ON THE NUMBER OF PARTITIONS INTO ODD PARTS
OR CONGRUENT TO $\pm 2 \text{ mod } 10$

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Abstract. Let $R_2(n)$ denote the number of partitions of $n$ into parts that are odd or congruent to $\pm 2 \pmod{10}$. In 2007, Andrews considered partitions with some negative parts and provided a second combinatorial interpretation for $R_2(n)$. In this paper, we give a collection of linear recurrence relations for the partition function $R_2(n)$. As a corollary, we obtain a simple criterion for deciding whether $R_2(n)$ is odd or even. Some identities involving overpartitions and partitions into distinct parts are derived in this context.

1. Introduction

We adopt the standard notations on partitions and $q$-series as used in [1, 7]. The $q$-shifted factorial $(a; q)_n$ is defined by

$$(a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}),$$

with $(a; q)_0 = 1$. For $|q| < 1$, we also define

$$(a; q)_\infty = \lim_{n \to \infty} (a; q)_n.$$

Because this infinite product diverges when $a \neq 0$ and $|q| \geq 1$, whenever $(a; q)_\infty$ appears in a formula, we shall assume that $|q| < 1$. Since products of $q$-shifted factorials occur so often, to simplify them we shall use the more compact notation

$$(a_1, a_2, \ldots, a_n; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_n; q)_\infty.$$

Let $R_2(n)$ be the function which enumerates the number of ordinary partitions of $n$ into parts that are either odd or congruent to $\pm 2 \pmod{10}$. The generating function for $R_2(n)$,

$$\sum_{n=0}^{\infty} R_2(n) q^n = \frac{1}{(q; q^2)_\infty (q^2, q^8; q^{10})_\infty},$$

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appears in the following \( q \)-identity of Rogers [2]  
\[
\sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(3n+1)/2}}{(q; q)_{2n+1}} = \frac{G(q^2)}{(q; q^2)_{\infty}},
\]
where  
\[
G(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_{\infty}}
\]
is the Rogers–Ramanujan function. Recall that \( G(q) \) is the generating function for the number of partition of \( n \) into parts of the form \( 5k+1 \) or \( 5k+4 \).

In 2007, Andrews [2] considered partitions with some negative parts and provided a second combinatorial interpretation for the numbers \( R_2(n) \). We remark that the sequence \( \{R_2(n)\}_{n \geq 0} \) is known and can be seen in [15, A133153].

In this paper, motivated by these results, we give a collection of recurrence relations for the partition function \( R_2(n) \). The first result states that the partition function \( R_2(n) \) satisfies Euler’s recurrence relation for the partition function \( p(n) \) unless \( n \) is the form \( k(5k+1) \) with \( k \in \mathbb{Z} \).

**Theorem 1.1.** For \( n \geq 0 \),  
\[
\sum_{j=-\infty}^{\infty} (-1)^j R_2(n - j(3j + 1)/2) = \begin{cases} 
(-1)^k, & \text{if } n = k(5k+1), k \in \mathbb{Z}, \\
0, & \text{otherwise}.
\end{cases}
\]

The second result provides a linear recurrence relation for \( R_2(n) \) that involves the triangular numbers, i.e., \( k(k+1)/2 \), \( k \in \mathbb{N}_0 \).

**Theorem 1.2.** For \( n \geq 0 \),  
\[
\sum_{j=0}^{\infty} (-1)^{[j/2]} R_2(n - j(j + 1)/2) = \begin{cases} 
(-1)^k, & \text{if } n = 30k^2 - 4k \text{ or } n = 30k^2 + 16k + 2, k \in \mathbb{Z}, \\
0, & \text{otherwise}.
\end{cases}
\]

In the following recurrence relation, the partition function \( R_2(n) \) is combined with the perfect square numbers.

**Theorem 1.3.** For \( n \geq 0 \),  
\[
\sum_{j=-\infty}^{\infty} (-1)^j R_2(n - j^2) = \begin{cases} 
1, & \text{if } n = k(15k+1)/2, k \in \mathbb{Z}, \\
-1, & \text{if } n = (3k-1)(5k-2)/2, k \in \mathbb{Z}, \\
0, & \text{otherwise}.
\end{cases}
\]

A more efficient recurrence relation for \( R_2(n) \) is given by the following result.

**Theorem 1.4.** For \( n \geq 0 \),  
\[
\sum_{j=-\infty}^{\infty} (-1)^j R_2(n - 2j^2) = \begin{cases} 
(-1)^{k(k-1)/2}, & \text{if } n = k(15k+1)/2 \text{ or } n = (3k-1)(5k-2)/2, k \in \mathbb{Z}, \\
0, & \text{otherwise}.
\end{cases}
\]
If $n$ is small, then it is relatively easy to evaluate $R_2(n)$, namely

\[
R_2(0) = 1, \\
R_2(1) = 1, \\
R_2(2) = 2R_2(0) = 2, \\
R_2(3) = 2R_2(1) - 1 = 3, \\
R_2(4) = 2R_2(2) = 4, \\
R_2(5) = 2R_2(3) = 6, \\
R_2(6) = 2R_2(4) = 8, \\
R_2(7) = 2R_2(5) - 1 = 11, \\
R_2(8) = 2R_2(6) - 2R_2(0) + 1 = 15, \\
R_2(9) = 2R_2(7) - 2R_2(1) = 20.
\]

As a corollary of Theorem 1.3 or 1.4, we derive a simple criterion for deciding whether the number $R_2(n)$ is odd or even.

**Corollary 1.5.** The number of ordinary partitions of $n$ into parts that are either odd or congruent to $\pm 2 \pmod{10}$ is odd if and only if $n = k(15k+1)/2$ or $n = (3k-1)(5k-2)/2$, $k \in \mathbb{Z}$.

As far as we know, this parity result is new. A collection of identities involving overpartitions and partitions into distinct parts are derived in this context.

## 2. Proof of Theorem 1.1

By the Jacobi triple product identity [4, Theorem 11],

\[
\sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n(n-1)/2} = (q, z, q/z; q)_\infty,
\]

with $q$ replaced by $q^{10}$ and $z$ replaced by $q^6$, we obtain

\[
\sum_{n=-\infty}^{\infty} (-1)^n q^{5n+1} = (q^4, q^6, q^{10}; q^{10})_\infty.
\]

On the other hand, considering Euler’s pentagonal number theorem

\[
\sum_{n=-\infty}^{\infty} (-1)^n q^{3n+1} = (q; q)_\infty,
\]

we can write

\[
(\sum_{n=0}^{\infty} R_2(n) q^n) \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \right) = \frac{(q; q)_\infty}{(q; q^2)_\infty (q^2; q^5; q^{10})_\infty}.
\]
Taking into account that
\[(q; q)_\infty \frac{(q_1; q_2)_\infty (q_1^2; q_3^8; q_4^{10})_\infty}{(q_1^3; q_4^5; q_5^7; q_6^9; q_7^{10})_\infty (q_2^2; q_3^8; q_4^{10})_\infty} = (q_4^4; q_5^6; q_6^{10}; q_7^{10})_\infty,
\]
by (2.2) and (2.4), we give the identity
\[\left( \sum_{n=0}^{\infty} R_2(n) q^n \right) \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n+1)/2} \right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+1)}.
\]

The proof follows easily applying the well known Cauchy product of two power series on this identity. \(\square\)

3. Proof of Theorem 1.2

By the Jacobi triple product identity (2.1), with \(q\) replaced by \(q^4\) and \(z\) replaced by \(q^3\), we obtain the relation
\[\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+n} = (q, q^3; q^4)_\infty,
\]
that can be written as
\[\sum_{n=0}^{\infty} (-1)^{\lceil n/2 \rceil} q^{n(n+1)/2} = (q, q^3, q^4)_\infty.
\]
It is clear that
\[(3.1) \quad \left( \sum_{n=0}^{\infty} R_2(n) q^n \right) \left( \sum_{n=0}^{\infty} (-1)^{\lceil n/2 \rceil} q^{n(n+1)/2} \right) = \frac{(q, q^3, q^4; q^4)_\infty}{(q; q^2)_\infty (q^2; q^8; q^{10})_\infty}.
\]

Watson’s quintuple product identity [5, 16] states that
\[(3.2) \quad \sum_{n=-\infty}^{\infty} z^{3n} q^{n(3n-1)/2} (1 - z q^n) = (q, z, q/z; q)_\infty (q z^2, q/z^2; q^2)_\infty.
\]
By this identity, with \(q\) replaced by \(q^{20}\) and \(z\) replaced by \(-q^2\), we get
\[\sum_{n=-\infty}^{\infty} (-1)^n \left( q^{30n^2-4n} + q^{30n^2+16n+2} \right)
\]
\[(3.3) = (-q^2, -q^18, q^{20}; q^{20})_\infty (q^{16}, q^{24}; q^{40})_\infty.
\]
Considering that
\[(−q^2, q^{18}, q^{20}; q^{20})_\infty (q^{16}, q^{24}; q^{40})_\infty \]
\[= \frac{(q^4, q^{20}, q^{36}, q^{40}; q^{40})_\infty}{(q^2, q^{18}; q^{20})_\infty} \cdot (q^{16}, q^{24}; q^{40})_\infty \]
\[= \frac{(q^4, q^{16}, q^{20}; q^{20})_\infty}{(q^2, q^{18}; q^{20})_\infty} \]
\[= \frac{(q^{16}, q^{20}, q^{20})_\infty}{(q^{8}, q^{12}; q^{20})_\infty (q^{2^2}, q^{18}; q^{20})_\infty} \]
\[= \frac{(q^4; q^4)_\infty}{(q^2, q^8; q^{10})_\infty} \]
\[= \frac{(q, q^3, q^4; q^4)_\infty}{(q; q^2)_\infty (q^2, q^8; q^{10})_\infty} \]

by (3.1) and (3.3), we obtain the relation
\[
\left(\sum_{n=0}^{\infty} R_2(n)q^n\right) \left(\sum_{n=0}^{\infty} (-1)^{\lceil n/2 \rceil} q^{n(n+1)/2}\right) = \sum_{n=-\infty}^{\infty} (-1)^n \left(q^{30n^2-4n+q^{30n^2+16n+2}}\right). \]

Applying the well known Cauchy product of two power series on this identity, we arrive at our conclusion. \(\square\)

4. PROOF OF THEOREM 1.3

By the Jacobi triple product identity (2.1) with \(q\) replaced by \(q^2\) and \(z\) replaced by \(q\), we obtain
\[(4.1) \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = (q, q, q^2; q^2)_\infty = (q; q^2)_\infty (q; q)_\infty. \]

So we can write
\[(4.2) \left(\sum_{n=0}^{\infty} R_2(n)q^n\right) \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}\right) = \frac{(q; q)_\infty}{(q^2, q^3; q^{10})_\infty}. \]

By Watson’s quintuple product identity (3.2), with \(q\) replaced by \(q^5\) and \(z\) replaced by \(q\), we get
\[(4.3) \sum_{n=-\infty}^{\infty} \left(q^{n(15n+1)/2} - q^{(3n-1)(5n-2)/2}\right) = (q, q^4, q^5; q^5)_\infty (q^3, q^7; q^{10})_\infty. \]

It is an easy exercise to show that
\[(4.4) \frac{(q; q)_\infty}{(q^2, q^3; q^{10})_\infty} = (q, q^4, q^5; q^5)_\infty (q^3, q^7; q^{10})_\infty. \]
Thus, by the identities (4.2) and (4.3), we deduce the relation
\[
\left(\sum_{n=0}^{\infty} R_2(n)q^n\right) \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}\right) = \sum_{n=-\infty}^{\infty} \left(q^{n(15n+1)/2} - q^{(3n-1)(5n-2)/2}\right).
\]

The proof follows easily considering the Cauchy product of two power series.

\[
\square
\]

5. PROOF OF THEOREM 1.4

By the Jacobi triple product identity (2.1), with \(q\) replaced by \(q^4\) and \(z\) replaced by \(q^2\), we obtain
\[
\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2} = (q^2, q^4; q^4)_\infty.
\]

So we can write
\[
\left(\sum_{n=0}^{\infty} R_2(n)q^n\right) \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}\right) = \frac{(q^2, q^4; q^4)_\infty}{(q; q^2)_\infty (q^2, q^8; q^{10})_\infty}.
\]

By Watson’s quintuple product identity (3.2), with \(q\) replaced by \(-q^5\) and \(z\) replaced by \(-q\), we get
\[
\sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2} \left(q^{n(15n+1)/2} + q^{(3n-1)(5n-2)/2}\right) = (-q, q^4, -q^5; -q^5)_\infty (-q^3, -q^7; q^{10})_\infty.
\]

Taking into account that
\[
(-q, q^4, -q^5; -q^5)_\infty (-q^3, -q^7; q^{10})_\infty = (q^2, q^4; q^4)_\infty (q^6, q^{14}; q^{20})_\infty (q^3, q^7; q^{10})_\infty
\]
\[
= \frac{(q^2, q^{10}, q^{18}; q^{20})_\infty}{(q^6, q^{10}, q^{18}; q^{20})_\infty} \cdot \frac{(q^2, q^4, q^6, q^8, q^{10}; q^{10})_\infty}{(q^2, q^8; q^{10})_\infty} \cdot \frac{(q^6, q^{14}; q^{20})_\infty}{(q^3, q^7; q^{10})_\infty}
\]
\[
= \frac{(q^2, q^6, q^{10}; q^{14}, q^{18}; q^{20})_\infty}{(q, q^2, q^4; q^4)_\infty} \cdot \frac{(q^2, q^2; q^7)_\infty}{(q^2, q^8; q^{10})_\infty}
\]
\[
= \frac{(q^2, q^4)_\infty}{(q; q^2)_\infty} \cdot \frac{(q^2, q^2; q^7)_\infty}{(q^2, q^8; q^{10})_\infty}
\]
\[
= \frac{(q^2, q^2, q^4; q^4)_\infty}{(q; q^2)_\infty (q^2, q^8; q^{10})_\infty}
\]

\[
\text{(5.4)}
\]
by (5.2) and (5.3), we deduce that
\[
\left( \sum_{n=0}^{\infty} R_2(n) q^n \right) \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2} \right) = \sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2} \left( q^{n(15n+1)/2} + q^{(3n-1)(5n-2)/2} \right).
\]
Applying again the Cauchy product of two power series, we arrive at our result. □

6. Connections with overpartitions and partitions into distinct parts

We start with two \( q \)-identities that involves the Rogers–Ramanujan function \( G(q) \). These follow easily from the previous sections. In order to simplify the writing, for any integer \( n \) we define
\[
a_n = \frac{n(15n+1)}{2} \quad \text{and} \quad b_n = \frac{(3n-1)(5n-2)}{2}.
\]

**Theorem 6.1.** For \( |q| < 1 \),
\[
\frac{G(q^2)}{(q^2; q^2)_{\infty}} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \left( q^{a_n} - q^{b_n} \right).
\]
**Proof.** To derive this identity, we consider the relations (4.1), (4.3), and (4.4). □

**Theorem 6.2.** For \( |q| < 1 \),
\[
\frac{G(q^2)}{(q^2; q^2)_{\infty}} = \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2} \left( q^{a_n} + q^{b_n} \right).
\]
**Proof.** This identity follows considering the relations (5.1), (5.3), and (5.4). □

Recall that an overpartition of the nonnegative integer \( n \) is a partition of \( n \) where the first occurrence of parts of each size may be overlined. Let \( \bar{p}(n) \) denote the number of overpartitions of \( n \). For example, the overpartitions of the integer 3 are:
\[
3, \ \bar{3}, \ 2 + 1, \ \bar{2} + 1, \ 2 + \bar{1}, \ \bar{2} + \bar{1}, \ 1 + 1 + 1 \ \text{and} \ \bar{1} + 1 + 1.
\]
We see that \( \bar{p}(3) = 8 \). The overpartition function \( \bar{p}(n) \) has the generating function
\[
\sum_{n=0}^{\infty} \bar{p}(n) q^n = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}}
\]
and its properties have been the subject of many recent studies [3, 6, 8, 9, 10, 11, 12, 13]. It is clear that, the partition function \( R_2(n) \) can be expressed in terms of the overpartition function \( \bar{p}(n) \) as combinatorial interpretations of Theorems 6.1 and 6.2.
Corollary 6.3. For any nonnegative integer \( n \),
\[
R_2(n) = \sum_{k=-\infty}^{\infty} [\bar{p}(n-a_k) - \bar{p}(n-b_k)].
\]

Proof. By Theorem 6.1, we obtain
\[
\sum_{n=0}^{\infty} R_2(n) q^n = \left( \sum_{n=0}^{\infty} \bar{p}(n) q^n \right) \left( \sum_{n=-\infty}^{\infty} \left( q^{an} - q^{bn} \right) \right).
\]
The proof follows easily applying the Cauchy product on this identity. \( \square \)

Corollary 6.4. For any nonnegative integer \( n \),
\[
R_2(n) = \sum_{k=-\infty}^{\infty} (-1)^{k(k-1)/2} \left[ \bar{p} \left( \frac{n}{2} - \frac{a_k}{2} \right) + \bar{p} \left( \frac{n}{2} - \frac{b_k}{2} \right) \right],
\]
with \( \bar{p}(x) = 0 \) if \( x \) is not an integer.

Proof. By Theorem 6.2, we can write
\[
\sum_{n=0}^{\infty} R_2(n) q^n = \left( \sum_{n=0}^{\infty} \bar{p}(n) q^{2n} \right) \left( \sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2} \left( q^{an} + q^{bn} \right) \right).
\]
The proof follows equating the coefficient of \( q^n \) on each side of this identity. \( \square \)

It is well known that
\[
\bar{p}(n) \equiv 0 \pmod{2} \quad \text{if and only if} \quad n \neq 0.
\]
This parity result can be deduced from the recurrence relation
\[
\sum_{j=-\infty}^{\infty} (-1)^j \bar{p}(n-j^2) = \delta_{0,n}
\]
and can be used together with Corollary 6.3 or 6.4 to derive another proof of Corollary 1.5.

Recall that the number of partitions of \( n \) into distinct parts is usually denoted by \( q(n) \) and the generating function of \( q(n) \) is given by
\[
\sum_{n=0}^{\infty} q(n) q^n = (-q; q)_\infty = \frac{1}{(q; q^2)_\infty}.
\]
We have the following results.

Corollary 6.5. For any nonnegative integer \( n \), the partition functions \( q(n) \) and \( R_2(n) \) are related by:
\[
(1) \sum_{k=0}^{n} (-1)^k R_2(n-k(3k-1)/2) = \sum_{k=-\infty}^{\infty} [q(n-a_k) - q(n-b_k)];
\]
\[
(2) \sum_{k=0}^{n} (-1)^k R_2(n-k(3k-1))
\]

\[ = \sum_{k=-\infty}^{\infty} (-1)^{k(k-1)/2} \left[ q \left( \frac{n}{2} - \frac{a_k}{2} \right) + q \left( \frac{n}{2} - \frac{b_k}{2} \right) \right], \]

where \( q(x) = 0 \) if \( x \) is not an integer.

**Proof.** Considering Theorems 6.1 and 6.2, we obtain

\[ (q; q)_{\infty} \sum_{n=0}^{\infty} R_2(n)q^n = (-q; q)_{\infty} \sum_{n=-\infty}^{\infty} \left( q^{a_n} - q^{b_n} \right) \]

and

\[ (q^2; q^2)_{\infty} \sum_{n=0}^{\infty} R_2(n)q^n = (-q^2; q^2)_{\infty} \sum_{n=-\infty}^{\infty} (-1)^{n(n+1)/2} \left( q^{a_n} + q^{b_n} \right), \]

respectively. The proof follows equating the coefficients of \( q^n \) in these relations. \( \square \)

As a consequence of Theorem 1.1 and Corollary 6.5, we derive the following recurrence relation for the partition function \( q(n) \).

**Corollary 6.6.** For \( n \geq 0 \),

\[ \sum_{j=-\infty}^{\infty} \left[ q \left( n - a_j \right) - q \left( n - b_j \right) \right] = \begin{cases} (-1)^k, & \text{if } n = k(5k+1), k \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \]

The number of partitions of \( n \) into distinct odd parts is denoted in this paper by \( q_{\text{odd}}(n) \) and the generating function of \( q_{\text{odd}}(n) \) is given by

\[ \sum_{n=0}^{\infty} q_{\text{odd}}(n)q^n = (-q^2; q^2)_{\infty} = \frac{1}{(q; q)_{\infty}}. \]

**Corollary 6.7.** For any nonnegative integer \( n \), the partition functions \( p(n) \), \( q_{\text{odd}}(n) \) and \( R_2(n) \) are related by:

1. \[ \sum_{k=0}^{n} (-1)^k q_{\text{odd}}(k)R_2(n - k) = \sum_{k=-\infty}^{\infty} \left[ p \left( n - a_k \right) - p \left( n - b_k \right) \right]; \]
2. \[ \sum_{k=0}^{n} (-1)^k q_{\text{odd}}(k)R_2(n - 2k) = \sum_{k=-\infty}^{\infty} (-1)^{k(k-1)/2} \left[ p \left( \frac{n}{2} - \frac{a_k}{2} \right) + p \left( \frac{n}{2} - \frac{b_k}{2} \right) \right], \]

where \( p(x) = 0 \) if \( x \) is not an integer;
3. \[ \sum_{k=0}^{n} q_{\text{odd}}(k)R_2(n - k) = \sum_{k=-\infty}^{\infty} (-1)^{k(k-1)/2} \left[ p \left( n - a_k \right) + p \left( n - b_k \right) \right]. \]

**Proof.** Rewriting Theorem 6.1 as

\[ \frac{1}{(q; q)_{\infty}} \sum_{n=0}^{\infty} R_2(n)q^n = \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \left( q^{a_n} - q^{b_n} \right), \]
we derive the first identity.

By Theorem 6.2, we give

\[
\frac{1}{(-q^2; q^2)_\infty} \sum_{n=0}^{\infty} R_2(n)q^n = \frac{1}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2} \left(q^{a_n} + q^{b_n}\right)
\]

and

\[
(-q; q^2) \sum_{n=0}^{\infty} R_2(n)q^n = \frac{1}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^{n(n-1)/2} \left(q^{a_n} + q^{b_n}\right).
\]

The proof of the next two identities follows easily. \(\square\)

On the other hand, we denote by \(q_{e-o}(n)\) the number of partitions of \(n\) into distinct parts with an even number of odd parts minus partitions of \(n\) into distinct parts with an odd number of odd parts. Because

\[
\sum_{n=0}^{\infty} q_{e-o}(n)q^n = \frac{1}{(-q; q^2)_\infty} = \sum_{n=0}^{\infty} (-1)^n q(n)q^n,
\]

Corollaries 6.5 and 6.6 can be rewritten in terms of the partition function \(q_{e-o}(n)\).

Similarly, we denote by \(p_{e-o}(n)\) the number of partitions of \(n\) into an even number of parts minus partitions of \(n\) into an odd number of parts. Taking into account that

\[
\sum_{n=0}^{\infty} p_{e-o}(n)q^n = \frac{1}{(-q; q)_\infty} = \sum_{n=0}^{\infty} (-1)^n q_{odd}(n)q^n,
\]

Corollary 6.7 can be restated in terms of the partition function \(q_{odd}(n)\).

7. Concluding remarks

Four linear recurrence relations for the number of ordinary partitions of \(n\) into parts that are either odd or congruent to \(\pm 2 \pmod{10}\) have been introduced in the paper. Two of these results allow us to derive a simple criterion for deciding whether the number of ordinary partitions of \(n\) into parts that are either odd or congruent to \(\pm 2 \pmod{10}\) is odd or even. We remark that Theorem 1.4 provides an extremely efficient algorithm for computing \(R_2(n)\). In [14], we carried out a similar set of recurrences for partitions where even parts do not repeat.

On the other hand, Theorems 6.1 and 6.2 allow us to derive a collection of identities involving partition into distinct parts. We should note that Theorem 6.2 can be derived from Theorem 6.1 replacing \(q\) by \(-q\). In fact Theorem 6.1 is a version of the following identity for the Rogers–Ramanujan function \(G(q)\).

**Theorem 7.1.** For \(|q| < 1\),

\[
G(q^2) = \frac{1}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \left(q^{n(15n+1)/2} - q^{(3n-1)(5n-2)/2}\right).
\]
A linear homogeneous recurrence relation for the partition function \( p(n) \) can be easily derived from this identity.

**Corollary 7.2.** For \( n \) odd,
\[
\sum_{k=-\infty}^{\infty} \left[ p \left( n - \frac{k(15k+1)}{2} \right) - p \left( n - \frac{(3k-1)(5k-2)}{2} \right) \right] = 0.
\]

Moreover, the case \( n \) even can be written as follows.

**Corollary 7.3.** The number of partitions of \( n \) into parts of the form \( 5k+1 \) or \( 5k+4 \) can be expressed in terms of the partition function \( p(n) \), i.e.,
\[
\sum_{k=-\infty}^{\infty} \left[ p \left( 2n - \frac{k(15k+1)}{2} \right) - p \left( 2n - \frac{(3k-1)(5k-2)}{2} \right) \right].
\]

**Corollary 7.4.** The number of partitions of \( n \) into parts of the form \( 5k+1 \) or \( 5k+4 \) can be expressed as a finite discrete convolution, i.e.,
\[
\sum_{k=0}^{2n} (-1)^{q_{odd}(k)} R_2(2n-k).
\]

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**References**


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