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SPLIT $(n+t)$–COLOR PARTITIONS AND 2–COLOR F–PARTITIONS

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Abstract. Andrews [Generalized Frobenius partitions. *Memoirs of the American Math. Soc.*, 301:1–44, 1984] defined the two classes of generalized F–partitions: F–partitions and k–color F–partitions. For many q–series and Rogers–Ramanujan type identities, the bijections are established between F–partitions and $(n+t)$–color partitions. Recently $(n+t)$–color partitions have been extended to split $(n+t)$–color partitions by Agarwal and Sood [Split $(n+t)$–color partitions and Gordon–McIntosh eight order mock theta functions. *Electron. J. Comb.*, 21(2):#P2.46, 2014]. The purpose of this paper is to study the k–color F–partitions as a combinatorial tool. The paper includes combinatorial proofs and bijections between split $(n+t)$–color partitions and 2–color F–partitions for some generalized q–series. Our results further give rise to infinite three way combinatorial identities in conjunction with some Rogers–Ramanujan type identities for some particular cases.

1. Introduction

The theory of partitions is a significant branch of additive number theory and combinatorics. It seems that Liebniz was the first to raise a question to Bernoulli about finding the count in which a given positive integer can be written as sum of more than one positive integers. It leads to defining a partition of a positive integer ν as 'a nonincreasing sequence of positive integers, whose sum is ν'. Euler in 1748, established the generating function for the partitions and some properties of the partition function. Using the generating function for the ordinary partitions, given in [10], Ramanujan [16] proved the following congruence relations for the partition function.

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

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In 1900, Frobenius [11] introduced the following representation for the ordinary partitions as a two rowed array,

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

of nonnegative integers such that parts in each row of the array are distinct and arranged in decreasing order, is known as a Frobenius representation or symbol of an ordinary partition of ν if $\nu = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i$.

Example. If $\nu = 20 = 4 + (5+3+1+0) + (4+2+1+0)$, then the corresponding Frobenius notation $\left(\begin{smallmatrix} 5 & 3 & 1 & 0 \\ 4 & 2 & 1 & 0 \end{smallmatrix}\right)$ is associated with the ordinary partition $6+5+4+4+1$ of ν.

Each ordinary partition of ν has a unique Frobenius symbol associated with it. A one to one correspondence between ordinary partitions and their Frobenius symbols is established in [11] with the help of Ferrers graphs. This representation was used by Frobenius [11] himself while studying the group representation theory.

More than eighty years later, Andrews [7] in 1984 generalized the concept of the Frobenius symbol in two ways which he called Frobenius partitions or simply F–partitions. Andrew's two classes of F–partitions are given below:

The first class contains F–partitions in which parts appear at most k times in any row. Let $\phi_k(\nu)$ denote the number of all such F–partitions of ν. Further, let $\Phi_k(q) = \sum_{\nu=0}^{\infty} \phi_k(\nu) q^\nu$ denote the generating function of $\phi_k(\nu)$. Then

$$\Phi_2(q) = 1 + q + 3q^2 + 5q^3 + 9q^4 + \cdots,$$
$$\Phi_3(q) = 1 + q + 3q^2 + 6q^3 + 11q^4 + \cdots.$$

Example. The F–partitions enumerated by $\phi_2(2)$ are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The second class contains colored F–partitions with k copies of the nonnegative integers

$$(j_i : 0 \le j \le n-1),\ 1 \le i \le k$$

where $j_i \neq j'_{i'}$, unless $j = j'$ and $i = i'$. There is also a strict decrease among the parts along the rows and the parts follow the order

$$0_1 < 0_2 < \cdots < 1_1 < 1_2 < 1_3 < \cdots < 2_1 <$$
$$2_2 < 2_3 < \cdots < 3_1 < 3_2 < 3_3 < \cdots.$$

Consider colored F–partitions of ν in which the parts in either row appear from k copies and are distinct. Let $c\phi_k(\nu)$ denote the number of all such partitions. Let $c\Phi_k(q) = \sum_{\nu=0}^{\infty} c\phi_k(\nu) q^\nu$ be the generating function of $c\phi_k(\nu)$. Then

$$c\Phi_2(q) = 1 + 4q + 9q^2 + 20q^3 + 42q^4 + \cdots.$$

In general, for a fixed k, the generating function for $c\phi_k(\nu)$ is given by the constant term (the z^0 term) in the following expression given in Andrews' AMS Memoir [7, Eq. (5.14)]:

$$\prod_{n=0}^{\infty}(1+zq^{n+1})^k(1+z^{-1}q^n)^k.$$

Example. F–partitions enumerated by $c\phi_2(2)$ are

$$\begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_1 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix},$$
$$\begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 0_2 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}.$$

Similar to the congruences established by Ramanujan [16], Andrews in his book [7] established the following congruence properties for the two classes of F–partitions.

$$\phi_2(5n+3) \equiv 0 \pmod{5},$$
$$c\phi_k(5n+3) \equiv 0 \pmod{k^2}.$$

The first class of F–partitions were used by many authors for interpreting q–series or identities combinatorially and the direct bijections between F–partitions and $(n+t)$–colored partitions are established, see [2, 3, 17, 18, 21, 23]. The second class of F–partitions has never been used as a combinatorial tool but some congruence relations have been established for the same, for instance see [6, 8, 12, 14]. The relationship between certain generalized colored F–partitions and ordinary partitions are shown by Kolitsch [15].

In 1987, Agarwal and Andrews [2] defined the $(n+t)$–color partitions, $(t \geq 0)$ as given below:

Definition 1.1 ([2])**.** *A partition with "$(n+t)$–copies of n", $t \geq 0$ is a partition in which a part of size $n, (n \geq 0)$, can come in $(n+t)$–different colors denoted by the subscripts, $n_1, n_2, n_3, \cdots, n_{n+t}$. The parts follow the order*

$$1_1 < 1_2 < 1_3 < \cdots < 2_1 < 2_2 < 2_3 < \cdots < 3_1 < 3_2 < 3_3 < \cdots.$$

Note that zeros are permitted if and only if $t \geq 1$. Also zeros are not permitted to repeat in any partition.

Definition 1.2. *The weighted difference of two parts m_i, n_j, $m \geq n$ is defined by $m-n-i-j$ and denoted by $((m_i - n_j))$.*

Agarwal and Sood [5] have extended the $(n+t)$–color partitions to 'split $(n+t)$–color partitions' as given below:

Definition 1.3. *Let m_i be a part in an $(n+t)$–color partition of a positive integer ν. Split the color 'i' into two parts - 'the green part' and 'the red part' and denote them by 'g' and 'r', respectively, such that $1 \le g \le i$, $0 \le r \le i-1$, and $i = g + r$. An $(n+t)$–color partition in which each part splits in this manner is called a split $(n+t)$–color partition.*

Example. The split n–color partitions of 2 are

$$2_1,\ 2_2,\ 2_{1+1},\ 1_1 + 1_1. \tag{1.1}$$

Note here that if red part is zero, it will not be written. Thus in the above example 2_{1+0} is written as 2_1.

The purpose of this paper is to introduce the Andrews' second class of F–partitions as an important combinatorial tool by providing combinatorial interpretations of many generalized q–series. In this paper, we also explicitly establish the bijections between 2–color F–partition functions enumerated by $cF_2^{(k,j)}(\nu)$ and split $(n+t)$–color partitions given by partition functions $A_j^k(\nu)$, $1 \le k \le 3$, in [5] and $P_j^k(\nu)$, $1 \le k \le 4$, in [20].

The paper is organized as follows: In Section 2, we enumerate two Gordon–McIntosh mock theta functions [13] by 2–color F–partitions combinatorially by establishing the bijections between 2–color F–partitions and split $(n+t)$–color partitions. In Section 3, we provide similar bijections for seven generalized q–series. In Section 4, direct proofs using 2–color F–partitions are given for the generalized q–series. We conclude in Section 5.

Remark. In the main results, given in the following sections, we use only those restricted k–color F–partitions in which the top and bottom row entries of each column appear with same subscripts and we enumerate such k–color F–partitions by $cF_k(\nu)$. In this context the relevant 2–color F–partitions are enumerated by $cF_2(\nu)$.

Example. The relevant 2–color F–partitions corresponding to $cF_2(2)$ are

$$\begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix}, \begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix}, \begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}, \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}, \begin{pmatrix} 0_2 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}.$$

2. Gordon–McIntosh mock theta functions and 2–color F–partitions

The authors of the present paper in [19], interpreted the following two Gordon–McIntosh mock theta functions $V_0(q)$ and $V_1(q)$, given in [13], using signed partitions:

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n},$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}.$$

These mock theta functions were also interpreted combinatorially by Agarwal and Sood [4] using split $(n+t)$–color partitions. Using the colored

F–partitions discussed in previous section, we interpret $V_0(q)$ and $V_1(q)$ by establishing bijections between the 2–color F–partitions and split $(n+t)$–color partitions. For more clarity, we here reproduce the results of Agarwal and Sood [4], given in Theorems 2.1 and 2.2.

Theorem 2.1. *For $\nu \geq 1$, let $A_1(\nu)$ denote the number of 'split n–color partitions' of ν such that*

(4.a) *the parts and their subscripts have the same parity,*
(4.b) *the red part of the subscripts cannot exceed 1,*
(4.c) *the least part is either k_k $(k \geq 1)$ or $k_{(k-1)+1}$ $(k \geq 2)$,*
(4.d) *the weighted difference of any two consecutive parts is 0.*

Then

$$V_0(q) = 1 + 2\sum_{\nu=1}^{\infty} A_1(\nu)q^{\nu}. \tag{2.1}$$

Remark. In conditions (4.a) and (4.d), the whole subscript i is considered, not its parts g and r, separately.

Theorem 2.2. *For $\nu \geq 1$, let $A_2(\nu)$ denote the number of 'split n–color partitions' of ν such that*

(5.a) *the parts and their subscripts have the same parity,*
(5.b) *the red part of the subscripts cannot exceed 1,*
(5.c) *the least part is k_k $(k \geq 1)$,*
(5.d) *the weighted difference of any two consecutive parts is 0.*

Then

$$V_1(q) = \sum_{\nu=1}^{\infty} A_2(\nu)q^{\nu}. \tag{2.2}$$

Remark. As in Theorem 2.1, here also, in conditions (5.a) and (5.d) the whole subscript i is considered, not its parts g and r, separately.

The following theorems give combinatorial interpretations of $V_0(q)$ and $V_1(q)$ using 2–color F–partitions.

Theorem 2.3. *Let $cF_2^1(\nu)$ denote the number of 2–color F–partitions of ν in which every column has the parts with same subscripts such that*

(6.e) *for each column $\binom{p_h}{q_h}$, $p \leq q - h + 1$,*
(6.f) *$h = 1$ or 2,*
(6.g) *for the last column, $p_h = 0_h$,*
(6.h) *for any two adjacent columns $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ and $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$, we have $p = q' + 1$, ignoring the subscripts.*

Then

$$V_0(q) = \sum_{\nu=1}^{\infty} A_1(\nu)q^{\nu} = \sum_{\nu=1}^{\infty} cF_2^1(\nu)q^{\nu}. \tag{2.3}$$

Theorem 2.4. *Let $cF_2^2(\nu)$ denote the number of 2–color F–partitions of ν in which every column has the parts with same subscripts such that*

(7.e) *for each column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 1$,
(7.f) $h = 1$ *or 2,*
(7.g) *for the last column,* $p_h = 0_1$,
(7.h) *for any two adjacent columns* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ *and* $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$, *we have* $p = q' + 1$, *ignoring the subscripts.*

Then

$$V_1(q) = \sum_{\nu=1}^{\infty} A_2(\nu) q^{\nu} = \sum_{\nu=1}^{\infty} cF_2^2(\nu) q^{\nu}. \tag{2.4}$$

Proof of Theorem 2.3. To prove the theorem, we establish a one to one correspondence between the 2–color F–partitions enumerated by $cF_2^1(\nu)$ and the split n–color partitions enumerated by $A_1(\nu)$. We do this by mapping each column $\binom{p_h}{q_h}$ of the 2–color F–partition to a single part m_{g+r} of a split n–color partition enumerated by $A_1(\nu)$. The mapping ϕ is

$$\phi : \begin{pmatrix} p_h \\ q_h \end{pmatrix} \to (p + q + 1)_{(q-p-h+2)+(h-1)}, \text{ if } p \leq q - h + 1 \tag{2.5}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_{g+r} \to \begin{pmatrix} \left(\frac{m-(g+r)}{2}\right)_{r+1} \\ \left(\frac{m+(g+r)-2}{2}\right)_{r+1} \end{pmatrix}, \text{ if } m \equiv g + r \pmod{2}. \tag{2.6}$$

Now suppose we have any two adjacent columns $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ and $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$ in a 2–color F–partition enumerated by $cF_2^1(\nu)$ with

$$\phi : \begin{pmatrix} p_h \\ q_h \end{pmatrix} = m_{g+r} \text{ and } \phi : \begin{pmatrix} p'_k \\ q'_k \end{pmatrix} = n_{g'+r'}.$$

Then since

$$\begin{pmatrix} p_h \\ q_h \end{pmatrix} \to (p + q + 1)_{(q-p-h+2)+(h-1)} = m_{g+r}$$

and

$$\begin{pmatrix} p'_k \\ q'_k \end{pmatrix} \to (p' + q' + 1)_{(q'-p'-k+2)+(k-1)} = n_{g'+r'},$$

we have

$$
\begin{aligned}
((m_{g+r}-n_{g+r})) &= m-n-(g+r)-(g'+r')\\
&= (p+q+1)-(p'+q'+1)-(q-p-h+2+(h-1))\\
&\quad -(q'-p'-k+2+(k-1))\\
&= 2(p-q'-1). \qquad (2.7)
\end{aligned}
$$

Clearly (2.7) and (6.h) imply (4.d).

Now using (2.5)

$$m-(g+r)=(p+q+1)-(q-p-h+2+(h-1))=2p \qquad (2.8)$$

which implies $(m-(g+r))\equiv 0 \pmod 2$, hence (4.$a$) holds. Together, (6.f) and (2.5) imply (4.b). Further, (6.f), (6.g), and (2.5) imply (4.c).

To see the reverse implication, we consider the inverse images of two consecutive parts m_{g+r}, $n_{g'+r'}$ of a split n–color partition enumerated by $A_1(\nu)$,

$$\phi^{-1}: m_{g+r} = \begin{pmatrix} (\frac{m-(g+r)}{2})_{r+1} \\ (\frac{m+(g+r)-2}{2})_{r+1} \end{pmatrix} \qquad (2.9)$$

and

$$\phi^{-1}: n_{g'+r'} = \begin{pmatrix} (\frac{n-(g'+r')}{2})_{r+1} \\ (\frac{n+(g'+r')-2}{2})_{r+1} \end{pmatrix}. \qquad (2.10)$$

That is,

$$p=\frac{m-(g+r)}{2}, \qquad (2.11)$$

$$q=\frac{m+(g+r)-2}{2}, \qquad (2.12)$$

$$p'=\frac{n-(g'+r')}{2}, \qquad (2.13)$$

$$q'=\frac{n+(g'+r')-2}{2} \qquad (2.14)$$

and so

$$m-(g+r)=2p, \qquad (2.15)$$

$$n+(g'+r')=2q'+2, \qquad (2.16)$$

$$2p-2q'-2=((m_i-n_j)). \qquad (2.17)$$

Together, (2.17) and (4.d) imply (6.h). Further, (2.15) and (4.c) implies (6.g); also (6.f) is obvious from (2.6) and (4.b). Now from (2.6), (2.11), and (2.12), we have

$$
\begin{aligned}
q-p-h+2 &= \frac{m+(g+r)-2}{2}-\frac{m-(g+r)}{2}-(r+1)+2\\
&= g. \qquad (2.18)
\end{aligned}
$$

Since the green part of the subscript, $g \geq 1$, therefore (2.18) implies (6.e). This completes the proof of Theorem 2.3. □

Example 2.5. *To illustrate the bijection we have constructed, an example for $\nu = 7$ is shown in the table below:*

Split n–color partitions relevant to $A_1(7)$	*2–color F–partitions relevant to* $cF_2^1(7)$
7_7	$\begin{pmatrix} 0_1 \\ 6_1 \end{pmatrix}$
7_{6+1}	$\begin{pmatrix} 0_2 \\ 6_2 \end{pmatrix}$
$6_4 1_1$	$\begin{pmatrix} 1_1 & 0_1 \\ 4_1 & 0_1 \end{pmatrix}$
$6_{3+1} 1_1$	$\begin{pmatrix} 1_2 & 0_1 \\ 4_2 & 0_1 \end{pmatrix}$
$5_1 2_2$	$\begin{pmatrix} 2_1 & 0_1 \\ 2_1 & 1_1 \end{pmatrix}$
$5_1 2_{1+1}$	$\begin{pmatrix} 2_1 & 0_2 \\ 2_1 & 1_2 \end{pmatrix}$

Hence

$$A_1(\nu) = cF_2^1(\nu) = 6. \tag{2.19}$$

Sketch of proof of Theorem 2.4. In this theorem, the only difference is that in (5.c) of Theorem 2.2, the least part is of the type k_k, therefore in the corresponding last column of 2–color F–partition, $a_s = 0_1$ and vice versa. □

Example 2.6. *To illustrate the bijection we have constructed, the example for $\nu = 7$ is shown in the table below:*

Split n–color partitions relevant to $A_2(7)$	*2–color F–partitions relevant to* $cF_2^2(7)$
7_7	$\begin{pmatrix} 0_1 \\ 6_1 \end{pmatrix}$
$6_4 1_1$	$\begin{pmatrix} 1_1 & 0_1 \\ 4_1 & 0_1 \end{pmatrix}$
$6_{3+1} 1_1$	$\begin{pmatrix} 1_2 & 0_1 \\ 4_2 & 0_1 \end{pmatrix}$
$5_1 2_2$	$\begin{pmatrix} 2_1 & 0_1 \\ 2_1 & 1_1 \end{pmatrix}$

Hence

$$A_2(\nu) = cF_2^2(\nu) = 4. \tag{2.20}$$

3. Bijections between split *(n+t)*–color partitions and 2–color F–partitions

Rana et. al. [20], interpreted the following four generalized q–series using split $(n+t)$–color partitions. Let $S = \{-1, 1, 3, 5, 7, \ldots\}$. For $|q| < 1, j \in S$ and $1 \leq k \leq 4$, define $f^{(k,j)}(q)$ by

$$f^{(1,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n-1)(j+3)/2]}}{(q;q^2)_n (q^4;q^4)_n}, \tag{3.1}$$

$$f^{(2,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[(n+1)(j+3)/2]}}{(q;q^2)_{n+1} (q^4;q^4)_n}, \tag{3.2}$$

$$f^{(3,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n+1)(j+3)/2]}}{(q;q^2)_{n+1} (q^4;q^4)_n}, \tag{3.3}$$

$$f^{(4,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n+1)(j+3)/2]}}{(q;q^2)_n (q^4;q^4)_n}. \tag{3.4}$$

We now interpret (3.1)–(3.4) using 2–color F–partitions by establishing bijections between split $(n+t)$–color partitions and 2–color F–partitions in the following theorems, respectively.

Theorem 3.1. *For $j \in S$, let $A^{(1,j)}(\nu)$ represent the number of split n–color partitions of ν such that*

(8.a) *the parts and their subscripts have the same parity,*
(8.b) *the value of the red part can be 0 or 1,*
(8.c) *if m_i is the least or only summand of partition, then $m - i \equiv 0 \pmod 4$,*
(8.d) *the weighted difference among any two consecutive summands is greater than j and is congruent to $(j+1) \pmod 4$.*

Further, let $cF_2^{(1,j)}(\nu)$ denote the number of 2–color F–partitions of ν in which the top and bottom row entries of each column appear with the same subscripts such that

(8.e) *for each column $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 1$, ignoring the subscripts,*
(8.f) *$h = 1$ or 2,*
(8.g) *for the last column, $p \equiv 0 \pmod 2$, ignoring the subscript,*
(8.h) *for any two adjacent columns $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ and $\begin{pmatrix} {p'}_k \\ {q'}_k \end{pmatrix}$, we have $p \geq q' + (j+3)/2$ and*

$$\begin{cases} p \equiv q' \pmod 2, & j \equiv 1 \pmod 4 \\ p \not\equiv q' \pmod 2, & j \equiv 3 \pmod 4 \end{cases}$$

ignoring the subscripts.

Then

$$\sum_{\nu=0}^{\infty} A^{(1,j)}(\nu)q^{\nu} = \sum_{\nu=o}^{\infty} cF_2^{(1,j)}(\nu)q^{\nu}.$$

Theorem 3.2. *For $j \in S$, let $A^{(2,j)}(\nu)$ represent the number of split $(n+1)$–color partitions of ν such that*

(9.a) *the parts and their subscripts have the opposite parity,*
(9.b) *the value of the red part can be 0 or 1,*
(9.c) *the smallest summand is of the form i_{i+1},*
(9.d) *the weighted difference among any two consecutive summands is greater than j and is congruent to $(j+1) \pmod 4$.*

Further, let $cF_2^{(2,j)}(\nu)$ denote the number of 2–color F–partitions of ν in which top and bottom row entries of each column appear with same subscripts such that

(9.e) *for each column $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 2$,*
(9.f) *$h = 1$ or 2,*
(9.g) *for the last column, either $p_h = 0_1$ or $p \geq (j+3)/2$, ignoring the subscript,*
(9.h) *for any two adjacent columns, $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ and $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$, we have $p \geq q' + (j+5)/2$ and*

$$\begin{cases} p \equiv q' \pmod 2, & j \equiv 3 \pmod 4 \\ p \not\equiv q' \pmod 2, & j \equiv 1 \pmod 4 \end{cases}$$

ignoring the subscripts.

Then

$$\sum_{\nu=0}^{\infty} A^{(2,j)}(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} cF_2^{(2,j)}(\nu)q^{\nu}.$$

Theorem 3.3. *For $j \in S$, let $A^{(3,j)}(\nu)$ represent the number of split $(n+2)$–color partitions of ν such that*

(10.a) *the parts and their subscripts have the same parity,*
(10.b) *the value of the red part can be 0 or 1,*
(10.c) *the smallest summand is of the form i_{i+2},*
(10.d) *the weighted difference among any two consecutive summands is greater than j and is congruent to $(j+1) \pmod 4$.*

Further, let $cF_2^{(3,j)}(\nu)$ denote the number of 2–color F–partitions of ν in which top and bottom row entries of each column appear with same subscripts such that

(10.e) *for each column $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 3$,*
(10.f) *$h = 1$ or 2,*

(10.g) *for the last column, either* $p_h = 0_1$ *or* $p \geq (j+5)/2$*, ignoring the subscript,*

(10.h) *for any two adjacent columns* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ *and* $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$*, we have* $p \geq q' + (j+7)/2$ *and*

$$\begin{cases} p \equiv q' \pmod 2, & j \equiv 1 \pmod 4 \\ p \not\equiv q' \pmod 2, & j \equiv 3 \pmod 4 \end{cases}$$

ignoring the subscripts.

Then

$$\sum_{\nu=0}^{\infty} A^{(3,j)}(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} cF_2^{(3,j)}(\nu)q^{\nu}.$$

Theorem 3.4. *For* $j \in S$*, let* $A^{(4,j)}(\nu)$ *represent the number of split n–color partitions of* ν *such that*

(11.a) *the parts and their subscripts have the same parity,*
(11.b) *the value of the red part can be 0 or 1,*
(11.c) *if* m_i *is the least or only summand of partition, then* $m \geq (j+4)$ *and* $m - i \equiv (j+3) \pmod 4$*,*
(11.d) *the weighted difference among any two consecutive summands is greater than* j *and is congruent to* $(j+1) \pmod 4$*.*

Further, let $cF_2^{(4,j)}(\nu)$ *denote the number of 2–color F–partitions of* ν *in which the top and bottom row entries of each column appear with same subscripts such that*

(11.e) *for each column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 1$*,*
(11.f) $h = 1$ *or 2,*
(11.g) *for last column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p + q - 3 \geq j$ *and* $p \equiv (j+3)/2 \pmod 2$*, ignoring the subscripts,*
(11.h) *for any two adjacent columns* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ *and* $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$*, we have* $p \geq q' + (j+3)/2$ *and*

$$\begin{cases} p \equiv q' \pmod 2, & j \equiv 1 \pmod 4 \\ p \not\equiv q' \pmod 2, & j \equiv 3 \pmod 4 \end{cases}$$

ignoring the subscripts.

Then

$$\sum_{\nu=0}^{\infty} A^{(4,j)}(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} cF_2^{(4,j)}(\nu).$$

Remark. The partition functions $A^{(k,j)}(\nu)$ of Theorems 3.1–3.4, are the same as partition functions $P_i^j(\nu)$, $1 \leq i \leq 4$ defined in Theorems 2.1–2.4 given in [20] and the partition functions $cF_2^{(k,j)}(\nu)$, $1 \leq k \leq 4$ are new. In

Theorems 3.1–3.4, conditions $(l.a)$, $(l.c)$, and $(l.d)$, $8 \leq l \leq 11$, the whole subscript i is considered irrespective of its parts g and r separately.

Agarwal and Sood [5] explored the split $(n+t)$–color partitions to interpret following three basic q–series combinatorially. For $|q| < 1, j \in S$ and $5 \leq k \leq 7$, define $f^{(k,j)}(q)$ by

$$f^{(5,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n-1)(j+3)/2]}}{(q;q)_{2n}}, \tag{3.5}$$

$$f^{(6,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+1)(j+3)/2]}}{(q;q)_{2n+1}}, \tag{3.6}$$

$$f^{(7,j)}(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n+1)(j+3)/2]}}{(q;q)_{2n+1}}. \tag{3.7}$$

We here extend the results of Agarwal and Sood [5] and use the 2–color F–partitions to interpret (3.5)–(3.7) in the following theorems, respectively.

Theorem 3.5. *For $j \in S$ and $\nu \geq 0$, let $A^{(5,j)}(\nu)$ denote the number of 'split n–color partitions' of ν such that*

(12.a) *the parts and their subscripts have the same parity,*
(12.b) *the red part of the subscripts cannot exceed 1,*
(12.c) *the weighted difference of any two consecutive parts is greater than j and even.*

Further, let $cF_2^{(5,j)}(\nu)$ denote the number of 2–color F–partitions of ν in which the top and bottom row entries of each column appear with same subscripts such that

(12.d) *for each column $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \leq q - h + 1$,*
(12.e) *$h = 1$ or 2,*
(12.f) *for any two adjacent columns $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ and $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$, we have $p \geq q' + (j+3)/2$, ignoring the subscripts.*

Then

$$\sum_{\nu=0}^{\infty} A^{(5,j)}(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} cF_2^{(5,j)}(\nu) q^{\nu}.$$

Remark. In Theorem 3.5, conditions $(12.a)$ and $(12.c)$, the whole subscript i is considered, not its parts g and r separately. Similarly, in Theorems 3.6 and 3.7 given below, conditions $(l.a)$, $(l.c)$, and $(l.d)$, $l = 13, 14$, the whole subscript i is considered, not its parts g and r separately.

Theorem 3.6. *For $j \in S$ and $\nu \geq 0$, let $A^{(6,j)}(\nu)$ denote the number of 'split $(n+1)$–color partitions' of ν such that*

(13.a) *the parts and their subscripts have the opposite parity,*
(13.b) *the red part of the subscripts cannot exceed 1,*

(13.c) *the smallest part is of the form* i_{i+1} *for some* i *and the red part of its subscript is 0,*
(13.d) *the weighted difference between any two consecutive parts is greater than* j *and even.*

Further, let $cF_2^{(6,j)}(\nu)$ *denote the number of 2–color* F*–partitions of* ν *in which the top and bottom row entries of each column appear with same subscripts such that*

(13.e) *for each column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \le q - h + 2$,
(13.f) $h = 1$ *or 2,*
(13.g) *for the last column, either* $p_h = 0_1$ *or* $p \ge (j+3)/2$*, ignoring the subscript,*
(13.h) *for any two adjacent columns* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ *and* $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$*, we have* $p \ge q' + (j+5)/2$*, ignoring the subscripts.*

Then

$$\sum_{\nu=0}^{\infty} A^{(6,j)}(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} cF_2^{(6,j)}(\nu) q^{\nu}.$$

Theorem 3.7. *For* $j \in S$ *and* $\nu \ge 0$*, let* $A^{(7,j)}(\nu)$ *denote the number of 'split* $(n+2)$*–color partitions' of* ν *such that*

(14.a) *the parts and their subscripts have the same parity,*
(14.b) *the red part of the subscripts cannot exceed 1,*
(14.c) *the smallest part is of the form* i_{i+2} *for some* i *and the red part of its subscript is 0,*
(14.d) *the weighted difference between any two consecutive parts is greater than* j *and even.*

Further, let $cF_2^{(7,j)}(\nu)$ *denote the number of 2–color* F*–partitions of* ν *in which top and bottom row entries of each column appear with same subscripts such that*

(14.e) *for each column* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$, $p \le q - h + 3$,
(14.f) $h = 1$ *or 2,*
(14.g) *for the last column, either* $p_h = 0_1$ *or* $p \ge (j+5)/2$*, ignoring the subscript,*
(14.h) *for any two adjacent columns* $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ *and* $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$*, we have* $p \ge q' + (j+7)/2$*, ignoring the subscripts.*

Then

$$\sum_{\nu=0}^{\infty} A^{(7,j)}(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} cF_2^{(7,j)}(\nu) q^{\nu}.$$

Note. The partition functions $A^{(k,j)}(\nu),\ 5 \le k \le 7$ of Theorems 3.5–3.7, respectively are due to Agarwal and Sood [5] and the partition functions $cF_2^{(k,j)}(\nu),\ 5 \le k \le 7$ are new.

We now give proofs of Theorems 3.5–3.7.

Proof of Theorem 3.5. We establish a one to one correspondence between the 2–color F–partitions enumerated by $cF_2^{(5,j)}(\nu)$ and the split n–color partitions enumerated by $A^{(5,j)}(\nu)$. We do this by mapping each column $\binom{p_h}{q_h}$ of the 2–color F–partition enumerated by $cF_2^{(5,j)}(\nu)$ to a single part m_{g+r} of a split n–color partition enumerated by $A^{(5,j)}(\nu)$. The mapping ϕ is

$$\phi : \begin{pmatrix} p_h \\ q_h \end{pmatrix} \to (p+q+1)_{(q-p-h+2)+(h-1)},\ \text{if } p \le q-h+1 \tag{3.8}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_{g+r} \to \begin{pmatrix} \left(\frac{m-(g+r)}{2}\right)_{r+1} \\ \left(\frac{m+(g+r)-2}{2}\right)_{r+1} \end{pmatrix},\ \text{if } m \equiv (g+r) \pmod 2, \tag{3.9}$$

Now suppose we have any two adjacent columns $\begin{pmatrix} p_h \\ q_h \end{pmatrix}$ and $\begin{pmatrix} p'_k \\ q'_k \end{pmatrix}$ in a 2–color F–partition enumerated by $cF_2^{(5,j)}(\nu)$ with

$$\phi : \begin{pmatrix} p_h \\ q_h \end{pmatrix} = m_{g+r} \text{ and } \phi : \begin{pmatrix} p'_k \\ q'_k \end{pmatrix} = n_{g'+r'}.$$

Then since

$$\begin{pmatrix} p_h \\ q_h \end{pmatrix} \to (p+q+1)_{(q-p-h+2)+(h-1)} = m_{g+r}$$

and

$$\begin{pmatrix} p'_k \\ q'_k \end{pmatrix} \to (p'+q'+1)_{(q'-p'-k+2)+(k-1)} = n_{g'+r'},$$

we have

$$\begin{aligned} ((m_i - n_j)) &= m - n - i - j \\ &= (p+q+1) - (p'+q'+1) - (q-p-h+2+(h-1)) \\ &\quad - (q'-p'-k+2+(k-1)) \\ &= 2(p-q'-1). \end{aligned} \tag{3.10}$$

Hence (3.10) and (12.f) imply (12.c).

Now using (3.8), we have

$$m-(g+r) = (p+q+1) - (q-p-h+2+(h-1)) = 2p$$

which implies $(m-(g+r)) \equiv 0 \pmod 2$, hence (12.$a$) holds. Further, (12.b) is obvious from (3.8) and (12.e).

To see the reverse implication, we consider the inverse images of two consecutive parts m_{g+r}, $n_{g'+r'}$ of a split n–color partition enumerated by $A^{(5,j)}(\nu)$

$$\phi^{-1} : m_{g+r} = \begin{pmatrix} \left(\frac{m-(g+r)}{2}\right)_{r+1} \\ \left(\frac{m+(g+r)-2}{2}\right)_{r+1} \end{pmatrix}$$

and

$$\phi^{-1} : n_{g'+r'} = \begin{pmatrix} \left(\frac{n-(g'+r')}{2}\right)_{r'+1} \\ \left(\frac{n+(g'+r')-2}{2}\right)_{r'+1} \end{pmatrix},$$

that is,

$$p = \frac{m-(g+r)}{2}, \tag{3.11}$$

$$q = \frac{m+(g+r)-2}{2}, \tag{3.12}$$

$$p' = \frac{n-(g'+r')}{2}, \tag{3.13}$$

$$q' = \frac{n+(g'+r')-2}{2}, \tag{3.14}$$

and so

$$m-(g+r) = 2p, \tag{3.15}$$

$$n+(g'+r') = 2q'+2, \tag{3.16}$$

hence

$$((m_i - n_j)) = 2p - 2q' - 2. \tag{3.17}$$

Together, (3.17) and (12.c) imply (12.f). Also (12.e) is obvious from (3.9) and (12.b). Now from (3.9), (3.11), and (3.12), we have

$$\begin{aligned} q-p-h+2 &= \frac{m+(g+r)-2}{2} - \frac{m-(g+r)}{2} - (r+1) + 2 \\ &= g. \end{aligned} \tag{3.18}$$

Hence (3.18) implies (12.d) using the fact that $g \geq 1$. This completes the proof of Theorem 3.5. □

Sketch of proof of Theorem 3.6. The map ϕ is given by

$$\phi : \begin{pmatrix} p_h \\ q_h \end{pmatrix} \to (p+q+1)_{(q-p-h+3)+(h-1)} \text{ if } p \leq q-h+2, \tag{3.19}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_i \to \begin{pmatrix} \left(\frac{m-(g+r)+1}{2}\right)_{r+1} \\ \left(\frac{m+(g+r)-3}{2}\right)_{r+1} \end{pmatrix}, \text{if } m \equiv (g+r+1) \pmod 2. \tag{3.20}$$

The part 0_1 in the split $(n+1)$–color partition corresponds to the phantom column $\left(\begin{smallmatrix} 0_1 \\ -1_1 \end{smallmatrix}\right)$ in the restricted 2–color F–partition. Since the part 0_1 does not have any contribution towards the total value of ν, the corresponding column in the 2–color F–partitions is eliminated.

Observe that if 0_1 appears as a part in the split $(n+1)$–color partition then after ignoring the corresponding phantom column of the 2–color F–partition, the next column satisfies the condition $p \geq (j+3)/2$. □

Sketch of proof of Theorem 3.7. The map ϕ is given by

$$\phi : \begin{pmatrix} p_h \\ q_h \end{pmatrix} \to (p+q+1)_{(q-p-h+4)+(h-1)} \text{ if } p \leq q-h+3, \tag{3.21}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_i \to \begin{pmatrix} \left(\frac{m-(g+r)+2}{2}\right)_{r+1} \\ \left(\frac{m+(g+r)-4}{2}\right)_{r+1} \end{pmatrix}, \text{if } m \equiv (g+r) \pmod 2. \tag{3.22}$$

As in the case of Theorem 3.6, the phantom column $\left(\begin{smallmatrix} 0_1 \\ -1_1 \end{smallmatrix}\right)$ in the colored F–partition corresponds to the part 0_2 in the split $(n+2)$–color partition as discussed in Theorem 3.6. Also the part 0_2 does not have any contribution towards the total value of ν, therefore the corresponding column in the colored F–partitions is eliminated.

Note that if 0_2 appears as a part in the split $(n+2)$–color partition then the next column satisfies the condition $p \geq (j+5)/2$, ignoring the corresponding phantom column of 2–color F–partitions. □

The proofs of Theorems 3.1–3.3 are similar to the proofs of Theorems 3.5–3.7, respectively. Using the same map one can proceed with similar steps to obtain the results. For the proof of Theorem 3.4, use the map given in Theorem 3.5 and the same steps to obtain the result.

In the next section, we use the method given in [1] and provide the direct proof of Theorems 3.1–3.7 by classifying the 2–color F–partitions into subclasses and use recurrence relations or q–functional equations to generate the desired q–series. Section 5 discusses the Roger–Ramanujan type identities as a particular case.

4. Direct proofs of generalized q–series using 2–color F–partitions

We now provide a direct proof of Theorem 3.5 in terms of 2–color F–partitions.

Let $cF_2^{(5,j)}(\nu)$ enumerate the 2–color F–partitions as described in Theorem 3.5. Our goal is to prove that

$$\sum_{\nu=0}^{\infty} cF_2^{(5,j)}(\nu) q^{\nu} = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n[1+(n-1)(j+3)/2]}}{(q;q)_{2n}}. \tag{4.1}$$

Proof of Theorem 3.5. Let $cF_2^{(5,j)}(m,\nu)$ represent the number of 2–color F–partitions of ν enumerated by $cF_2^{(5,j)}(\nu)$ into m columns. Split the partitions enumerated by $cF_2^{(5,j)}(m,\nu)$ into following four classes:

(i) contains $\binom{p_h}{q_h}$ as the mth column, $p \neq 0$,
(ii) contains $\binom{0_1}{0_1}$ as the mth column,
(iii) contains $\binom{0_2}{1_2}$ as the mth column,
(iv) contains $\binom{0_1}{q_1}$, $q \geq 1$ or $\binom{0_2}{q_2}$, $q \geq 2$ as the mth column.

Transform the partitions of class (i) by subtracting 1 from the top and bottom row entries of each column, we get partitions of $\nu-2m$ into m parts without disturbing the restrictions on the columns as described in Theorem 3.5. Thus the transformed partitions are enumerated by $cF_2^{(5,j)}(m,\nu-2m)$.

Transform the partitions of class (ii) by deleting the column $\binom{0_1}{0_1}$ and then subtracting $(j+3)/2$ from the top and bottom row entries of each column. The transformed partitions are enumerated by $cF_2^{(5,j)}(m-1,\nu-m(j+3)+j+2)$ as this transformation does not effect the conditions of Theorem 3.5.

Next, transform the partitions of class (iii) by deleting the column $\binom{0_2}{1_2}$ and then subtracting $(j+5)/2$ from the top and bottom row entries of each column. The transformed partitions are enumerated by $cF_2^{(5,j)}(m-1,\nu-m(j+5)+j+3)$ as this transformation does not effect the conditions of Theorem 3.5.

Finally, transform the partitions of class (iv) by subtracting 1 from the top row entries of each column except the last column and 1 from the bottom row entries of each column. We get the partitions enumerated by $cF_2^{(5,j)}(m,\nu-2m+1)$ having the mth column as $\binom{0_h}{(q-1)_h}$. Thus the actual number of partitions in class (iv) are obtained by subtracting the number of those partitions which are enumerated by $cF_2^{(5,j)}(m,\nu-2m+1)$ with the last column as $\binom{p_h}{q_h}, p \neq 0$ from $cF_2^{(5,j)}(m,\nu-2m+1)$. Thus the transformed partitions are enumerated by $cF_2^{(5,j)}(m,\nu-2m+1)-cF_2^{(5,j)}(m,\nu-4m+1)$. Hence we get the following recurrence formula for $cF_2^{(5,j)}(m,\nu)$:

$$\begin{aligned} cF_2^{(5,j)}(m,\nu) = {} & cF_2^{(5,j)}(m,\nu-2m) + cF_2^{(5,j)}(m-1,\nu-m(j+3)+j+2) \\ & + cF_2^{(5,j)}(m-1,\nu-m(j+5)+j+3) \\ & + cF_2^{(5,j)}(m,\nu-2m+1) - cF_2^{(5,j)}(m,\nu-4m+1), \end{aligned} \tag{4.2}$$

where $cF_2^{(5,j)}(0,0)=1$ and $cF_2^{(5,j)}(m,\nu)=0$ for $\nu<0$.

For $|q|<1$ and $|z|<|q|^{-1}$, let $g^j(z,q)$ be defined by

$$g^j(z,q) = \sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} cF_2^{(5,j)}(m,\nu) z^m q^\nu, \text{ for all } j \in S. \tag{4.3}$$

Substituting $cF_2^{(5,j)}(m,\nu)$ from (4.2) into (4.3), we get the q–functional equation

$$
\begin{aligned}
g^j(z,q) &= g^j(zq^2,q) + zqg^j(zq^{j+3},q) + zq^2g^j(zq^{j+5},q)\\
&\quad + q^{-1}g^j(zq^2,q) - q^{-1}g^j(zq^4,q). \qquad (4.4)
\end{aligned}
$$

Setting

$$g^j(z,q) = \sum_{n=0}^{\infty} \alpha(n,q)z^n \qquad (4.5)$$

and using (4.4) in (4.5) then examining the coefficients of z^n, we get

$$\alpha(n,q) = \frac{q^{1+(j+3)(n-1)}(1+q^{2n-1})}{(1-q^{2n})(1-q^{2n-1})}\alpha(n-1,q). \qquad (4.6)$$

Iterating (4.6) n times and noting that $\alpha(0,q)=1$, we find

$$\alpha(n,q) = \frac{(-q;q^2)_n q^{n(1+(j+3)(n-1)/2)}}{(q^2;q^2)_n(q;q^2)_n}. \qquad (4.7)$$

Therefore,

$$
\begin{aligned}
g^j(z,q) &= \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(1+(j+3)(n-1)/2)}}{(q;q)_{2n}} z^n \qquad (4.8)\\
&= f^{(5,j)}(z,q) \qquad (4.9)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{\nu=0}^{\infty} cF_2^{(5,j)}(\nu)q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=0}^{\infty} cF_2^{(5,j)}(m,\nu)\right)q^\nu\\
&= f^{(5,j)}(1,q)\\
&= f^{(5,j)}(q).
\end{aligned}
$$

This proves Theorem 3.5. □

Proof of Theorem 3.6. Let $\rho^j(\nu)$ represent the number of 2–color F–partitions of ν enumerated by $cF_2^{(6,j)}(\nu)$ with the additional condition that the last column is of the type $\binom{0_h}{q_h}$ and let $\rho^j(m,\nu)$ represent the number of 2–color F–partitions of ν enumerated by $\rho^j(\nu)$ into m columns. Further, let

$$h^j(q) = \sum_{\nu=0}^{\infty} \rho^j(\nu)q^\nu \qquad (4.10)$$

and

$$h^j(z,q) = \sum_{\nu,m=0}^{\infty} \rho^j(m,\nu)z^m q^\nu. \qquad (4.11)$$

With the help of (4.2), we have

$$\rho^j(m,\nu) = cF_2^{(5,j)}(m-1,\nu-m(j+3)+j+2) + \frac{1}{2}\{cF_2^{(5,j)}(m-1,\nu-m(j+5)+j+3) + cF_2^{(5,j)}(m,\nu-2m+1) - cF_2^{(5,j)}(m,\nu-4m+1)\}, \tag{4.12}$$

where $cF_2^{(5,j)}(0,0)=1$ and $cF_2^{(5,j)}(m,\nu)=0$ for $\nu<0$.

Transforming (4.12) into a q–functional equation, we get

$$h^j(z,q) = zqg^j(zq^{j+3},q) + \frac{1}{2}zq^2g^j(zq^{j+5},q) + \frac{1}{2}q^{-1}g^j(zq^2,q) - \frac{1}{2}q^{-1}g^j(zq^4,q). \tag{4.13}$$

Setting

$$h^j(z,q) = \sum_{n=0}^{\infty}\beta(n,q)z^n \tag{4.14}$$

and then examining the coefficients of z^n in the above expression (4.13) we get

$$2\beta(n,q) = 2q^{(j+3)(n-1)+1}\alpha(n-1,q) + q^{(j+5)(n-1)+2}\alpha(n-1,q) + q^{2n-1}\alpha(n,q) - q^{4n-1}\alpha(n,q). \tag{4.15}$$

Substituting $\alpha(n,q)$ from (4.7) into (4.15) and then simplifying, we get

$$\beta(n,q) = \frac{(-q;q^2)_{n-1}q^{n(1+(j+3)(n-1)/2)}}{(q^2;q^2)_{n-1}(q;q^2)_n}. \tag{4.16}$$

Thus

$$h^j(z,q) = \sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{(n+1)[1+(j+3)n/2]}}{(q^2;q^2)_n(q;q^2)_{n+1}}z^{n+1} = zqf^{(6,j)}(zq,q). \tag{4.17}$$

Define $\psi^j(m,\nu)$ by

$$f^{(6,j)}(z,q) = \sum_{m,\nu=0}^{\infty}\psi^j(m,\nu)z^mq^\nu.$$

By examining the coefficients of (4.17), we get

$$\rho^j(m+1,\nu+m+1) = \psi^j(m,\nu).$$

If 1 is subtracted from the bottom row entry of each column which is enumerated by $\rho^j(m+1,\nu+m+1)$ ignoring the subscripts, we have the final partitions enumerated by $cF_2^{(6,j)}(m+1,\nu)$. Note here that as illustrated in the bijective proof of Theorem 3.6, we discard the phantom column $\binom{0}{-1}$. Thus

$$\psi^j(m,\nu) = cF_2^{(6,j)}(m+1,\nu)$$

and so

$$\sum_{m,\nu=0}^{\infty} cF_2^{(6,j)}(m+1,\nu)z^m q^\nu = f^{(6,j)}(z,q). \tag{4.18}$$

Now

$$\begin{aligned}\sum_{\nu=0}^{\infty} cF_2^{(6,j)}(\nu)q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=1}^{\infty} cF_2^{(6,j)}(m,\nu)\right)q^\nu \\ &= \sum_{m,\nu=0}^{\infty} cF^2_{(6,j)}(m+1,\nu)q^\nu \\ &= f^{(6,j)}(1,q) \\ &= f^{(6,j)}(q).\end{aligned}$$

This proves Theorem 3.6. □

Proof of Theorem 3.7. Rewrite equation (4.17) as

$$h^j(z,q) = zqf^{(7,j)}(z,q). \tag{4.19}$$

Define $\eta^j(m,\nu)$ by

$$f^{(7,j)}(z,q) = \sum_{m,\nu=0}^{\infty} \eta^j(m,\nu)z^m q^\nu. \tag{4.20}$$

By examining the coefficients of (4.19), we get

$$\rho^j(m+1,\nu+1) = \eta^j(m,\nu). \tag{4.21}$$

If the last column $\binom{0_h}{q_h}$ is replaced by $\binom{0_h}{q_h-1}$ which is enumerated by $\rho^j(m+1,\nu+1)$, we have the final partitions enumerated by $cF_2^{(7,j)}(m+1,\nu)$. Noting that as illustrated in bijective proof of Theorem 3.7, we discard the phantom column $\binom{0}{-1}$. Thus

$$\eta^j(m,\nu) = cF_2^{(7,j)}(m+1,\nu)$$

and introduce

$$\sum_{m,\nu=0}^{\infty} cF_2^{(7,j)}(m+1,\nu)z^m q^\nu = f^{(7,j)}(z,q). \tag{4.22}$$

Now

$$\begin{aligned}\sum_{\nu=0}^{\infty} cF_2^{(7,j)}(\nu)q^{\nu} &= \sum_{\nu=0}^{\infty}\left(\sum_{m=1}^{\infty} cF_2^{(7,j)}(m,\nu)\right)q^{\nu}\\ &= \sum_{m,\nu=0}^{\infty} cF_2^{(7,j)}(m+1,\nu)q^{\nu}\\ &= f^{(7,j)}(1,q)\\ &= f^{(7,j)}(q).\end{aligned}$$

This proves Theorem 3.7. □

Sketch of Proof of Theorem 3.1. The proof of Theorem 3.1 proceeds in the same manner as Theorem 3.5; we obtain the following recurrence relations and hence the q–functional equation.

$$\begin{aligned}cF_2^{(1,j)}(m,\nu) = {} & cF_2^{(1,j)}(m,\nu-4m) + cF_2^{(1,j)}(m-1,\nu-m(j+3)+j+2)\\ & + cF_2^{(1,j)}(m-1,\nu-m(j+5)+j+3)\\ & + cF_2^{(1,j)}(m,\nu-2m+1) - cF_2^{(1,j)}(m,\nu-6m+1),\end{aligned}\tag{4.23}$$

$$\begin{aligned}g_1^j(z,q) = {} & g^{(1,j)}(zq^4,q) + zqg_1^j(zq^{j+3},q) + zq^2g_1^j(zq^{j+5},q)\\ & + q^{-1}g_1^j(zq^2,q) - q^{-1}g_1^j(zq^6,q).\end{aligned}\tag{4.24}$$

Setting

$$g_1^j(z,q) = \sum_{n=0}^{\infty}\alpha'(n,q)z^n$$

in (4.24) and then examining the coefficients of z^n in the above expression we get

$$\alpha'(n,q) = \frac{q^{1+(j+3)(n-1)}(1+q^{2n-1})}{(1-q^{4n})(1-q^{2n-1})}\alpha'(n-1,q).$$

Iterating the above expression n times and noting that $\alpha'(0,q)=1$, we find that

$$\alpha'(n,q) = \frac{(-q;q^2)_n q^{n(1+(j+3)(n-1)/2)}}{(q^4;q^4)_n(q;q^2)_n}.\tag{4.25}$$

Therefore,

$$\begin{aligned}g_1^j(z,q) &= \sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(1+(j+3)(n-1)/2)}}{(q^4;q^4)_n(q;q^2)_n}z^n\\ &= f^{(1,j)}(z,q)\end{aligned}$$

and

$$\sum_{\nu=0}^{\infty} cF_2^{(1,j)}(\nu)q^\nu = \sum_{\nu=0}^{\infty}\left(\sum_{m=0}^{\infty} cF_2^{(1,j)}(m,\nu)\right)q^\nu = f^{(1,j)}(1,q) = f^{(1,j)}(q).$$

This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. Let $D^j(\nu)$ represent the number of 2–color F–partitions of ν enumerated by $cF_2^{(2,j)}(\nu)$ with the additional constraint that the last column is of the type $\binom{0_h}{q_h}$ and let $D^j(m,\nu)$ represent the number of 2–color F–partitions of ν enumerated by $D^j(\nu)$ into m columns. Further, let

$$h_1^j(q) = \sum_{\nu=0}^{\infty} D^j(\nu)q^\nu \tag{4.26}$$

and

$$h_1^j(z,q) = \sum_{\nu,m=0}^{\infty} D^j(m,\nu)z^m q^\nu. \tag{4.27}$$

With the help of 4.23, we have

$$\begin{aligned} D^j(m,\nu) = {} & cF_2^{(1,j)}(m-1,\nu-m(j+3)+j+2) \\ & + \frac{1}{2}[cF_2^{(1,j)}(m-1,\nu-m(j+5)+j+3) \\ & + cF_2^{(1,j)}(m,\nu-2m+1) - cF_2^{(1,j)}(m,\nu-6m+1)], \end{aligned} \tag{4.28}$$

where $cF_2^{(1,j)}(0,0)=1$ and $cF_2^{(1,j)}(m,\nu)=0$ for $\nu<0$. Transforming (4.28) into a q–functional equation, we get

$$\begin{aligned} h_1^j(z,q) = {} & zqf^{(1,j)}(zq^{j+3},q) + \frac{1}{2}zq^2 f^{(1,j)}(zq^{j+5},q) \\ & + \frac{1}{2}q^{-1}f^{(1,j)}(zq^2,q) - \frac{1}{2}q^{-1}f^{(1,j)}(zq^6,q). \end{aligned} \tag{4.29}$$

Setting

$$h_1^j(z,q) = \sum_{n=0}^{\infty}\gamma'(n,q)z^n, \tag{4.30}$$

and examining the coefficients of z^n in (4.30) we get

$$\begin{aligned} 2\gamma'(n,q) = {} & 2q^{(j+3)(n-1)+1}\alpha'(n-1,q) + q^{(j+5)(n-1)+2}\alpha'(n-1,q) \\ & + q^{2n-1}\alpha'(n,q) - q^{6n-1}\alpha'(n,q). \end{aligned} \tag{4.31}$$

Substituting $\alpha'(n,q)$ from (4.25) into (4.31) and then simplifying, we get

$$\gamma'(n,q) = \frac{(-q;q^2)_{n-1} q^{n(1+(j+3)(n-1)/2)}}{(q^4;q^4)_{n-1}(q;q^2)_n}. \tag{4.32}$$

Thus

$$h_1^j(z,q) = \sum_{\nu=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)[1+(j+3)n/2]}}{(q^4;q^4)_n (q;q^2)_{n+1}} z^{n+1} = zqf^{(2,j)}(zq,q). \tag{4.33}$$

Define $Q^j(m,\nu)$ by

$$f^{(2,j)}(z,q) = \sum_{m,\nu=0}^{\infty} Q^j(m,\nu) z^m q^\nu. \tag{4.34}$$

By examining the coefficients of 4.35, we get

$$D^j(m+1, \nu+m+1) = Q^j(m,\nu).$$

If each summand is subtracted by 1 which is enumerated by $D^j(m+1,\nu+m+1)$ ignoring the subscripts, we have the final partitions enumerated by $cF_2^{(2,j)}(m+1,\nu)$. Thus

$$Q^j(m,\nu) = cF_2^{(2,j)}(m+1,\nu)$$

and so

$$\sum_{m,\nu=0}^{\infty} cF_2^{(2,j)}(m+1,\nu) z^m q^\nu = f^{(2,j)}(z,q).$$

Now

$$\begin{aligned}
\sum_{n=0}^{\infty} cF_2^{(2,j)}(\nu) q^\nu &= \sum_{\nu=0}^{\infty} \left(\sum_{m=1}^{\infty} cF_2^{(2,j)}(m,\nu) \right) q^\nu \\
&= \sum_{m,\nu=0}^{\infty} cF_2^{(2,j)}(m+1,\nu) q^\nu \\
&= f^{(2,j)}(1,q) \\
&= f^{(2,j)}(q).
\end{aligned}$$

This completes the proof of Theorem 3.2. □

Proof of Theorem 3.3. Rewrite (4.31) as

$$h_1^j(z,q) = zqf^{(3,j)}(z,q). \tag{4.35}$$

Define $R^j(m,\nu)$ by

$$f^{(3,j)}(z,q) = \sum_{m,\nu=0}^{\infty} R^j(m,\nu) z^m q^\nu. \tag{4.36}$$

By examining the coefficients of (4.35), we get

$$D^j(m+1,\nu+1) = R^j(m,\nu). \tag{4.37}$$

If last column $\binom{0_h}{q_h}$ is replaced by $\binom{0_h}{q_h-1}$ which is enumerated by $D^j(m+1,\nu+1)$, we have the final partitions enumerated by $cF_2^{(3,j)}(m+1,\nu)$. Thus

$$R^j(m,\nu) = cF_2^{(3,j)}(m+1,\nu)$$

and

$$\sum_{m,\nu=0}^{\infty} cF_2^{(3,j)}(m+1,\nu)z^m q^\nu = f^{(3,j)}(z,q). \tag{4.38}$$

Now

$$\begin{aligned}\sum_{\nu=0}^{\infty} cF_2^{(3,j)}(\nu)q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=1}^{\infty} cF_2^{(3,j)}(m,\nu)\right)q^\nu \\ &= \sum_{m,\nu=0}^{\infty} cF_2^{(3,j)}(m+1,\nu)q^\nu \\ &= f^{(3,j)}(1,q) \\ &= f^{(3,j)}(q).\end{aligned}$$

This completes the proof of Theorem 3.3. □

Proof of Theorem 3.4. Let $cF_2^{(4,j)}(m,\nu)$ represent the number of 2–color F–partitions of ν enumerated by $cF_2^{(4,j)}(\nu)$ into m columns. Split the partitions enumerated by $cF_2^{(4,j)}(m,\nu)$ into following four classes:

(i) contains $\binom{p_h}{q_h}$ as the mth column, $p \neq (j+3)/2$,
(ii) contains $\binom{((j+3)/2)_1}{((j+3)/2)_1}$ as the mth column,
(iii) contains $\binom{((j+3)/2)_2}{((j+5)/2)_2}$ as the mth column,
(iv) contains $\binom{((j+3)/2)_1}{q_1}$, $q \geq (j+5)/2$ or $\binom{((j+3)/2)_2}{q_2}$, $q \geq (j+7)/2$ as the mth column.

Transform the partitions of class (i) by subtracting 2 from the top and bottom row entries of each column, we get partitions of $(\nu-4m)$ into m parts without disturbing the restrictions on the columns as described in Theorem 3.4. Thus the transformed partitions are enumerated by $cF_2^{(4,j)}(m,\nu-4m)$.

Further, transform the partitions of class (ii) by deleting the column $\binom{((j+3)/2)_1}{((j+3)/2)_1}$ and then subtracting $(j+3)/2$ from the top and bottom row entries of each column. The transformed partitions are enumerated by $cF_2^{(4,j)}(m-1,\nu-m(j+3)-1)$ as this transformation does not effect the conditions of Theorem 3.4.

Next, transform the partitions of class (iii) by deleting the column $\binom{((j+3)/2)_2}{((j+5)/2)_2}$ and then subtracting $(j+5)/2$ from the top and bottom row entries of each column. The transformed partitions are enumerated by $cF_2^{(4,j)}(m-1, \nu - m(j+5))$ as this transformation does not effect the conditions of Theorem 3.4.

Finally, transform the partitions of class (iv) by subtracting 1 from the top row entries of each column except the last column and 1 from the bottom row entries of each column. We get the partitions enumerated by $cF_2^{(4,j)}(m, \nu - 2m+1)$ having the m^{th} column as $\binom{((j+3)/2)_1}{q_1}$, $q \geq (j+5)/2$ or $\binom{((j+3)/2)_2}{q_2}$, $q \geq (j+7)/2$. Thus the actual number of partitions in class (iv) are obtained by subtracting the number of those partitions which are enumerated by $cF_2^{(4,j)}(m, \nu - 2m+1)$ with the last column as $\binom{p_h}{q_h}$, $p \neq (j+3)/2$ from $cF_2^{(4,j)}(m, \nu - 2m+1)$. Thus the transformed partitions are enumerated by $cF_2^{(4,j)}(m, \nu - 2m+1) - cF_2^{(4,j)}(m, \nu - 6m+1)$. Hence we get the following recurrence formula for $cF_2^{(4,j)}(m, \nu)$:

$$
\begin{aligned}
(4.39)\quad cF_2^{(4,j)}(m,\nu) = {} & cF_2^{(4,j)}(m, \nu - 4m) \\
& + cF_2^{(4,j)}(m-1, \nu - m(j+3) - 1) \\
& + cF_2^{(4,j)}(m-1, \nu - m(j+5)) \\
& + cF_2^{(4,j)}(m, \nu - 2m+1) - cF_2^{(4,j)}(m, \nu - 6m+1),
\end{aligned}
$$

where $cF_2^{(4,j)}(0,0) = 1$ and $cF_2^{(4,j)}(m,\nu) = 0$ for $\nu < 0$.

For $|q| < 1$ and $|z| < |q|^{-1}$, let $g_1^j(z,q)$ be defined by

$$
(4.40)\qquad g_1^j(z,q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} cF_2^{(4,j)}(m,\nu) z^m q^\nu, \text{ for all } j \in S
$$

Substituting $cF_2^{(4,j)}(m,\nu)$ from (4.39) into (4.40), we get the q–functional equation

$$
\begin{aligned}
g_1^j(z,q) = {} & g_1^j(zq^4, q) + zq^{j+4} g_1^j(zq^{j+3}, q) + zq^{j+5} g_1^j(zq^{j+5}, q) \\
(4.41)\qquad & + q^{-1} g_1^j(zq^2, q) - q^{-1} g_1^j(zq^6, q).
\end{aligned}
$$

Set

$$
(4.42)\qquad g_1^j(z,q) = \sum_{n=0}^{\infty} \gamma(n,q) z^n.
$$

Using (4.41) in (4.42) and then examining the coefficients of z^n, we get

$$
(4.43)\qquad \gamma(n,q) = \frac{q^{1+n(j+3)}(1+q^{2n-1})}{(1-q^{4n})(1-q^{2n-1})} \gamma(n-1,q).
$$

Iterating (4.43) n times and noting that $\gamma(0,q)=1$, we find

$$\gamma(n,q)=\frac{(-q;q^2)_n q^{n(1+(j+3)(n+1)/2)}}{(q^4;q^4)_n(q;q^2)_n}. \tag{4.44}$$

Therefore,

$$g_1^j(z,q)=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(1+(j+3)(n+1)/2)}}{(q^4;q^4)_n(q;q^2)_n}z^n \tag{4.45}$$

$$=f^{(4,j)}(z,q) \tag{4.46}$$

and

$$\begin{aligned}\sum_{\nu=0}^{\infty} cF_2^{(4,j)}(\nu)q^\nu &= \sum_{\nu=0}^{\infty}\left(\sum_{m=0}^{\infty} cF_2^{(4,j)}(m,\nu)\right)q^\nu\\ &= f^{(4,j)}(1,q)\\ &= f^{(4,j)}(q).\end{aligned}$$

This completes the proof of Theorem 3.4. □

5. Particular cases

Theorems 3.1–3.7 translate into Theorems 5.1–5.7 and provide the combinatorial interpretations to following seven Rogers–Ramanujan type identities, respectively, which are listed in Chu and Zhang's Compendium [9] and

Slater's compendium [22].

([9, I(29)]) $$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_nq^{n^2}}{(q^4,q^4)_n(-q,q^2)_n}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q^2,-q^3;q^5]_\infty,$$

([9, I(113)]) $$\sum_{n=0}^{\infty}\frac{(-q,q^2)_nq^{2n(n+1)}}{(q^4,q^4)_n(q,q^2)_{n+1}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[q^{12},q^3,q^9;q^{12}]_\infty,$$

([9, I(25)]) $$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_nq^{n(n+2)}}{(q^4,q^4)_n(-q,q^2)_{n+1}}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q^5,-q^5;q^5]_\infty,$$

([9, I(27)]) $$\sum_{n=0}^{\infty}\frac{(-1)^n(q,q^2)_nq^{n(n+2)}}{(q^4,q^4)_n(-q,q^2)_n}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[q^5,-q,-q^4;q^5]_\infty,$$

([22, I(29)]) $$\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{n^2}}{(q,q)_{2n}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^2,-q^4,q^6;q^6]_\infty,$$

([9, I(104)], [22, I(51)])
$$\sum_{n=0}^{\infty}\frac{(-q,q^2)_nq^{n(n+1)}}{(q,q)_{2n+1}}=\frac{1}{(q;q)_\infty}[q^4,q^8,q^{12};q^{12}]_\infty,$$

([9, I(102)]. [22, I(50)])
$$\sum_{n=0}^{\infty}\frac{(-q,q^2)_nq^{n(n+2)}}{(q,q)_{2n+1}}=\frac{1}{(q;q)_\infty}[q^2,q^{10},q^{12};q^{12}]_\infty,$$

Theorem 5.1. *Let $C_1(\nu)$ represent the number of n–color partitions of ν which contain distinct parts such that the first two copies of parts congruent to* $5 \pmod{10}$ *are allowed; only first copy of parts congruent to* $\pm 1 \pmod{10}$ *is allowed; and let $D_1(\nu)$ represent the number of n–color partitions of ν which contain the first two copies of parts congruent to* $\pm 2 \pmod{10}$. *Further, let*

$$B_1(\nu)=\sum_{l=0}^{\nu}C_1(l)D_1(\nu-l),$$

then

$$B_1(\nu)=A^{(1,-1)}(\nu)=cF_2^{(1,-1)}(\nu),\ \textit{for all}\ \nu,$$

where $A^{(1,-1)}(\nu)$ and $cF_2^{(1,-1)}$ are defined in Theorem 3.1 for $j=-1$.

Theorem 5.2. *Let $C_2(\nu)$ represent the number of partitions of ν which contain distinct parts congruent to* $\pm 1, \pm 5 \pmod{12}$ *and let $D_2(\nu)$ represent the partitions of ν which contain parts congruent to* $\pm 2, \pm 4 \pmod{12}$. *Further, let*

$$B_2(\nu)=\sum_{l=0}^{\nu}C_2(l)D_2(\nu-l),$$

then

$$B_2(\nu)=A^{(2,1)}(\nu)=cF_2^{(2,1)}(\nu),\ \textit{for all}\ \nu,$$

where $A^{(2,1)}(\nu)$ *and* $cF_2^{(2,1)}$ *are defined in Theorem 3.2 for* $j = 1$.

Theorem 5.3. *Let* $C_3(\nu)$ *represent the number partitions of* ν *which contain the parts congruent to* $\pm1, \pm3 \pmod{10}$ *and let* $D_3(\nu)$ *represent the number of partitions of* ν *which contain the parts congruent to* $\pm2, \pm4 \pmod{10}$. *Further, let*

$$B_3(\nu) = \sum_{l=0}^{\nu} C_3(l) D_3(\nu - l),$$

then

$$B_3(\nu) = A^{(3,-1)}(\nu) = cF_2^{(3,-1)}(\nu), \text{ for all } \nu,$$

where $A^{(3,-1)}(\nu)$ *and* $cF_2^{(3,-1)}$ *are defined in Theorem 3.3 for* $j = -1$.

Theorem 5.4. *Let* $C_4(\nu)$ *represent the number of* n*–color partitions of* ν *which contain the first two copies of distinct parts congruent to* $5 \pmod{10}$ *and only first copy of parts congruent to* $\pm3 \pmod{10}$ *and let* $D_4(\nu)$ *represent the number of* n*–color partitions of* ν *which contain the first two copies of parts congruent to* $\pm4 \pmod{10}$. *Further, let*

$$B_4(\nu) = \sum_{l=0}^{\nu} C_4(l) D_4(\nu - l),$$

then

$$B_4(\nu) = A^{(4,-1)}(\nu) = cF_2^{(4,-1)}(\nu), \text{ for all } \nu,$$

where $A^{(4,-1)}(\nu)$ *and* $cF_2^{(4,-1)}(\nu)$ *are defined in Theorem 3.4 for* $j = -1$.

Theorem 5.5. *Let* $B_5(\nu)$ *represent the number of partitions of* ν *which contain distinct odd parts, even parts congruent to* $\pm2, \pm4 \pmod{12}$ *and the parts which are congruent to* $\pm2 \pmod{12}$, *appear in two copies. Then*

$$B_5(\nu) = A^{(1,-1)}(\nu) = cF_2^{(5,-1)}(\nu), \text{ for all } \nu,$$

where $A^{(5,-1)}(\nu)$ *and* $cF_2^{(5,-1)}$ *are defined in Theorem 3.5 for* $j = -1$.

Example. The relevant partitions corresponding to $B_5(7) = 13$ are,

$$7, 5 + 2_1, 5 + 2_2, 4 + 3, 4 + 2_1 + 1, 4 + 2_2 + 1, 3 + 2_1 + 2_1,$$
$$3 + 2_2 + 2_2, 3 + 2_2 + 2_1, 2_2 + 2_2 + 2_2 + 1, 2_1 + 2_1 + 2_1 + 1,$$
$$2_2 + 2_2 + 2_1 + 1, 2_2 + 2_1 + 2_1 + 1.$$

To illustrate the bijection, example for $\nu = 7$ is shown in the table below:

Split n–color partitions relevant to $A^{(5,-1)}(7)$	2–color F–partitions relevant to $cF_2^{(5,-1)}(7)$

7_1	$\begin{pmatrix} 3_1 \\ 3_1 \end{pmatrix}$
7_3	$\begin{pmatrix} 2_1 \\ 4_1 \end{pmatrix}$
7_5	$\begin{pmatrix} 1_1 \\ 5_1 \end{pmatrix}$
7_7	$\begin{pmatrix} 0_1 \\ 6_1 \end{pmatrix}$
7_{2+1}	$\begin{pmatrix} 2_2 \\ 4_2 \end{pmatrix}$
7_{4+1}	$\begin{pmatrix} 1_2 \\ 5_2 \end{pmatrix}$
7_{6+1}	$\begin{pmatrix} 0_2 \\ 6_2 \end{pmatrix}$
$6_4 1_1$	$\begin{pmatrix} 1_1 & 0_1 \\ 4_1 & 0_1 \end{pmatrix}$
$6_{3+1} 1_1$	$\begin{pmatrix} 1_2 & 0_1 \\ 4_2 & 0_1 \end{pmatrix}$
$6_2 1_1$	$\begin{pmatrix} 2_1 & 0_1 \\ 3_1 & 0_1 \end{pmatrix}$
$6_{1+1} 1_1$	$\begin{pmatrix} 2_2 & 0_1 \\ 3_2 & 0_1 \end{pmatrix}$
$5_1 2_2$	$\begin{pmatrix} 2_1 & 0_1 \\ 2_1 & 1_1 \end{pmatrix}$
$5_1 2_{1+1}$	$\begin{pmatrix} 2_1 & 0_2 \\ 2_1 & 1_2 \end{pmatrix}$

Theorem 5.6. *Let $B_6(\nu)$ represent the number of partitions of ν into parts congruent to $\pm1, \pm2, \pm3, \pm5, 6 \pmod{12}$. Then*

$$B_6(\nu) = A^{(6,-1)}(\nu) = cF_2^{(6,-1)}(\nu), \text{ for all } \nu,$$

where $A^{(6,-1)}(\nu)$ and $cF_2^{(6,-1)}$ are defined in Theorem 3.6 for $j = -1$.

Theorem 5.7. *Let $B_7(\nu)$ represent the number of partitions of ν into parts congruent to $\pm1, \pm3, \pm4, \pm5, 6 \pmod{12}$. Then*

$$B_7(\nu) = A^{(7,-1)}(\nu) = cF_2^{(7,-1)}(\nu), \text{ for all } \nu,$$

where $A^{(7,-1)}(\nu)$ and $cF_2^{(7,-1)}$ are defined in Theorem 3.7 for $j = -1$.

6. Conclusion

Our results provide the infinite two–way combinatorial identities of the generalized q–series (3.1)–(3.7) using 2–color F–partitions and include results in [5, 20]. In some particular cases we obtain the three–way combinatorial interpretations of some Rogers–Ramanujan type identities.

$$A^{(k,-1)}(\nu) = cF_2^{(k,-1)}(\nu) = B_k(\nu), \quad 1 \leq k \leq 7,\ k \neq 2 \tag{6.1}$$

$$A^{(2,1)}(\nu) = cF_2^{(2,1)}(\nu) = B_2(\nu), \tag{6.2}$$

where $A^{(k,j)}(\nu)$ and $cF_2^{(k,j)}(\nu)$, $1 \leq k \leq 7$ are defined in Theorems 3.1–3.7 and $B_k(\nu)$, $1 \leq k \leq 7$ are defined in Theorems 5.1–5.7, respectively. In total, we obtain twenty one new combinatorial identities in the usual sense. Out of which five are obtained in [20], three are given by Agarwal and Sood [5] and are reproduced in Theorems 3.5–3.7, the rest are given as below:

$$A^{(k,-1)}(\nu) = cF_2^{(k,-1)}(\nu), \quad 1 \leq k \leq 7,\ k \neq 2 \tag{6.3}$$

$$A^{(2,1)}(\nu) = cF_2^{(2,1)}(\nu), \tag{6.4}$$

$$B_k(\nu) = cF_2^{(k,-1)}(\nu), \quad 1 \leq k \leq 7,\ k \neq 2 \tag{6.5}$$

$$B_2(\nu) = cF_2^{(2,1)}(\nu). \tag{6.6}$$

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