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ON THE METRIC DIMENSION OF CIRCULANT GRAPHS WITH 4 GENERATORS

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Abstract. Circulant graphs are Cayley graphs of cyclic groups and the metric dimension of circulant graphs with at most 3 generators has been extensively studied in the last decade. We extend known results in the area by presenting lower and upper bounds on the metric dimension of circulant graphs with 4 generators.

1. Introduction

The concept of metric dimension was introduced by Slater [11] and studied independently by Harary and Melter [4]. Slater referred to a metric dimension of a graph as its location number and motivated the study of this invariant by its application to the placement of minimum number of loran/sonar detecting devices in a network so that the position of every vertex in the network can be uniquely represented in terms of its distances to the devices in the set. In this paper we study the metric dimension of Cayley graphs of cyclic groups known as circulant graphs.

Let G be a connected graph with the vertex set $V(G)$. The distance $d(u,v)$ between two vertices $u, v \in V(G)$ is the number of edges in a shortest path between them. A vertex w resolves a pair of vertices u, v if $d(u,w) \neq d(v,w)$. For an ordered set of vertices $W = \{w_1, w_2, \ldots, w_z\}$, the representation of distances of a vertex v with respect to W is the ordered z-tuple

$$r(v|W) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_z)).$$

A set of vertices $W \subset V(G)$ is a resolving set of G if every two vertices of G have distinct representations (i.e., if every pair of vertices of G is resolved by some vertex of W). The cardinality of a smallest resolving set is called the metric dimension and it is denoted by $\dim(G)$. Note that the i-th coordinate in $r(v|W)$ is 0 if and only if $v = w_i$. This means that in order to show that W is a resolving set of G, it suffices to verify that $r(u|W) \neq r(v|W)$ for every pair of distinct vertices $u, v \in V(G) \setminus W$.

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From [3], it follows that the question whether $\dim(G) < z$ is an NP-complete problem. In [2], it was shown that a connected graph G has $\dim(G) = 1$ if and only if G is a path. Cycles have metric dimension 2. The metric dimension of various classes of graphs has been investigated for four decades. For example, the metric dimension of trees was studied in [11], convex polytopes in [10], and graphs of given diameter were considered in [8].

We introduce a circulant graph. Let n, m and $a_1, a_2, \ldots, a_m$ be positive integers, such that $1 \leq a_1 < a_2 < \cdots < a_m \leq n/2$. The circulant graph $C_n(a_1, a_2, \ldots, a_m)$ consists of the vertices $v_0, v_1, \ldots, v_{n-1}$ and the edges $v_i v_{i+a_j}$ where $0 \leq i \leq n-1$, $1 \leq j \leq m$, the indices are taken modulo n. The numbers $a_1, a_2, \ldots, a_m$ are called generators. The graph $C_n(a_1, a_2, \ldots, a_m)$ is a regular graph either of degree $2m$ if all generators are smaller than $n/2$, or of degree $2m-1$ if $n/2$ is one of the generators.

The distance between two vertices v_i and v_j in $C_n(1,2,3,4)$, where $0 \leq i \leq j < n$, is

$$d(v_i, v_j) = \min\left\{ \left\lceil \frac{j-i}{4} \right\rceil, \left\lceil \frac{n-(j-i)}{4} \right\rceil \right\}. \tag{1.1}$$

This equation can be simplified as

$$d(v_i, v_j) = \begin{cases} \lceil \frac{j-i}{4} \rceil, & \text{if } 0 \leq j-i \leq \frac{n}{2}, \\ \lceil \frac{n-(j-i)}{4} \rceil, & \text{if } \frac{n}{2} \leq j-i < n. \end{cases}$$

The circulant graph $C_n(a_1, a_2, \ldots, a_m)$ is isomorphic to the Cayley graph for the cyclic group Z_n and the generating set $X = \{a_1, a_2, \ldots, a_m, -a_1, -a_2, \ldots, -a_m\}$. A Cayley graph $C(\Gamma, X)$ is specified by a group Γ and an identity-free generating set X for this group such that $X = X^{-1}$. The vertices of $C(\Gamma, X)$ are the elements of Γ and there is an edge between two vertices u and v in $C(\Gamma, X)$ if and only if there is a generator $a \in X$ such that $v = ua$.

The metric dimension of circulant graphs has been extensively studied. Javaid, Rahim and Ali [7] showed that $\dim(C_n(1,2)) = 3$ if $n \equiv 0, 2, 3 \pmod 4$. Imran et. al. [5] showed that $\dim(C_n(1,2,3)) = 4$ if $n \equiv 2, 3, 4, 5 \pmod 6$, $n \geq 14$. Borchert and Gosselin [1] found the values of $\dim(C_n(1,2))$ and $\dim(C_n(1,2,3))$ for any n. They proved that $\dim(C_n(1,2)) = 4$ if $n \equiv 1 \pmod 4$, and for $n \geq 8$ we have $\dim(C_n(1,2,3)) = 5$ if $n \equiv 1 \pmod 6$ and $\dim(C_n(1,2,3)) = 4$ otherwise. The metric dimension of the circulant graphs $C_n(1,3)$ was studied in [6] and the circulant graphs $C_n(1, n/2)$ for even n were considered in [9]. We extend known results on the metric dimension of circulant graphs by presenting lower and upper bounds on the metric dimension of circulant graphs with 4 generators.

2. Resolving sets of $C_n(1,2,3,4)$

We state upper bounds on the metric dimension of the circulant graphs $C_n(1,2,3,4)$ by finding resolving sets having at most 6 vertices for any $n \geq$

17. First let us show that there is an infinite set of circulant graphs with 4 generators, which can be resolved by 4 vertices.

Theorem 2.1. *Let* $n \equiv 4 \pmod 8$ *where* $n \geq 20$. *Then*

$$\dim(C_n(1,2,3,4)) \leq 4.$$

Proof. Let $n = 8k+4$ where $k \geq 2$. We show that $W = \{v_0, v_2, v_{4k}, v_{4k+2}\}$ is a resolving set of $C_n(1,2,3,4)$. Representations of distances of all the vertices in $V(C_n(1,2,3,4)) \setminus W$ with respect to W are given in Table 1.

Representation	v_0	v_2	v_{4k}	v_{4k+2}
v_1	1	1	k	$k+1$
v_{4i-1} $(1 \leq i \leq k)$	i	i	$k-i+1$	$k-i+1$
v_{4i} $(1 \leq i \leq k-1)$	i	i	$k-i$	$k-i+1$
v_{4i+1} $(1 \leq i \leq k-1)$	$i+1$	i	$k-i$	$k-i+1$
v_{4i+2} $(1 \leq i \leq k-1)$	$i+1$	i	$k-i$	$k-i$
v_{4k+1}	$k+1$	k	1	1
$v_{8k-4i+4}$ $(1 \leq i \leq k)$	i	$i+1$	$k-i+1$	$k-i+1$
$v_{8k-4i+5}$ $(1 \leq i \leq k)$	i	$i+1$	$k-i+2$	$k-i+1$
$v_{8k-4i+6}$ $(1 \leq i \leq k)$	i	i	$k-i+2$	$k-i+1$
$v_{8k-4i+7}$ $(1 \leq i \leq k+1)$	i	i	$k-i+2$	$k-i+2$

TABLE 1

Since any two vertices have different representations, $\dim(C_n(1,2,3,4)) \leq 4$. $\square$

Let us prove that if $n \equiv p \mod 8$ where $p = 2,3,5,6$, then there exists a set of 5 vertices which resolves the graphs $C_n(1,2,3,4)$.

Theorem 2.2. *Let* $n \equiv p \mod 8$ *where* $n \geq 18$ *and* $p = 2,3,5,6$. *Then*

$$\dim(C_n(1,2,3,4)) \leq 5.$$

Proof. Let $n = 8k+p$ where $k \geq 2$ and $p = 2,3,5,6$. Let us show that $W = \{v_0, v_1, v_2, v_3, v_4\}$ is a resolving set of $C_n(1,2,3,4)$. First we give representations of distances of the vertices v_i for $5 \leq i \leq 4k+1$ and $4k+p+3 \leq i \leq 8k+p-1$ with respect to W.

Representation	v_0	v_1	v_2	v_3	v_4
v_{4i-3} $(2 \leq i \leq k+1)$	i	$i-1$	$i-1$	$i-1$	$i-1$
v_{4i-2} $(2 \leq i \leq k)$	i	i	$i-1$	$i-1$	$i-1$
v_{4i-1} $(2 \leq i \leq k)$	i	i	i	$i-1$	$i-1$
v_{4i} $(2 \leq i \leq k)$	i	i	i	i	$i-1$
$v_{8k-4i+p}$ $(1 \leq i \leq k-1)$	i	$i+1$	$i+1$	$i+1$	$i+1$
$v_{8k-4i+p+1}$ $(1 \leq i \leq k-1)$	i	i	$i+1$	$i+1$	$i+1$
$v_{8k-4i+p+2}$ $(1 \leq i \leq k-1)$	i	i	i	$i+1$	$i+1$
$v_{8k-4i+p+3}$ $(1 \leq i \leq k)$	i	i	i	i	$i+1$

TABLE 2

Note that no two vertices in Table 2 have the same representations of distances. We distinguish the following cases:

Case 1: $p = 2$.

Representations of distances of v_{4k+2}, v_{4k+3} and v_{4k+4} are in the following table:

Representation	v_0	v_1	v_2	v_3	v_4
v_{4k+2}	k	$k+1$	k	k	k
v_{4k+3}	k	k	$k+1$	k	k
v_{4k+4}	k	k	k	$k+1$	k

TABLE 3

It can be checked that any two distinct vertices of $C_n(1,2,3,4)$ have different representations of distances with respect to W, therefore W is a resolving set of $C_n(1,2,3,4)$.

Case 2: $p = 3$.

We give representations of distances of v_{4k+2}, v_{4k+3}, v_{4k+4}, and v_{4k+5}. The representation of v_{4k+2} can be obtained by using $i = k+1$ for v_{4i-2} in Table 2. Similarly, the representation of v_{4k+5} can be obtained by using $i = k$ for $v_{8k-4i+p+2}$. The remaining two vertices have the following representations:

Representation	v_0	v_1	v_2	v_3	v_4
v_{4k+3}	k	$k+1$	$k+1$	k	k
v_{4k+4}	k	k	$k+1$	$k+1$	k

TABLE 4

No two vertices of $C_n(1,2,3,4)$ have the same representations, therefore $\dim(C_n(1,2,3,4)) \leq 5$.

Case 3: $p = 5$.

It remains to give representations of distances of the vertices v_i for $4k + 2 \leq i \leq 4k + 7$. Representations of $v_{4k+2}, v_{4k+3}, v_{4k+4}$ can be obtained by using $i = k + 1$ for $v_{4i-2}, v_{4i-1}, v_{4i}$ in Table 2 and representations of $v_{4k+5}, v_{4k+6}, v_{4k+7}$ can be obtained by using $i = k$ for $v_{8k-4i+p}, v_{8k-4i+p+1}$, $v_{8k-4i+p}$, respectively. No two representations are the same.

Case 4: $p = 6$.

As in the previous case, consider representations of distances of v_{4i-3}, $v_{4i-2}, v_{4i-1}, v_{4i}$ presented in Table 2 for $2 \leq i \leq k+1$, and representations of $v_{8k-4i+p}, v_{8k-4i+p+1}, v_{8k-4i+p+2}, v_{8k-4i+p+3}$ for $1 \leq i \leq k$ to obtain representations of all the vertices in $V(C_n(1, 2, 3, 4)) \setminus W$, except for v_{4k+5}. Since the distance between v_{4k+5} and any vertex in W is $k + 1$, all the vertices have different representations. The proof is complete. □

In the next three theorems, we present upper bounds on the metric dimension of $C_n(1, 2, 3, 4)$ for $n \equiv p \pmod 8$ where $p = 0, 1, 7$.

Theorem 2.3. *Let* $n \equiv 7 \pmod 8$ *where* $n \geq 23$. *Then*

$$\dim(C_n(1, 2, 3, 4)) \leq 6.$$

Proof. Let $n = 8k + 7$ where $k \geq 2$, and let $W' = \{v_0, v_1, v_2, v_3, v_4\}$. We consider representations of distances of vertices $v_{4i-3}, v_{4i-2}, v_{4i-1}, v_{4i}$ with respect to the vertices in W' given in Table 2 for $2 \leq i \leq k + 1$. Similarly, consider the representations of vertices $v_{8k-4i+p}$, $v_{8k-4i+p+1}$, $v_{8k-4i+p+2}$, $v_{8k-4i+p+3}$ in Table 2 for $1 \leq i \leq k$ and $p = 7$. It remains to give representations of the vertices v_{4k+5} and v_{4k+6} with respect to W'. The distance between any of these two vertices and any vertex in W' is $k + 1$. The vertices v_{4k+5} and v_{4k+6} are the only vertices which are not resolved by W'. However it is easy to find a vertex, for example v_5, which resolves v_{4k+5} and v_{4k+6} (since $d(v_5, v_{4k+5}) = k$ and $d(v_5, v_{4k+6}) = k + 1$). Hence the set $W = \{v_0, v_1, v_2, v_3, v_4, v_5\}$ is a resolving set of $C_n(1, 2, 3, 4)$. □

Theorem 2.4. *Let* $n \equiv 0 \pmod 8$ *where* $n \geq 24$. *Then*

$$\dim(C_n(1, 2, 3, 4)) \leq 6.$$

Proof. Let $n = 8k$ where $k \geq 3$. Let us show that $W = \{v_0, v_1, v_3, v_4, v_6, v_9\}$ is a resolving set of $C_n(1, 2, 3, 4)$. We state representations of distances of all the vertices in $V(C_n(1, 2, 3, 4)) \setminus W$ with respect to $W' = \{v_0, v_1, v_3, v_6\}$.

Representation	v_0	v_1	v_3	v_6
v_2	1	1	1	1
v_5	2	1	1	1
v_{4i-3} $(4 \leq i \leq k)$	i	$i-1$	$i-1$	$i-2$
v_{4i-2} $(3 \leq i \leq k)$	i	i	$i-1$	$i-2$
v_{4i-1} $(2 \leq i \leq k)$	i	i	$i-1$	$i-1$
v_{4i} $(2 \leq i \leq k)$	i	i	i	$i-1$
v_{4k+1}, v_{4k+2}	k	k	k	$k-1$
v_{4k+3}	k	k	k	k
v_{4k+4}	$k-1$	k	k	k
v_{4k+5}	$k-1$	$k-1$	k	k
v_{8k-4i} $(1 \leq i \leq k-2)$	i	$i+1$	$i+1$	$i+2$
$v_{8k-4i+1}$ $(1 \leq i \leq k-2)$	i	i	$i+1$	$i+2$
$v_{8k-4i+2}$ $(1 \leq i \leq k-1)$	i	i	$i+1$	$i+1$
$v_{8k-4i+3}$ $(1 \leq i \leq k-1)$	i	i	i	$i+1$

TABLE 5

Let us present the only vertices which have the same representations:

$$r(v_{4k}|W') = r(v_{4k+1}|W') = r(v_{4k+2}|W'),$$
$$r(v_{4k+5}|W') = r(v_{4k+6}|W').$$

However,

$$\begin{aligned} d(v_4, v_{4k}) &= k-1, & d(v_4, v_{4k+1}) &= k, \\ d(v_9, v_{4k+1}) &= k-2, & d(v_9, v_{4k+2}) &= k-1, \\ d(v_9, v_{4k+5}) &= k-1, & d(v_9, v_{4k+6}) &= k, \end{aligned}$$

which means that all the vertices in $V(C_n(1,2,3,4)) \setminus W$ are resolved by W. □

Theorem 2.5. *Let* $n \equiv 1 \pmod 8$ *where* $n \geq 17$*. Then*

$$\dim(C_n(1,2,3,4)) \leq 6.$$

Proof. Let $n = 8k+1$ where $k \geq 2$, and let $W = \{v_0, v_1, v_4, v_7, v_{4k+2}, v_{4k+3}\}$. We give representations of distances of all the vertices in $V(C_n(1,2,3,4)) \setminus W$ with respect to $W' = \{v_0, v_1, v_7, v_{4k+2}, v_{4k+3}\}$.

Representation	v_0	v_1	v_7	v_{4k+2}	v_{4k+3}
v_2	1	1	2	k	k
v_3	1	1	1	k	k
v_5	2	1	1	k	k
v_6	2	2	1	$k-1$	k
v_{4i-1} $(3 \leq i \leq k)$	i	i	$i-2$	$k-i+1$	$k-i+1$
v_{4i} $(2 \leq i \leq k)$	i	i	$i-1$	$k-i+1$	$k-i+1$
v_{4i-3} $(3 \leq i \leq k)$	i	$i-1$	$i-2$	$k-i+2$	$k-i+2$
v_{4i-2} $(3 \leq i \leq k)$	i	i	$i-2$	$k-i+1$	$k-i+2$
v_{4k+1}	k	k	$k-1$	1	1
v_{4k+4}	k	k	k	1	1
v_{4k+5}	$k-1$	k	k	1	1
v_{4k+6}	$k-1$	$k-1$	k	1	1
v_{4k+7}	$k-1$	$k-1$	k	2	1
$v_{8k-4i+1}$ $(1 \leq i \leq k-2)$	i	$i+1$	$i+2$	$k-i$	$k-i$
$v_{8k-4i+2}$ $(1 \leq i \leq k-2)$	i	i	$i+2$	$k-i$	$k-i$
$v_{8k-4i+3}$ $(1 \leq i \leq k-2)$	i	i	$i+2$	$k-i+1$	$k-i$
$v_{8k-4i+4}$ $(1 \leq i \leq k-1)$	i	i	$i+1$	$k-i+1$	$k-i+1$

TABLE 6

The only vertices which have the same representations are the pairs v_2, v_{8k} and v_{4k}, v_{4k+1}. Since the vertex v_4 resolves the pair v_2, v_{8k} and the vertex v_9 resolves the pair v_{4k}, v_{4k+1},

$$\begin{aligned} d(v_4, v_2) &= k-1, \quad d(v_4, v_{8k}) = 2, \\ d(v_9, v_{4k}) &= k-1, \quad d(v_9, v_{4k+1}) = k, \end{aligned}$$

all the vertices in $V(C_n(1,2,3,4)) \setminus W$ are resolved by W. $\square$

3. Lower bounds on $C_n(1,2,3,4)$

We prove that a resolving set of any circulant graph $C_n(1,2,3,4)$ consists of at least 4 vertices. Note that $d(v_i, v_{i+1}) = d(v_i, v_{i+2}) = d(v_i, v_{i+3}) = d(v_i, v_{i+4}) = 1$, $d(v_i, v_{i+5}) = d(v_i, v_{i+6}) = d(v_i, v_{i+7}) = d(v_i, v_{i+8}) = 2$, $d(v_i, v_{i+9}) = d(v_i, v_{i+10}) = d(v_i, v_{i+11}) = d(v_i, v_{i+12}) = 3, \dots,$ if n is sufficiently large. This implies that if $1 \leq j \leq n/2 - 4$ and $d(v_0, v_j) = p$, then $d(v_0, v_{j+4}) = p+1$ and

$$p = d(v_0, v_j) \leq d(v_0, v_{j+1}) \leq d(v_0, v_{j+2}) \leq d(v_0, v_{j+3}) \leq d(v_0, v_{j+4}) = p+1.$$

Similarly, if $n/2 + 4 \leq j \leq n-1$ and $d(v_0, v_j) = p$, then $d(v_0, v_{j-4}) = p+1$ and

$$p = d(v_0, v_j) \leq d(v_0, v_{j-1}) \leq d(v_0, v_{j-2}) \leq d(v_0, v_{j-3}) \leq d(v_0, v_{j-4}) = p+1.$$

Let $d^o(v_i, v_j)$ be the number of edges in the path $v_i v_{i+1} v_{i+2} \dots v_j$. So if $i \leq j$, then $d^o(v_i, v_j) = j - i$, and if $i > j$, then $d^o(v_i, v_j) = n + (j - i)$. It follows that for i, j, l, such that $0 \le i \le j \le l \le n-1$, we have

$$d^o(v_i, v_j) + d^o(v_j, v_l) + d^o(v_l, v_i) = n. \tag{3.1}$$

Theorem 3.1. *For every $n \ge 24$, we have* $\dim(C_n(1,2,3,4)) \ge 4$.

Proof. We prove the result by contradiction. Suppose that $C_n(1,2,3,4)$ contains a resolving set W' which consists of 3 different vertices v_i, v_j, v_l, where $0 \le i < j < l \le n-1$. Note that (at least) one of the numbers $d^o(v_i, v_j)$, $d^o(v_j, v_l)$, $d^o(v_l, v_i)$ is at most $n/3$, otherwise if all $d^o(v_i, v_j)$, $d^o(v_j, v_l)$, $d^o(v_l, v_i)$ are greater than $n/3$, then $d^o(v_i, v_j) + d^o(v_j, v_l) + d^o(v_l, v_i) > n$ which contradicts (3.1). Similarly, one of $d^o(v_i, v_j)$, $d^o(v_j, v_l)$, $d^o(v_l, v_i)$ must be at least $n/3$, otherwise if all $d^o(v_i, v_j)$, $d^o(v_j, v_l)$, $d^o(v_l, v_i)$ are smaller than $n/3$, then $d^o(v_i, v_j) + d^o(v_j, v_l) + d^o(v_l, v_i) < n$ (which means that the order of $C_n(1,2,3,4)$ is smaller than n).

Without loss of generality we can assume that $d^o(v_i, v_j) \le n/3$ and $d^o(v_l, v_i) \ge n/3$. Due to the symmetry in the graph, we can assume that $v_i = v_0$. Then $1 \le j \le n/3$. Clearly, $d(v_0, v_1) = d(v_0, v_2) = d(v_0, v_3) = d(v_0, v_4) = 1$ and $d(v_0, v_{n-1}) = d(v_0, v_{n-2}) = d(v_0, v_{n-3}) = d(v_0, v_{n-4}) = 1$. We consider two main cases:

Case 1: $l \ge n/2$.

We show that the vertices $v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}$ cannot be resolved by W'. More specifically, we show that one of the three pairs (v_{n-4}, v_{n-3}), (v_{n-3}, v_{n-2}), (v_{n-2}, v_{n-1}) cannot be resolved by v_j and v_l.

We have $d(v_j, v_{n-1}) = p$ for some positive integer p (the shortest path between v_j and v_{n-1} is $v_{n-1}v_0v_1 \dots v_j$), then

$$p = d(v_j, v_{n-1}) \le d(v_j, v_{n-2}) \le d(v_j, v_{n-3}) \le d(v_j, v_{n-4}) \le p+1.$$

This means that v_j can resolve at most one of the pairs (v_{n-4}, v_{n-3}), (v_{n-3}, v_{n-2}), (v_{n-2}, v_{n-1}).

Similarly, the shortest path between v_l and v_{n-4} is $v_l v_{l+1} v_{l+2} \dots v_{n-4}$, so for some positive integer r, we have $r = d(v_l, v_{n-4}) \le d(v_l, v_{n-3}) \le d(v_l, v_{n-2}) \le d(v_l, v_{n-1}) \le r+1$. Thus, v_l can resolve at most one of the pairs (v_{n-4}, v_{n-3}), (v_{n-3}, v_{n-2}), (v_{n-2}, v_{n-1}). It follows that one of the pairs (v_{n-4}, v_{n-3}), (v_{n-3}, v_{n-2}), (v_{n-2}, v_{n-1}) is not resolved by W'. Therefore W' is not a resolving set of $C_n(1,2,3,4)$.

Case 2: $l \le \frac{n}{2}$.

We show that we have two vertices in the set $V' = \{v_1, v_2, v_3, v_4\}$, which are not resolved by W'. Note that the distance between any two vertices in V' is 1. We distinguish three subcases.

Subcase 1: $v_j, v_l \in V'$.

Then for two vertices $v', v'' \in V' \setminus \{v_j, v_l\}$ we have $d(v_j, v') = d(v_j, v'') = 1$, $d(v_l, v') = d(v_l, v'') = 1$, which means that the vertices v' and v'' are not resolved by W'.

Subcase 2: $v_j \in V'$ and $v_l \notin V'$.

Let us denote the vertices in $V' \setminus \{v_j\}$ by v_a, v_b, v_c, where $a < b < c$. We have $d(v_j, v_a) = d(v_j, v_b) = d(v_j, v_c) = 1$ and $d(v_l, v_c) = r$

for some positive integer r (the shortest path between v_l and v_c is $v_c v_{c+1} v_{c+2} \dots v_l$). Then $r = d(v_l, v_c) \le d(v_l, v_b) \le d(v_l, v_a) \le r+1$, which means that v_l can resolve at most one of the pairs (v_a, v_b), (v_b, v_c), so there are 2 vertices in the set $\{v_a, v_b, v_c\}$ having the same representations with respect to W'.

Subcase 3: $v_j, v_l \notin V'$.

Since for some positive integer p, we have $p = d(v_j, v_4) \le d(v_j, v_3) \le d(v_j, v_2) \le d(v_j, v_1) \le p+1$, it follows that the vertex v_j can resolve at most one of the pairs (v_1, v_2), (v_2, v_3), (v_3, v_4). Similarly, $r = d(v_l, v_4) \le d(v_l, v_3) \le d(v_l, v_2) \le d(v_l, v_1) \le r+1$ for some r, which means that v_l can resolve at most one of the pairs (v_1, v_2), (v_2, v_3), (v_3, v_4). It follows that one of the pairs (v_1, v_2), (v_2, v_3), (v_3, v_4) is not resolved by W'.

Thus W' is not a resolving set of $C_n(1,2,3,4)$, a contradiction. Hence $\dim(C_n(1,2,3,4)) \ge 4$. □

We improve the bound given in the previous theorem by showing that if $n \equiv 0,1,6,7 \pmod 8$, then 4 vertices cannot resolve the graphs $C_n(1,2,3,4)$.

Theorem 3.2. *Let $n \equiv p \pmod 8$ where $n \ge 6$ and $p = 0,1,6,7$. Then*

$$\dim(C_n(1,2,3,4)) \ge 5.$$

Proof. Let $n = 8k+q$ where $k \ge 0$ is an integer and $q = 6,7,8,9$. We prove that $\dim(C_n(1,2,3,4)) \ge 5$. Suppose to the contrary that the graph $C_n(1,2,3,4)$ can be resolved by 4 vertices. Let $W' = \{w_1, w_2, w_3, w_4\}$ be a resolving set of $C_n(1,2,3,4)$. Without loss of generality we can assume that $w_1 = v_0$. Let $V' = \{v_{4k+1}, v_{4k+2}, \dots, v_{4k+5}\}$. From (1.1), we know that $d(v_0, v_i) = k+1$ for every $v_i \in V'$ (in general, we have $q-1$ vertices v_{4k+1}, $v_{4k+2}, \dots, v_{4k+q-1}$, which are at distance $k+1$ from v_0, but we consider only the vertices of V' in this proof). Any two vertices $v_i, v_j \in V'$ are adjacent, which means that $d(v_i, v_j) = 1$. For $v_i \notin V'$, we have

$$\begin{aligned} p = d(v_i, v_{4k+1}) &\le d(v_i, v_{4k+2}) \le d(v_i, v_{4k+3}) \\ &\le d(v_i, v_{4k+4}) \le d(v_i, v_{4k+5}) \le p+1 \end{aligned}$$

if $0 \le i \le 4k$, and

$$\begin{aligned} p = d(v_i, v_{4k+5}) &\le d(v_i, v_{4k+4}) \le d(v_i, v_{4k+3}) \\ &\le d(v_i, v_{4k+2}) \le d(v_i, v_{4k+1}) \le p+1 \end{aligned}$$

if $4k+6 \le i \le n-1$, where p is some positive integer. This means that out of the 4 pairs of vertices (v_{4k+1}, v_{4k+2}), (v_{4k+2}, v_{4k+3}), (v_{4k+3}, v_{4k+4}), (v_{4k+4}, v_{4k+5}), at most one pair can be resolved by the vertex $v_i \notin V'$. We distinguish the following cases:

Case 1: $w_2, w_3, w_4 \in V'$.

There are two distinct vertices $v', v'' \in V' \setminus \{w_2, w_3, w_4\}$ and we have $d(w_i, v') = d(w_i, v'') = 1$ for $i = 2,3,4$, which means that the vertices v' and v'' are not resolved by the vertices in W'.

Case 2: $|V' \cap W'| = 2$.

Without loss of generality we can assume that $w_2, w_3 \in V'$ and $w_4 \notin V'$. We have $d(w_2, v') = d(w_3, v') = 1$ for every vertex $v' \in V' \setminus \{w_2, w_3\}$ which means that there are three distinct vertices in V', say v'_1, v'_2, v'_3, which are not resolved by the vertices w_1, w_2, w_3. Since the distances $d(w_4, v'_1)$, $d(w_4, v'_2)$, $d(w_4, v'_3)$ are p or $p+1$ for some positive integer p, it follows that there are two different vertices in V', which cannot be resolved by w_4.

Case 3: $|V' \cap W'| = 1$.

We can assume that $w_2 \in V'$ and $w_3, w_4 \notin V'$. Note that $d(w_2, v') = 1$ for every $v' \in V' \setminus \{w_2\}$. Let us write $V' = \{w_2, v_{4k+a}, v_{4k+b}, v_{4k+c}, v_{4k+d}\}$ where $a < b < c < d$. Since the vertex w_i, $i = 3, 4$, can resolve at most one of the pairs $(v_{4k+a}, v_{4k+b}), (v_{4k+b}, v_{4k+c}), (v_{4k+c}, v_{4k+d})$, at least one pair of vertices of V' cannot be resolved by w_3 and w_4 (and by W').

Case 4: $w_2, w_3, w_4 \notin V'$.

Any vertex w_i, $i = 2, 3, 4$, can resolve at most one of the pairs $(v_{4k+1}, v_{4k+2}), (v_{4k+2}, v_{4k+3}), (v_{4k+3}, v_{4k+4}), (v_{4k+4}, v_{4k+5})$, which means that at least one pair of vertices of V' is not resolved by W'.

Therefore W' is not a resolving set of $C_n(1, 2, 3, 4)$, a contradiction. Hence $\dim(C_n(1, 2, 3, 4)) \geq 5$. □

4. Conclusion

By [1], [5], and [7], for $n \geq 5$ we have

$$\dim(C_n(1,2)) = \begin{cases} 4 & \text{if } n \equiv 1 \pmod{4}, \\ 3 & \text{otherwise,} \end{cases}$$

and for $n \geq 8$

$$\dim(C_n(1,2,3)) = \begin{cases} 5 & \text{if } n \equiv 1 \pmod{6}, \\ 4 & \text{otherwise.} \end{cases}$$

One could have an impression that these results can be generalized to obtain the following: For sufficiently large n

$$\dim(C_n(1,2,\ldots,t)) = \begin{cases} t+2 & \text{if } n \equiv 1 \pmod{2t}, \\ t+1 & \text{otherwise.} \end{cases}$$

However, from our results presented in the previous two sections it follows that this statement does not hold for $t = 4$. The bounds on the metric dimension of $C_n(1, 2, 3, 4)$ for any $n \geq 22$ are presented in Table 7.

$\dim(C_n(1,2,3,4))$	Lower bound	Upper bound
$n \equiv 0 \pmod 8$	5	6
$n \equiv 1 \pmod 8$	5	6
$n \equiv 2 \pmod 8$	4	5
$n \equiv 3 \pmod 8$	4	5
$n \equiv 4 \pmod 8$	4	4
$n \equiv 5 \pmod 8$	4	5
$n \equiv 6 \pmod 8$	5	5
$n \equiv 7 \pmod 8$	5	6

Table 7

In most cases $\dim(C_n(1,2,3,4)) \leq 5$. We conjecture that for any $n \geq 22$ the metric dimension of $C_n(1,2,3,4)$ is equal to the upper bounds presented in Table 7, which would mean that the resolving sets given in Section 2 are the best possible. Note that for $n = 19$, $\dim(C_n(1,2,3,4)) = 4$ and values of $\dim(C_n(1,2,3,4))$ for $n \leq 22$ were presented in [1].

References

1. A. Borchert and S. Gosselin, *The metric dimension of circulant graphs and Cayley hypergraph*, preprint. `http://ion.uwinnipeg.ca/~sgosseli/Borchert%20Gosselin%20August%202014%20web.pdf`.
2. G. Chartrand, L. Eroh, M.A. Johnson, and O.R. Oellermann, *Resolvability in graphs and the metric dimension of a graph*, Discrete Appl. Math. **105** (2000), 99–113.
3. M.R. Garey and D.S. Johnson, *Computers and intractability: A guide to the theory of NP-completeness*, W.H. Freeman and Company, San Francisco, 1979.
4. F. Harary and R.A. Melter, *On the metric dimension of a graph*, Ars Comb. **2** (1976), 191–195.
5. M. Imran, A.Q. Baig, S.A. Bokhary, and I. Javaid, *On the metric dimension of circulant graphs*, Appl. Math. Lett. **25** (2012), 320–325.
6. I. Javaid, M.N. Azhar, and M. Salman, *Metric dimension and determining number of Cayley graphs*, World Applied Sciences Journal **18** (2012), 1800–1812.
7. I. Javaid, M.T. Rahim, and K. Ali, *Families of regular graphs with constant metric dimension*, Util. Math. **75** (2008), 21–33.
8. I. Kousar, I. Tomescu, and M. Husnine, *Graphs with same diameter and metric dimension*, J. Prime Res. Math. **6** (2010), 22–31.
9. M. Salman, I. Javaid, and M.A. Chaudhry, *Resolvability in circulant graphs*, Acta Math. Sin. **28** (2012), 1851–1864.
10. H.M.A. Siddiqui and M. Imran, *Computing the metric dimension of wheel related graphs*, Appl. Math. Comput. **242** (2014), 624–632.
11. P.J. Slater, *Leaves of trees*, Congress. Numer. **14** (1975), 549–559.

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