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EULERIAN POLYNOMIALS AND POLYNOMIAL CONGRUENCES

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Abstract. We prove that the Eulerian polynomial satisfies certain polynomial congruences. Furthermore, these congruences characterize the Eulerian polynomial.

1. Introduction

The Eulerian polynomial $A_\ell(x)$ ($\ell \geq 1$) was introduced by Euler in the study of sums of powers [5]. In this paper, we define the Eulerian polynomial $A_\ell(x)$ as the numerator of the rational function

$$F_\ell(x) = \sum_{k=1}^{\infty} k^\ell x^k = \left(x\frac{d}{dx}\right)^\ell \frac{1}{1-x} = \frac{A_\ell(x)}{(1-x)^{\ell+1}}. \tag{1.1}$$

The first few examples are $A_1(x) = x, A_2(x) = x + x^2, A_3(x) = x + 4x^2 + x^3, A_4(x) = x + 11x^2 + 11x^3 + x^4$, etc. The Eulerian polynomial $A_\ell(x)$ is a monic of degree ℓ with positive integer coefficients. Write $A_\ell(x) = \sum_{k=1}^{\ell} A(\ell, k)x^k$. The coefficient $A(\ell, k)$ is called an Eulerian number.

In the 1950s, Riordan [10] discovered a combinatorial interpretation of Eulerian numbers in terms of descents and ascents of permutations, and Carlitz [3] defined q-Eulerian numbers. Since then, Eulerian numbers are actively studied in enumerative combinatorics. (See [4, 11, 8].)

Another combinatorial application of the Eulerian polynomial was found in the theory of hyperplane arrangements [15, 14]. The characteristic polynomial of the so-called Linial arrangement [9] can be expressed in terms of the root system generalization of Eulerian polynomials introduced by Lam and Postnikov [6]. The comparison of expressions in [9] and [15] yields the following.

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Theorem 1.1 ([15, Proposition 5.5])**.** *For $\ell, m \geq 2$, the Eulerian polynomial $A_\ell(x)$ satisfies the following*

$$A_\ell(x^m) \equiv \left(\frac{1+x+x^2+\cdots+x^{m-1}}{m}\right)^{\ell+1} A_\ell(x) \mod (x-1)^{\ell+1}. \tag{1.2}$$

The purpose of this paper is two-fold. First, we give a direct and simpler proof of Theorem 1.1. Second, we prove the converse of the above theorem. Namely, the congruence (1.2) characterizes the Eulerian polynomial as follows.

Theorem 1.2. *Let $f(x)$ be a monic of degree ℓ. Then, $f(x) = A_\ell(x)$ if and only if the congruence (1.2) holds for some $m \geq 2$. (See Theorem 5.1.)*

The remainder of this paper is organized as follows. After recalling classical results on Eulerian polynomials in §2, we briefly describe in §3 the proof of the congruence (1.2) in [15] that is based on the expression of characteristic polynomials of Linial hyperplane arrangements. In §4, we give a direct proof of the congruence. In §5, we give the proof of Theorem 1.2.

Remark. The right-hand side of (1.2) is discussed also in [12, Proposition 2.5].

2. Brief review of Eulerian polynomials

In this section, we recall classical results on the Eulerian polynomial and the Eulerian numbers $A(\ell, k)$. By definition (1.1), the Eulerian polynomial $A_\ell(x)$ satisfies the relation

$$\frac{A_\ell(x)}{(1-x)^{\ell+1}} = x\frac{d}{dx}\frac{A_{\ell-1}(x)}{(1-x)^\ell},$$

which yields the following recursive relation.

$$A(\ell, k) = k \cdot A(\ell-1, k) + (\ell-k+1) \cdot A(\ell-1, k-1). \tag{2.1}$$

Consider the coordinate change $w = \frac{1}{x}$. Then, the Euler operator is transformed as $x\frac{d}{dx} = -w\frac{d}{dw}$. The direct computation using the relation $\frac{1}{1-\frac{1}{w}} = 1 - \frac{1}{1-w}$ yields $x^{\ell+1}A_\ell(\frac{1}{x}) = A_\ell(x)$. Equivalently, $A(\ell, k) = A(\ell, \ell + 1 - k)$.

Definition (1.1) is also equivalent to

$$A_\ell(x) = (1-x)^{\ell+1} \cdot \sum_{k=0}^{\infty} k^\ell x^k \tag{2.2}$$

and

$$\sum_{k=0}^{\infty} k^\ell x^k = A_\ell(x) \cdot \sum_{k=0}^{\infty} (-x)^k \binom{-\ell-1}{k}. \tag{2.3}$$

Then, (2.2) yields

$$A(\ell, k) = \sum_{j=0}^{k} (-1)^j \binom{\ell+1}{j} (k-j)^\ell, \tag{2.4}$$

and (2.3) yields

$$k^\ell = \sum_{j=1}^{\ell} A(\ell, j) \binom{k+\ell-j}{\ell}. \tag{2.5}$$

Note that both sides of (2.5) are polynomials of degree ℓ in k, and it holds for any $k > 0$. Hence the equality holds at the level of polynomials in t. Thus, we have

$$t^\ell = \sum_{j=1}^{\ell} A(\ell, j) \binom{t+\ell-j}{\ell}, \tag{2.6}$$

which is called the Worpitzky identity [13]. Using the shift operator $S : t \longmapsto t-1$ (see §3), the Worpitzky identity can be written as

$$t^\ell = A_\ell(S) \binom{t+\ell}{\ell}. \tag{2.7}$$

Next, we consider exponential generating series of $A_\ell(x)$, and describe relations with Bernoulli numbers. First, using (2.2), we have

$$\begin{aligned} \sum_{\ell=0}^{\infty} \frac{A_\ell(x)}{\ell!} t^\ell &= \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} (1-x)^{\ell+1} \sum_{n=0}^{\infty} n^\ell x^n \\ &= (1-x) \sum_{n=0}^{\infty} x^n e^{nt(1-x)} \\ &= \frac{1-x}{1-xe^{t(1-x)}}. \end{aligned} \tag{2.8}$$

Replacing x by -1 in (2.8), we have

$$\sum_{\ell=0}^{\infty} \frac{A_\ell(-1)}{\ell!} t^\ell = \frac{2}{1+e^{2t}}. \tag{2.9}$$

Recall that the Bernoulli polynomial $B_\ell(x)$ is defined by

$$\sum_{\ell=0}^{\infty} \frac{B_\ell(x)}{\ell!} t^\ell = \frac{te^{xt}}{e^t - 1}, \tag{2.10}$$

($B_0(x) = 1, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6}, B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \cdots$) and the constant term $B_\ell(0)$ is called the Bernoulli number. Replacing x by 0 and t by at with $a \in \mathbb{C}$ in (2.10), we have

$$\sum_{\ell=0}^{\infty} \frac{B_\ell(0)}{\ell!} (at)^\ell = \frac{at}{e^{at} - 1}. \tag{2.11}$$

Using the identity $\frac{2t}{e^{2t}+1} = \frac{2t}{e^{2t}-1} - \frac{4t}{e^{4t}-1}$, the Bernoulli number $B_\ell(0)$ can be expressed as

$$B_\ell(0) = \frac{\ell}{2^\ell(1-2^\ell)} A_{\ell-1}(-1). \tag{2.12}$$

There is another relation between Eulerian polynomials and Bernoulli polynomials. Let $\ell > 0$. Using (2.5) and the famous formula $\sum_{x=0}^{N-1} x^\ell = \frac{B_{\ell+1}(N)-B_{\ell+1}(0)}{\ell+1}$, we have

$$B_{\ell+1}(N) - B_{\ell+1}(0) = (\ell+1) \cdot \sum_{k=1}^{\ell} A(\ell,k)\binom{\ell+N-k}{\ell+1}. \tag{2.13}$$

With the shift operator S, (2.13) can also be expressed as

$$B_{\ell+1}(t) - B_{\ell+1}(0) = (\ell+1)A_\ell(S)\binom{t+\ell}{\ell+1}. \tag{2.14}$$

(This formula appeared in [13, page 209] as "The second form of Bernoulli function.")

3. Background on hyperplane arrangements

In this section, we recall the proof of the congruence (1.2) presented in [15].

Let $\mathcal{A} = \{H_1, \dots, H_k\}$ be a finite set of affine hyperplanes in a vector space V. We denote the set of all intersections of $\mathcal{A}$ by $L(\mathcal{A}) = \{\cap S \mid S \subset \mathcal{A}\}$. The set $L(\mathcal{A})$ is partially ordered by reverse inclusion, which has a unique minimal element $\hat{0} = V$. The characteristic polynomial of $\mathcal{A}$ is defined by

$$\chi(A, q) = \sum_{X \in L(\mathcal{A})} \mu(X) q^{\dim X},$$

where μ is the Möbius function on $L(\mathcal{A})$, defined by

$$\mu(X) = \begin{cases} 1, & \text{if } X = \hat{0} \\ -\sum_{Y<X} \mu(Y), & \text{otherwise.} \end{cases}$$

Let $V = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{\ell+1} \mid \sum x_i = 0\} \subset \mathbb{R}^{\ell+1}$. For integers $0 \le i < j \le \ell$ and $s \in \mathbb{Z}$, denote by $H_{ij,s}$ the affine hyperplane $\{(x_0, \dots, x_\ell) \in V \mid x_i - x_j = s\}$.

Let $m \ge 1$ be a positive integer. The arrangement

$$\mathcal{L}^m = \{H_{ij,s} \mid 0 \le i < j \le \ell, 1 \le s \le m\}$$

is called the (extended) Linial arrangement (of type A_ℓ). The Linial arrangement has several intriguing enumerative properties [9]. Postnikov and Stanley [9] (see also [1]) gave the following expression for the characteristic polynomial $\chi(\mathcal{L}^m, t)$.

$$\chi(\mathcal{L}^m, t) = \left(\frac{1 + S + S^2 + \cdots + S^m}{m+1}\right)^{\ell+1} t^\ell, \tag{3.1}$$

where S acts on a function $f(t)$ by $Sf(t) = f(t-1)$ (naturally $S^k f(t) = f(t-k)$) as the shift operator. Using the Worpitzky identity (2.7), (3.1) can be written as

$$\chi(\mathcal{L}^m, t) = \left(\frac{1+S+S^2+\cdots+S^m}{m+1}\right)^{\ell+1} A_\ell(S)\binom{t+\ell}{\ell}. \tag{3.2}$$

On the other hand, using the lattice points interpretation of the Worpitzky identity, the following formula was obtained in [15]

$$\chi(\mathcal{L}^m, t) = A_\ell(S^{m+1})\binom{t+\ell}{\ell}. \tag{3.3}$$

The formulas (3.2) and (3.3) imply that the operator

$$\left(\frac{1+S+S^2+\cdots+S^m}{m+1}\right)^{\ell+1} A_\ell(S) - A_\ell(S^{m+1}) \tag{3.4}$$

annihilates the polynomial $\binom{t+\ell}{\ell}$ of degree ℓ, which means that (3.4) is divisible by $(S-1)^{\ell+1}$ (see [15, Prop. 2.8]). Hence the congruence (1.2) follows.

4. Direct proof of the congruence

4.1. **Special case: $m = 2$.** We first handle the case $m = 2$. By considering $F_\ell(x) + F_\ell(-x)$, it is easily seen that the formal power series $F_\ell(x) = \sum_{k=1}^{\infty} k^\ell x^k$ satisfies

$$F_\ell(x) - 2^{\ell+1} F_\ell(x^2) = -F_\ell(-x). \tag{4.1}$$

Using the Eulerian polynomial, (4.1) can be written as

$$(1+x)^{\ell+1}\cdot A_\ell(x) - 2^{\ell+1}\cdot A_\ell(x^2) = -(1-x)^{\ell+1}\cdot A_\ell(-x), \tag{4.2}$$

which implies the congruence (1.2) for $m = 2$.

Remark. Substituting formally $x = 1$ into (4.1), we obtain the formula "$F_\ell(1) = \frac{A_\ell(-1)}{2^{\ell+1}(2^{\ell+1}-1)}$." Then, (2.12) implies "$F_\ell(1) = -\frac{B_{\ell+1}(0)}{\ell+1}$", which gives the correct value $\zeta(-\ell) = -\frac{B_{\ell+1}(0)}{\ell+1}$ of the Riemann zeta function for $\ell \geq 1$.

4.2. **General case.** Let $m \geq 2$. Denote by $\zeta_m = e^{2\pi\sqrt{-1}/m}$ the primitive mth root of unity. We will use the following fact

$$\sum_{j=1}^{m-1} \zeta_m^{jk} = \begin{cases} m-1, & \text{if } m|k, \\ -1, & \text{if } m \not| k, \end{cases} \tag{4.3}$$

for $k \in \mathbb{Z}$.

Using definition (1.1) (or (2.2)), the polynomial

$$A_\ell(x^m) - \left(\frac{1+x+x^2+\cdots+x^{m-1}}{m}\right)^{\ell+1} A_\ell(x)$$

can be expressed as

$$(1-x^m)^{\ell+1}\sum_{k=1}^{\infty}k^\ell x^{mk}-\left(\frac{1+x+x^2+\cdots+x^{m-1}}{m}\right)^{\ell+1}(1-x)^{\ell+1}\sum_{k=1}^{\infty}k^\ell x^k$$
$$=\left(\frac{1-x^m}{m}\right)^{\ell+1}\cdot\left\{m\cdot\sum_{k=1}^{\infty}(mk)^\ell x^{mk}-\sum_{k=1}^{\infty}k^\ell x^k\right\}.$$

It is enough to show that

$$P(x):=(1+x+\cdots+x^{m-1})^{\ell+1}\left\{m\cdot\sum_{k=1}^{\infty}(mk)^\ell x^{mk}-\sum_{k=1}^{\infty}k^\ell x^k\right\} \tag{4.4}$$

becomes a polynomial. Applying (4.3), we have

$$\begin{aligned}P(x)&=(1+x+\cdots+x^{m-1})^{\ell+1}\sum_{k=1}^{\infty}\sum_{j=1}^{m-1}\zeta_m^{jk}k^\ell x^k\\&=\prod_{i=1}^{m-1}(1-\zeta_m^i x)^{\ell+1}\cdot\sum_{k=1}^{\infty}\sum_{j=1}^{m-1}k^\ell(\zeta_m^j x)^k\\&=\sum_{j=1}^{m-1}\left(\prod_{\substack{1\le i\le m-1\\ i\ne j}}(1-\zeta_m^i x)^{\ell+1}\right)\cdot A_\ell(\zeta_m^j x),\end{aligned}$$

which is a polynomial in x. This completes the proof of Theorem 1.1.

Remark. The congruence (1.2) is not optimal when ℓ is even. Indeed, if ℓ is even, the congruence (1.2) holds modulo $(1-x)^{\ell+2}$, which follows from the symmetry $A(\ell,k)=A(\ell,\ell+1-k)$ and $A_\ell(-1)=0$.

5. A characterization of the Eulerian polynomial

In this section, we prove the following.

Theorem 5.1. *Let $f(x)=x^\ell+a_1x^{\ell-1}+\cdots+a_\ell\in\mathbb{C}[x]$ be a monic complex polynomial of degree $\ell>0$. Then, the following are equivalent.*

(a) $f(x)=A_\ell(x)$.

(b) *For any $m\ge 2$, $f(x)$ satisfies the congruence (1.2). Namely,*

$$f(x^m)\equiv\left(\frac{1+x+\cdots+x^{m-1}}{m}\right)^{\ell+1}f(x)\mod (1-x)^{\ell+1} \tag{5.1}$$

is satisfied.

(c) *The congruence for $m=2$ holds, namely,*

$$f(x^2)\equiv\left(\frac{1+x}{2}\right)^{\ell+1}f(x)\mod (1-x)^{\ell+1}$$

is satisfied.

(d) *There exists an integer $m\ge 2$ such that the congruence (5.1) holds.*

Proof. $(a) \Longrightarrow (b)$ is nothing but Theorem 1.1. The implications $(b) \Longrightarrow (c) \Longrightarrow (d)$ are obvious.

Let us assume (d). We shall prove (a). Choose an integer $m \geq 2$ such that (5.1) is satisfied. There exists a polynomial $g(x) \in \mathbb{C}[x]$ that satisfies

$$f(x^m) - \left(\frac{1+x+\cdots+x^{m-1}}{m}\right)^{\ell+1} \cdot f(x) = (1-x)^{\ell+1} \cdot g(x). \tag{5.2}$$

Note that $\deg g = m\ell + m - \ell - 2 < (m-1)(\ell+1)$. Dividing this equation by $(1-x^m)^{\ell+1}$, we have

$$\frac{f(x^m)}{(1-x^m)^{\ell+1}} - \frac{1}{m^{\ell+1}} \cdot \frac{f(x)}{(1-x)^{\ell+1}} = \frac{g(x)}{(1+x+\cdots+x^{m-1})^{\ell+1}}. \tag{5.3}$$

We expand $\frac{g(x)}{(1+\cdots+x^{m-1})^{\ell+1}}$ into a partial fraction,

$$\frac{g(x)}{(1+\cdots+x^{m-1})^{\ell+1}} = \sum_{j=1}^{m-1} \frac{R_j(x)}{(1-\zeta_m^j x)^{\ell+1}}, \tag{5.4}$$

where $R_j(x) \in \mathbb{C}[x]$ with $\deg R_j(x) \leq \ell$. As is well known in the theory of formal power series [11], there exist polynomials $\alpha(t), \beta_j(t) \in \mathbb{C}[t]$ with $\deg \alpha(t), \deg \beta_j(t) \leq \ell$ such that

$$\frac{f(x)}{(1-x)^{\ell+1}} = \sum_{k=0}^{\infty} \alpha(k) x^k, \text{ and } \frac{R_j(\zeta_m^{-j}x)}{(1-x)^{\ell+1}} = \sum_{k=0}^{\infty} \beta_j(k) x^k. \tag{5.5}$$

Then, the right hand side of (5.3) is

$$\begin{aligned} \text{RHS of (5.3)} &= \sum_{j=1}^{m-1} \sum_{k\geq 0} \beta_j(k) \zeta_m^{jk} x^k \\ &= \sum_{j=1}^{m-1} \sum_{r=0}^{m-1} \sum_{q=0}^{\infty} \beta_j(qm+r) \zeta_m^{j(qm+r)} x^{qm+r} \\ &= \sum_{q=0}^{\infty} \sum_{r=0}^{m-1} \left(\sum_{j=1}^{m-1} \zeta_m^{jr} \beta_j(qm+r) \right) x^{qm+r}. \end{aligned} \tag{5.6}$$

On the other hand, the left hand side of (5.3) is

(5.7)

$$\begin{aligned} \text{LHS of (5.3)} &= \sum_{k\geq 0} \alpha(k) x^{km} - \frac{1}{m^{\ell+1}} \sum_{k\geq 0} \alpha(k) x^k \\ &= \sum_{q=0}^{\infty} \left(\alpha(q) - \frac{1}{m^{\ell+1}} \alpha(mq) \right) x^{mq} - \sum_{q=0}^{\infty} \sum_{r=1}^{m-1} \frac{\alpha(mq+r)}{m^{\ell+1}} x^{mq+r}. \end{aligned}$$

Comparison of (5.6) and (5.7) gives

$$
\begin{aligned}
\sum_{j=1}^{m-1} \beta_j(qm) &= \alpha(q) - \frac{1}{m^{\ell+1}}\alpha(mq) \\
\sum_{j=1}^{m-1} \beta_j(qm+1)\zeta_m^j &= -\frac{1}{m^{\ell+1}}\alpha(mq+1) \\
&\vdots \\
\sum_{j=1}^{m-1} \beta_j(qm+m-1)\zeta_m^{j(m-1)} &= -\frac{1}{m^{\ell+1}}\alpha(mq+m-1),
\end{aligned}
\tag{5.8}
$$

for any $q \geq 0$. Since both sides of (5.8) are polynomials in q, we have the following polynomial identities.

$$
\begin{aligned}
\sum_{j=1}^{m-1} \beta_j(t) &= \alpha\left(\frac{t}{m}\right) - \frac{1}{m^{\ell+1}}\alpha(t) \\
\sum_{j=1}^{m-1} \beta_j(t)\zeta_m^j &= -\frac{1}{m^{\ell+1}}\alpha(t) \\
&\vdots \\
\sum_{j=1}^{m-1} \beta_j(t)\zeta_m^{j(m-1)} &= -\frac{1}{m^{\ell+1}}\alpha(t).
\end{aligned}
\tag{5.9}
$$

By summing up all identities in (5.9), we obtain a functional equation

$$\alpha\left(\frac{t}{m}\right) = \frac{1}{m^{\ell}}\alpha(t).$$

This relation is satisfied only by the polynomial of the form $\alpha(t) = c_0 \cdot t^{\ell}$, where $c_0 \in \mathbb{C}$. Again comparing (1.1) and (5.5), $f(x) = c_0 \cdot A_\ell(x)$. Since $f(x)$ is a monic, we have $f(x) = A_\ell(x)$. $\square$

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