=\sum_{n=0}^{\infty}(-1)^n q(n)q^n,$$

Corollaries 6.5 and 6.6 can be rewritten in terms of the partition function $q_{e-o}(n)$.

Similarly, we denote by $p_{e-o}(n)$ the number of partitions of n into an even number of parts minus partitions of n into an odd number of parts. Taking into account that

$$\sum_{n=0}^{\infty}p_{e-o}(n)q^n=\frac{1}{(-q;q)_\infty}=\sum_{n=0}^{\infty}(-1)^n q_{odd}(n)q^n,$$

Corollary 6.7 can be restated in terms of the partition function $q_{odd}(n)$.

7. Concluding remarks

Four linear recurrence relations for the number of ordinary partitions of n into parts that are either odd or congruent to $\pm 2 \pmod{10}$ have been introduced in the paper. Two of these results allow us to derive a simple criterion for deciding whether the number of ordinary partitions of n into parts that are either odd or congruent to $\pm 2 \pmod{10}$ is odd or even. We remark that Theorem 1.4 provides an extremely efficient algorithm for computing $R_2(n)$. In [14], we carried out a similar set of recurrences for partitions where even parts do not repeat.

On the other hand, Theorems 6.1 and 6.2 allow us to derive a collection of identities involving partition into distinct parts. We should note that Theorem 6.2 can be derived from Theorem 6.1 replacing q by $-q$. In fact Theorem 6.1 is a version of the following identity for the Rogers–Ramanujan function $G(q)$.

Theorem 7.1. *For $|q|<1$,*

$$G(q^2)=\frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\left(q^{n(15n+1)/2}-q^{(3n-1)(5n-2)/2}\right).$$

A linear homogeneous recurrence relation for the partition function $p(n)$ can be easily derived from this identity.

Corollary 7.2. *For n odd,*

$$\sum_{k=-\infty}^{\infty} \left[p\left(n - \frac{k(15k+1)}{2} \right) - p\left(n - \frac{(3k-1)(5k-2)}{2} \right) \right] = 0.$$

Moreover, the case n even can be written as follows.

Corollary 7.3. *The number of partitions of n into parts of the form $5k+1$ or $5k+4$ can be expressed in terms of the partition function $p(n)$, i.e.,*

$$\sum_{k=-\infty}^{\infty} \left[p\left(2n - \frac{k(15k+1)}{2} \right) - p\left(2n - \frac{(3k-1)(5k-2)}{2} \right) \right].$$

Corollary 7.4. *The number of partitions of n into parts of the form $5k+1$ or $5k+4$ can be expressed as a finite discrete convolution, i.e.,*

$$\sum_{k=0}^{2n} (-1)^k q_{odd}(k) R_2(2n-k).$$

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