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STRING C-GROUPS OF ORDER 1024

YASUSHI GOMI, MARK L. LOYOLA,
AND MA. LOUISE ANTONETTE N. DE LAS PEÑAS

Abstract. This paper determines the nondegenerate string C-groups of order 1024. For groups of rank 3, we use the technique of central extension of string C-groups of order 512. For groups of rank at least 4, we compute for quotients of universal string C-groups.

1. Introduction

In the literature, string C-groups have been well-studied in connection with the field of abstract regular polytopes. One of the basic results of this field is the correspondence between regular abstract polytopes and string C-groups acting as their groups of automorphisms [12].

Most known works pertaining to the classification of string C-groups have been for special groups such as [4] for symmetric groups, [5] for alternating groups, and [3, 8, 10] for (almost) simple groups. Studies on string C-groups of small order have been possible with the aid of computational algebra software. In [7], Hartley created an online atlas of regular polytopes with automorphism groups of order at most 2000, excluding 1024 and 1536, via an algorithm implemented using the software GAP [6]. Even with the aid of a computer, the task of determining string C-groups of order 1024 by enumerating and checking the almost 50 billion groups of this order is impracticable. Consequently, it was suggested in [7, 14] that string C-groups of order equal to a power of 2 be characterized first prior to the enumeration of these groups. In 2012, Conder [1], using a procedure that runs in Magma [13], enumerated all non-degenerate string C-groups of order up to 2000 by finding and analyzing all normal subgroups of index at most 2000 in relevant Coxeter groups.

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This paper determines the nondegenerate string C-groups of order 1024 using the following approach: for groups of rank 3, we use the technique of central extension of string C-groups of order 512; for groups of rank at least 4, on the other hand, we compute for quotients of universal string C-groups coming from suitably chosen classes. This approach, however, is not only limited to generating string C-groups of order 1024. It can also be applied to generate string C-groups of order equal to any power of 2, in general.

The paper is organized as follows. In **Section 2**, we recall basic definitions and give known results that will be used in succeeding discussions. In **Sections 3** and **4**, we present the method to obtain the string C-groups of order 1024 with rank 3 and rank at least 4, respectively, and outline the results. Some examples are also presented. In **Section 5**, we discuss separately the case pertaining to an infinite universal string C-group. Finally, in the last section, we present the concluding remarks.

2. Preliminaries

A set $S = \{s_0, s_1, \ldots, s_{n-1}\}$, $n \geq 1$, of distinct involutions in a group Γ can be assigned an edge-labeled graph or diagram $\mathcal{D}(S)$ consisting of n vertices. Each vertex of $\mathcal{D}(S)$ represents an involution and an edge connects the vertices representing s_j and s_k and is labeled $p_{j,k} = \mathrm{ord}(s_j s_k)$ if and only if $p_{j,k} \geq 3$. By convention, we omit the label in the diagram when $p_{j,k} = 3$. If $p_{j,k} = 2$ for $0 \leq j, k \leq n-1$ whenever $|j - k| \geq 2$, then $\mathcal{D}(S)$ is a *string diagram*. We call this property the *string condition*. If, in addition to the string condition, we also have $p_{j,j+1} > 2$ for $0 \leq j \leq n-2$, then $\mathcal{D}(S)$ is a *connected string diagram* (see **Figure 1**).

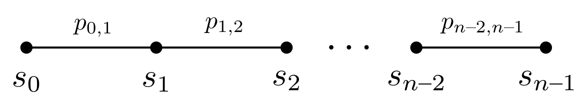

Figure 1. A connected string diagram.

Definition 2.1 (String C-Group). *Let Γ be a group generated by a set $S = \{s_0, s_1, \ldots, s_{n-1}\}$, $n \geq 1$, of involutions called its* distinguished generators*. The pair (Γ, S) is called a* string C-group *of* rank n *with respect to S if $\mathcal{D}(S)$ is a string diagram and S satisfies the property that*

for all $J, K \subseteq \{0, 1, \ldots, n-1\}$, $\langle s_j \mid j \in J\rangle \cap \langle s_k \mid k \in K\rangle = \langle s_i \mid i \in J \cap K\rangle$.

The latter property is called the intersection condition*. Further, we call (Γ, S)* nondegenerate *if $\mathcal{D}(S)$ is a connected string diagram, or* degenerate*, otherwise. If, in addition, the diagram of (Γ, S) is the string diagram in **Figure 1**, we say (Γ, S) has* type $\{p_{0,1}, p_{1,2}, \ldots, p_{n-2,n-1}\}$ *and* dual type $\{p_{n-2,n-1}, p_{n-3,n-2}, \ldots, p_{0,1}\}$.

It is important to remark that a string C-group (Γ, S) always involves a pair consisting of a group Γ and a set S of distinguished generators. If we refer simply to Γ as a string C-group, then it is implied that either a set S of distinguished generators for Γ has been specified or that we can find a set S of generating involutions that satisfies both the string condition and the intersection condition.

For the purposes of verifying the intersection condition for groups generated by 3, 4, or 5 involutions, the following lemmas, which are particular cases of a general result provided in [2], are useful.

Lemma 2.2. *The intersection condition holds for* $\Gamma = \langle s_0, s_1, s_2\rangle$ *if* $\langle s_0, s_1\rangle \cap \langle s_1, s_2\rangle = \langle s_1\rangle$.

Lemma 2.3. *The intersection condition holds for* $\Gamma = \langle s_0, s_1, s_2, s_3\rangle$ *if all the following are true:*

(i) $\langle s_0, s_1, s_2\rangle$ *satisfies the intersection condition (**Lemma 2.2**);*
(ii) $\langle s_0, s_1, s_2\rangle \cap \langle s_1, s_2, s_3\rangle = \langle s_1, s_2\rangle$;
(iii) $\langle s_0, s_1, s_2\rangle \cap \langle s_2, s_3\rangle = \langle s_2\rangle$; *and*
(iv) $\langle s_0, s_1, s_2\rangle \cap \langle s_3\rangle = \langle e\rangle$.

Lemma 2.4. *The intersection condition holds for* $\Gamma = \langle s_0, s_1, s_2, s_3, s_4\rangle$ *if all the following are true:*

(i) $\langle s_0, s_1, s_2, s_3\rangle$ *satisfies the intersection condition (**Lemma 2.3**);*
(ii) $\langle s_0, s_1, s_2, s_3\rangle \cap \langle s_1, s_2, s_3, s_4\rangle = \langle s_1, s_2, s_3\rangle$;
(iii) $\langle s_0, s_1, s_2, s_3\rangle \cap \langle s_2, s_3, s_4\rangle = \langle s_2, s_3\rangle$;
(iv) $\langle s_0, s_1, s_2, s_3\rangle \cap \langle s_3, s_4\rangle = \langle s_3\rangle$; *and*
(v) $\langle s_0, s_1, s_2, s_3\rangle \cap \langle s_4\rangle = \langle e\rangle$.

Below is another helpful result which gives a lower bound for the order of a string C-group given its type.

Lemma 2.5. *Let* $\Gamma = \langle s_0, s_1, \ldots, s_{n-1}\rangle$ *be a string C-group of type* $\{p_{0,1}, p_{1,2}, \ldots, p_{n-2,n-1}\}$. *Then the following right cosets of* $\Gamma_0 = \langle s_i \mid i \neq 0\rangle$:

$$\Gamma_0, \Gamma_0(s_0s_1), \Gamma_0(s_0s_1)^2, \ldots, \Gamma_0(s_0s_1)^{p_{0,1}-1}$$

are all distinct; $|\Gamma| \geq p_{0,1}|\Gamma_0|$; *and* $|\Gamma| \geq 2p_{0,1}p_{1,2}\ldots p_{n-2,n-1}$.

The classification of string C-groups of any order is up to a more stringent isomorphism, which we define below.

Definition 2.6. *The string C-groups* (Γ_1, S_1) *and* (Γ_2, S_2) *are* isomorphic *if there exists a group isomorphism* $\alpha : \Gamma_1 \to \Gamma_2$ *such that* $\alpha(S_1) = S_2$. *In this case, we write* $(\Gamma_1, S_1) \simeq (\Gamma_2, S_2)$ *or* $\Gamma_1 \simeq \Gamma_2$ *if both* S_1 *and* S_2 *are understood.*

In particular, a string C-group Γ generated by $S = \{s_0, s_1, \ldots, s_{n-1}\}$ is isomorphic to its *dual*, the string C-group Γ_d generated by $S_d = \{s'_0 := s_{n-1}, s'_1 := s_{n-2}, \ldots, s'_{n-1} := s_0\}$ of dual type. Note that under this definition of isomorphism, two groups which are isomorphic as abstract groups may not be isomorphic as string C-groups.

If the rank of a string C-group is 1, then it is isomorphic to the cyclic group $\mathbf{Z}_2$. If its rank is 2 and has order $2m$, then it is isomorphic to the dihedral group $\mathbf{D}_m$. For instance, the string C-group of rank 2 and order 1024 must be isomorphic to $\mathbf{D}_{512}$. Hence, in any classification theorem involving string C-groups, one may work with the assumption that a string C-group (Γ, S) has rank at least 3. In addition, we can further assume that $\mathcal{D}(S)$ is connected, since any degenerate string C-group may be expressed as a direct product of proper subgroups corresponding to the connected components of $\mathcal{D}(S)$.

Examples of string C-groups are symmetry groups of regular convex and star-polytopes, automorphism groups of regular abstract polytopes [12], and string Coxeter groups (see [9, p. 113]). The connection between string C-groups and string Coxeter groups is made much deeper and precise in the following known result.

Theorem 2.7. *A string C-group* $\Gamma = \langle s_0, s_1, \ldots, s_{n-1} \rangle$ *of type* $\{p_{0,1}, p_{1,2}, \ldots, p_{n-2,n-1}\}$ *is a* smooth *quotient of the string Coxeter group*

$$W = [p_{0,1}, p_{1,2}, \ldots, p_{n-2,n-1}]$$
$$= \left\langle w_0, w_1, \ldots, w_{n-1} \;\middle|\; \begin{array}{c} w_0^2 = w_1^2 = \ldots = w_{n-1}^2 = e, \\ (w_j w_{j+1})^{p_{j,j+1}} = e \;\; \textit{for } j = 0, 1, \ldots, n-2, \\ (w_j w_k)^2 = e \;\; \textit{for } 0 \leq j, k \leq n-1, \; |j-k| \geq 2 \end{array} \right\rangle.$$

By smooth, we mean that $\mathrm{ord}(s_j s_k) = \mathrm{ord}(w_j w_k) = p_{j,k}$.

Consequently, there exists a group presentation $\mathcal{P}$ for Γ that includes all relations in the presentation for W with w_i replaced by s_i. We call these the *Coxeter relations* in $\mathcal{P}$. Meanwhile, we call the non-Coxeter relations, if any, in $\mathcal{P}$ the *additional relations*. Thus, to specify a string C-group of a given type, we only need to specify its additional relations. In particular, if all relations in $\mathcal{P}$ are Coxeter relations, then $\Gamma \simeq W$.

In this paper, each finite string C-group of type $\{p_{0,1}, p_{1,2}, \ldots, p_{n-2,n-1}\}$ will be denoted by $[p_{0,1}, p_{1,2}, \ldots, p_{n-2,n-1}]^* \,\mathrm{ord}_i$, where ord is the order of the group and i is an index which uniquely identifies that group among other groups (if there is more than one) of the same type and order. On the other hand, the string Coxeter group of this type will be denoted by $[p_{0,1}, p_{1,2}, \ldots, p_{n-2,n-1}]$. The previous notation is almost similar to the one used in *The Atlas of Small Regular Polytopes* [7], which we refer to as *The Atlas* from hereon, a web-based database of string Coxeter groups of order at most 2000, except 1024 and 1536. The only difference is that we use square brackets rather than curly brackets. For instance, the group $[4, 8]^*512_a$ of type $\{4, 8\}$ and order 512, which we will encounter later in the next section, is precisely the group $\{4, 8\}^*512_a$ appearing in *The Atlas* with additional relations

$$(s_0 s_1 s_2 s_1)^4 = (s_2 s_0 (s_1 s_2)^3 s_0 s_1)^2 = e.$$

Finally, we state and prove a result on finitely generated 2-groups that will be used a number of times in this paper.

Lemma 2.8. *Let Γ be a 2-group with minimal generating set $S = \{s_0, s_1, \ldots, s_{n-1}\}$. Then for any nonempty subset $J \subseteq \{0, 1, \ldots, n-1\}$, the map $\varphi_J : \Gamma \to \{-1, 1\} \simeq \mathbf{Z}_2$ sending a generator $s_i \in S$ to*

$$\varphi_J(s_i) = \begin{cases} -1 & \text{if } i \in J, \\ 1 & \text{if } i \notin J \end{cases}$$

defines a surjective group homomorphism.

Proof. Let $\Phi(\Gamma)$ be the Frattini subgroup of Γ. Recall that for a finite p-group Γ, the Frattini subgroup is the smallest normal subgroup N such that Γ/N is an elementary abelian group. In particular, $\Gamma/\Phi(\Gamma) \simeq \mathbf{Z}_2^n$. Define $\rho : \Gamma \to \Gamma/\Phi(\Gamma)$ to be the canonical projection sending s_i to $\overline{s_i} = s_i\Phi(\Gamma)$. Then $\overline{S} = \{\overline{s_0}, \overline{s_1}, \ldots, \overline{s_{n-1}}\}$ is a minimal generating set for $\Gamma/\Phi(\Gamma)$. For any nonempty $J \subseteq \{0, 1, \ldots, n-1\}$, define the map $\psi_J : \Gamma/\Phi(\Gamma) \to \{-1, 1\}$ which sends $\overline{s_i}$ to

$$\psi_J(\overline{s_i}) = \begin{cases} -1 & \text{if } i \in J, \\ 1 & \text{if } i \notin J. \end{cases}$$

Then ψ_J defines a surjective group homomorphism. Consequently, the composition $\varphi_J = (\psi_J \circ \rho)$ also defines a surjective homomorphism from Γ to $\{-1, 1\}$. □

We shall write φ_j instead of $\varphi_{\{j\}}$ for the homomorphism defined in **Lemma 2.8** when $J = \{j\}$.

3. Rank three

We begin this section with two lemmas pertaining to a string C-group of rank 3. We remark that these lemmas may be generalized to a string C-group of any rank. For brevity, we omit their respective proofs.

In what follows, for a string C-group $\Gamma = \langle s_0, s_1, \ldots, s_{n-1} \rangle$ and for $J \subset \{0, 1, \ldots, n-1\}$, we write $\Gamma_J = \langle s_j \mid j \notin J \rangle$. In case $J = \{j\}$, we write Γ_j for $\Gamma_{\{j\}}$. An element in the center $Z(\Gamma)$ of Γ will be described as *central.* A known result in finite group theory states that a finite 2-group has a nontrivial center and, hence, by Cauchy's Theorem, has a central involution.

Lemma 3.1 ([12, p. 57])**.** *Let $\Gamma = \langle s_0, s_1, s_2 \rangle$ be a string C-group of type $\{p_{0,1}, p_{1,2}\}$ and let $N \trianglelefteq \Gamma$ such that $N \cap \Gamma_0\Gamma_2 = \{e\}$. Then $\Gamma/N = \langle \overline{s_0}, \overline{s_1}, \overline{s_2} \rangle$, where $\overline{s_i} = s_iN$ for $i = 0, 1, 2$, is also a string C-group of type $\{p_{0,1}, p_{1,2}\}$.*

Lemma 3.2. *Let $\Gamma = \langle s_0, s_1, s_2 \rangle$ be a string C-group and let $N \trianglelefteq \Gamma$ such that $N \leq \Gamma_0$ (resp. $N \leq \Gamma_2$) and the quotient Γ_0/N (resp. Γ_2/N) is a string C-group. Then $\Gamma/N = \langle \overline{s_0}, \overline{s_1}, \overline{s_2} \rangle$, where $\overline{s_i} = s_iN$ for $i = 0, 1, 2$, is also a string C-group.*

The proof of the following result is immediate from the previous lemma.

Corollary 3.3. *Let $\Gamma = \langle s_0, s_1, s_2\rangle$ be a string C-group of order 2^m and type $\{2k, 2l\}$, where $k, l \geq 2$. If $\tau = (s_0s_1)^k \in Z(\Gamma)$ (resp. $\tau = (s_1s_2)^l \in Z(\Gamma)$), then the quotient $\Gamma/\langle\tau\rangle = \langle\overline{s_0}, \overline{s_1}, \overline{s_2}\rangle$, where $\overline{s_i} = s_i\langle\tau\rangle$ for $i = 0, 1, 2$, is a string C-group of type $\{k, 2l\}$ (resp. $\{2k, l\}$).*

The lemma below characterizes central elements of Γ in $\Gamma_0\Gamma_2$.

Lemma 3.4. *Let $\Gamma = \langle s_0, s_1, s_2\rangle$ be a string C-group of order 2^m and type $\{2k, 2l\}$, where $k, l \geq 2$.*

(i) If $g \in Z(\Gamma) \cap \Gamma_0\Gamma_2$ is not the identity, then g is equal to one of the following:

$$(s_1s_2)^l,\ (s_0s_1)^k,\ (s_1s_2)^l(s_0s_1)^k,\ (s_1s_2)^l s_1(s_0s_1)^k.$$

(ii) If $(s_1s_2)^l(s_0s_1)^k \in Z(\Gamma)$, then $(s_1s_2)^l, (s_0s_1)^k \in Z(\Gamma)$.

(iii) If $(s_1s_2)^l s_1(s_0s_1)^k \in Z(\Gamma)$, then $(s_0s_1)^k$, $(s_1s_2)^l, (s_1s_2)^l(s_0s_1)^k \notin Z(\Gamma)$.

Proof. First observe that since $\Gamma_0 \simeq \mathbf{D}_{2l}$ and $\Gamma_2 \simeq \mathbf{D}_{2k}$, we have $Z(\Gamma_0) = \langle(s_1s_2)^l\rangle$ and $Z(\Gamma_2) = \langle(s_0s_1)^k\rangle$.

(i) From **Lemma 2.5**, we have the following right coset decomposition

$$\Gamma_0\Gamma_2 = \Gamma_0 \cup \Gamma_0(s_0s_1) \cup \Gamma_0(s_0s_1)^2 \cup \ldots \cup \Gamma_0(s_0s_1)^{2k-1}.$$

Let $g \in Z(\Gamma) \cap \Gamma_0\Gamma_2$ be a nonidentity element. Then $g \in \Gamma_0(s_0s_1)^i$ for some $0 \leq i \leq 2k-1$. Write $g = g_0(s_0s_1)^i$, where $g_0 \in \Gamma_0$. Then, since $s_1 \in \Gamma_0$, we have

$$gs_1 = g_0(s_0s_1)^i s_1 = g_0(s_1s_0)^{2k-i}s_1 = g_0s_1(s_0s_1)^{2k-i} \in \Gamma_0(s_0s_1)^{2k-i}$$

and

$$s_1g = s_1g_0(s_0s_1)^i \in \Gamma_0(s_0s_1)^i.$$

Because g is central, $gs_1 = s_1g$ and, hence, $\Gamma_0(s_0s_1)^{2k-i} = \Gamma_0(s_0s_1)^i$. It follows that $i = 0$ or $i = k$. If $i = 0$, then $g = g_0 \in \Gamma_0 \simeq \mathbf{D}_{2l}$ and, hence, $g = (s_1s_2)^l$. If $i = k$, on the other hand, then $g = g_0(s_0s_1)^k$. We then have

$$g_0s_1(s_0s_1)^k = g_0(s_0s_1)^k s_1 = gs_1 = s_1g = s_1g_0(s_0s_1)^k,$$

which implies that g_0 commutes with s_1. Thus, g_0 must be one of $e, s_1, (s_1s_2)^l$, or $(s_1s_2)^l s_1$. If $g_0 = s_1$, then $g = s_1(s_0s_1)^k$ and we have

$$\begin{aligned}(s_0s_1)^{k+1} &= s_0(s_1(s_0s_1)^k) = s_0g\\ &= gs_0 = (s_1(s_0s_1)^k)s_0 = (s_1s_0)^{k+1} = (s_0s_1)^{k-1}\end{aligned}$$

implying that $(s_0s_1)^2 = e$, which is absurd. Thus, we obtain the result.

(ii) If $(s_1s_2)^l(s_0s_1)^k \in Z(\Gamma)$, then we have

$$(s_1s_2)^l s_0(s_0s_1)^k = (s_1s_2)^l(s_0s_1)^k s_0 = s_0(s_1s_2)^l(s_0s_1)^k$$

implying that $(s_1s_2)^l$ commutes with s_0, which means that $(s_1s_2)^l \in Z(\Gamma)$. An analogous argument shows that $(s_0s_1)^k \in Z(\Gamma)$.

(iii) Assume $(s_1s_2)^l s_1(s_0s_1)^k \in Z(\Gamma)$. Suppose $(s_1s_2)^l \in Z(\Gamma)$. Then we have $s_1(s_0s_1)^k \in Z(\Gamma)$, which contradicts *(i)*. We get an analogous contradiction if we assume that $(s_0s_1)^k \in Z(\Gamma)$. Suppose $(s_1s_2)^l(s_0s_1)^k \in Z(\Gamma)$, then by *(ii)*, we have $(s_1s_2)^l, (s_0s_1)^k \in Z(\Gamma)$, which is impossible as we have just seen.

□

Lemma 3.5. *Let $\Gamma = \langle s_0, s_1, s_2\rangle$ be a string C-group of order 2^m and type $\{2k, 2l\}$, where $k, l \geq 2$, then $(s_1s_2)^l s_1(s_0s_1)^k \notin Z(\Gamma)$.*

Proof. Let $\varphi_{\{0,1,2\}} : \Gamma \to \{-1, 1\}$ be the homomorphism defined in **Lemma 2.8**. Then $\varphi_{\{0,1,2\}}((s_1s_2)^l s_1(s_0s_1)^k) = -1$. If K is the kernel of this homomorphism, then $|K| = 2^{m-1}$ and $(s_1s_2)^l s_1(s_0s_1)^k \notin K$. Suppose $(s_1s_2)^l s_1(s_0s_1)^k \in Z(\Gamma)$. It follows that

$$\Gamma = \langle (s_1s_2)^l s_1(s_0s_1)^k\rangle \times K$$

and, hence, $\mathrm{rank}(\Gamma) = \mathrm{rank}(\langle (s_1s_2)^l s_1(s_0s_1)^k\rangle) + \mathrm{rank}(K)$. This gives us $\mathrm{rank}(K) = 2$ which implies that $K \simeq \mathbf{D}_{2^{m-2}}$. Thus, $\Gamma \simeq \mathbf{Z}_2 \times \mathbf{D}_{2^{m-2}}$ is degenerate for any choice of three generating involutions. This contradicts the assumption that Γ has type $\{2k, 2l\}$, where $k, l \geq 2$. □

Theorem 3.6. *Let $\Gamma = \langle s_0, s_1, s_2\rangle$ be a string C-group of order 2^m and type $\{2k, 2l\}$, where $k, l \geq 2$. Then Γ has a central involution τ such that the quotient $\Gamma/\langle\tau\rangle = \langle \overline{s_0}, \overline{s_1}, \overline{s_2}\rangle$, where $\overline{s_i} = s_i\langle\tau\rangle$ for $i = 0, 1, 2$, is also a string C-group.*

Proof. Let $\Gamma = \langle s_0, s_1, s_2\rangle$ be a string C-group of order 2^m and type $\{2k, 2l\}$. If $(s_1s_2)^l$ or $(s_0s_1)^k$ is central, then we take $\tau = (s_1s_2)^l$ or $(s_0s_1)^k$ and obtain a string C-group $\Gamma/\langle\tau\rangle$ by **Corollary 3.3**. Now, we assume that neither $(s_1s_2)^l$ nor $(s_0s_1)^k$ is central. Let τ be any central involution in Γ. Then by **Lemma 3.4** and **Lemma 3.5**, we have $\tau \notin \Gamma_0\Gamma_2$, and by **Lemma 3.1**, $\Gamma/\langle\tau\rangle$ is a string C-group. □

The next corollary is immediate from the previous theorem.

Corollary 3.7. *For any string C-group Γ of order 2^m and rank 3, there is a string C-group $\overline{\Gamma}$ of order 2^{m-1} and rank 3 such that Γ is obtained from a cental extension of $\overline{\Gamma}$ by $\mathbf{Z}_2$.*

Suppose we want to enumerate the string C-groups Γ of order 2^m, rank 3, and type $\{2k, 2l\}$. The restrictions for the values of k and l for a fixed m are implied by **Lemma 2.5**. If $\Gamma = \langle s_0, s_1, s_2\rangle$, then its Coxeter relations are

$$s_0^2 = s_1^2 = s_2^2 = e \text{ and } (s_0s_1)^{2k} = (s_0s_2)^2 = (s_1s_2)^{2l} = e.$$

It then suffices to find the non-Coxeter defining relations for Γ.

If Γ is degenerate, then Γ has type $\{2, 2^{m-2}\}$ and is isomorphic to $\mathbf{Z}_2 \times \mathbf{D}_{2^{m-2}}$. In this case, Γ is defined only by Coxeter relations. It then suffices to consider the case when Γ is nondegenerate. By **Corollary 3.7**, Γ is a central extension by the cyclic group $\mathbf{Z}_2$ of a string C-group $\overline{\Gamma}$ of order 2^{m-1} and rank 3. That is, there exists a central involution $\tau \in \Gamma$ such that $\overline{\Gamma} \simeq \Gamma/\langle\tau\rangle = \langle\overline{s_0}, \overline{s_1}, \overline{s_2}\rangle$, where $\overline{s_i} = s_i\langle\tau\rangle$ for $i = 0, 1, 2$. In addition, $\overline{\Gamma}$ has one of the following three types: $\{k, 2l\}$, $\{2k, l\}$, $\{2k, 2l\}$.

Since τ is a central involution,

$$\tau^2 = (s_0\tau)^2 = (s_1\tau)^2 = (s_2\tau)^2 = e$$

are additional relations for Γ. Now, let

$$R_1(\overline{s_0}, \overline{s_1}, \overline{s_2}) = R_2(\overline{s_0}, \overline{s_1}, \overline{s_2}) = \ldots = R_b(\overline{s_0}, \overline{s_1}, \overline{s_2}) = e$$

be the additional defining relations for $\overline{\Gamma}$. Since $R_i(\overline{s_0}, \overline{s_1}, \overline{s_2}) = e$ is equivalent to $R_i(s_0, s_1, s_2)\langle\tau\rangle = \langle\tau\rangle$, we must have, for each $i = 1, 2, \ldots, b$, an additional relation for Γ given by

$$R_i(s_0, s_1, s_2) = e \text{ or } \tau.$$

The choice for the central involution τ depends on the type of $\overline{\Gamma}$. If $\overline{\Gamma}$ has type $\{k, 2l\}$ or $\{2k, l\}$, then by **Corollary 3.3**, $\tau = (s_0s_1)^k$ or $(s_1s_2)^l$, respectively. If $\overline{\Gamma}$ has type $\{2k, 2l\}$, we assert that $\tau = R_j(s_0, s_1, s_2)$ for some j. Suppose that this is not the case. Then $R_i(s_0, s_1, s_2) = e$ for each $i = 1, 2, \ldots, b$. Thus, except for the defining relations of Γ involving τ, Γ, and $\overline{\Gamma}$ are defined by the exact same relations. It then follows that if $\tau \in \langle\overline{s_0}, \overline{s_1}, \overline{s_2}\rangle$, then Γ is a quotient of $\overline{\Gamma}$. On the other hand, if $\tau \notin \langle\overline{s_0}, \overline{s_1}, \overline{s_2}\rangle$, then $\Gamma \simeq \overline{\Gamma} \times \langle\tau\rangle$. Since $\mathrm{rank}(\Gamma) = \mathrm{rank}(\overline{\Gamma}) + \mathrm{rank}(\langle\tau\rangle)$ and $\mathrm{rank}(\Gamma) = \mathrm{rank}(\overline{\Gamma}) = 3$, τ must be the identity, whence $\Gamma \simeq \overline{\Gamma}$. In either case, the assumption that $R_i(s_0, s_1, s_2) = e$ for each $i = 1, 2, \ldots, m$ leads to the conclusion that $|\Gamma| \leq |\overline{\Gamma}|$. This proves the assertion.

We summarize the results on the defining relations in the presentation of Γ in the following theorem.

Theorem 3.8. *Let $\overline{\Gamma} = \langle\overline{s_0}, \overline{s_1}, \overline{s_2}\rangle$ be a nondegenerate string C-group of order 2^{m-1} defined by b additional relations $R_i(\overline{s_0}, \overline{s_1}, \overline{s_2}) = e$ for $i = 1, 2, \ldots, b$. Suppose $\Gamma = \langle s_0, s_1, s_2\rangle$ is a string C-group of order 2^m and type $\{2k, 2l\}$ obtained from a central extension by $\langle\tau\rangle \simeq \mathbf{Z}_2$ of $\overline{\Gamma}$. Then Γ belongs to one of the three classes of defining relations in **Table 1**.*

Note that a group Γ resulting from the method presented above may not necessarily be a string C-group. In fact, we may sometimes encounter the case when $|\Gamma| < 2^m$. Thus, it is imperative to check that all relations defining Γ will yield a group of order 2^m and will satisfy the intersection condition (refer to **Lemma 2.2**). These conditions are easily verifiable using the computational group theory software GAP [6].

Class of Γ	Type of $\overline{\Gamma}$	Relations for Γ
I	$\{k, 2l\}$	$s_0^2 = s_1^2 = s_2^2 = e,$ $(s_0s_1)^{2k} = (s_0s_2)^2 = (s_1s_2)^{2l} = e,$ $\tau = (s_0s_1)^k,\ \tau^2 = (s_0\tau)^2 = (s_1\tau)^2 = (s_2\tau)^2 = e,$ $R_i(s_0, s_1, s_2) = e$ or τ for $i = 1, 2, \ldots, b$
II	$\{2k, l\}$	$s_0^2 = s_1^2 = s_2^2 = e,$ $(s_0s_1)^{2k} = (s_0s_2)^2 = (s_1s_2)^{2l} = e,$ $\tau = (s_1s_2)^l,\ \tau^2 = (s_0\tau)^2 = (s_1\tau)^2 = (s_2\tau)^2 = e,$ $R_i(s_0, s_1, s_2) = e$ or τ for $i = 1, 2, \ldots, b$
III	$\{2k, 2l\}$	$s_0^2 = s_1^2 = s_2^2 = e,$ $(s_0s_1)^{2k} = (s_0s_2)^2 = (s_1s_2)^{2l} = e,$ $\tau = R_j(s_0, s_1, s_2)$ for some j, $\tau^2 = (s_0\tau)^2 = (s_1\tau)^2 = (s_2\tau)^2 = e,$ $R_i(s_0, s_1, s_2) = e$ or τ for $i \neq j$

TABLE 1. Classes and defining relations of nondegenerate string C-groups of order 2^m, rank 3, and type $\{2k, 2l\}$.

We now use the method described above to enumerate string C-groups Γ of order 1024 and rank 3. If Γ is degenerate, then Γ has type $\{2, 256\}$ (up to duality) and is isomorphic to $\mathbf{Z}_2 \times \mathbf{D}_{256}$. Otherwise, by **Lemma 2.5**, we have the following possible types (up to duality) for Γ: $\{4, 2^a\}$ for $2 \leq a \leq 7$; $\{8, 2^a\}$ for $3 \leq a \leq 6$; and $\{16, 2^a\}$ for $4 \leq a \leq 5$. We illustrate in **Example 3.9** how to proceed with the method when Γ has type $\{4, 8\}$.

Example 3.9. *Suppose $\Gamma = \langle s_0, s_1, s_2 \rangle$ has type $\{4, 8\}$. From **Theorem 3.8**, we determine the class of relations of Γ by identifying τ and specifying for each i whether $R_i(s_0, s_1, s_2) = e$ or τ. We split the possibilities depending on the type of the string C-group $\overline{\Gamma}$ of order 512 and rank 3. The Atlas gives the defining relations for each $\overline{\Gamma}$.*

Class I. *If $\overline{\Gamma}$ has type $\{2, 8\}$, then $\overline{\Gamma} \simeq \mathbf{Z}_2 \times \mathbf{D}_8$. Since $|\overline{\Gamma}| = 32$, Γ cannot be a central extension of a string C-group of type $\{2, 8\}$.*

Class II. *If $\overline{\Gamma}$ has type $\{4, 4\}$, then $\tau = (s_1s_2)^4$. Only one group $\overline{\Gamma}$ has this type. Its additional relation is given by $R_1(\overline{s_0}, \overline{s_1}, \overline{s_2}) = (\overline{s_0}\ \overline{s_1}\ \overline{s_2}\ \overline{s_1})^8 = e$. Thus, depending on whether $R_1(s_0, s_1, s_2) = e$ or τ, we obtain the groups $[4, 8]^*1024_a$ and $[4, 8]^*1024_b$, respectively. It can be checked through GAP that these two groups are indeed non-isomorphic string C-groups of order 1024.*

Class III. *If $\overline{\Gamma}$ has type $\{4, 8\}$, then $\overline{\Gamma}$ is one of the following four groups with additional relations $R_1(\overline{s_0}, \overline{s_1}, \overline{s_2}) = e$ and $R_2(\overline{s_0}, \overline{s_1}, \overline{s_2}) = e$ given in **Table 2**.*

*Thus, depending on whether $R_1(s_0, s_1, s_2) = e$ or τ and $R_2(s_0, s_1, s_2) = e$ or τ (see assignments in **Table 3**), we obtain the six additional* *groups* $[4,8]^*1024_c$, $[4,8]^*1024_d$, $[4,8]^*1024_e$, $[4,8]^*1024_f$, $[4,8]^*1024_g$, *and* $[4,8]^*1024_h$. *Likewise, it can be checked through GAP that these six groups are indeed nonisomorphic string C-groups of order 1024. A summary of the results of the computations for this case is provided in* ***Table 4***.

$\overline{\Gamma}$	$R_1(\overline{s_0},\overline{s_1},\overline{s_2})$	$R_2(\overline{s_0},\overline{s_1},\overline{s_2})$
$[4,8]^*512_a$	$(\overline{s_0}\ \overline{s_1}\ \overline{s_2}\ \overline{s_1})^4$	$(\overline{s_2}\ \overline{s_0}\ (\overline{s_1}\ \overline{s_2})^3\ \overline{s_0}\ \overline{s_1})^2$
$[4,8]^*512_b$	$(\overline{s_2}\ \overline{s_0}\ \overline{s_1})^8$	$(\overline{s_0}\ (\overline{s_1}\ \overline{s_2})^3\ \overline{s_1})^2$
$[4,8]^*512_c$	$(\overline{s_2}\ (\overline{s_0}\ \overline{s_1}\ \overline{s_2}\ \overline{s_1})^2)^2$	$(\overline{s_2}\ \overline{s_0}\ (\overline{s_1}\ \overline{s_2})^3\ \overline{s_0}\ \overline{s_1})^2$
$[4,8]^*512_d$	$(\overline{s_0}\ (\overline{s_1}\ \overline{s_2})^3\ \overline{s_1})^2$	$(\overline{s_2}\ \overline{s_0}\ \overline{s_1})^5\ \overline{s_2}\ (\overline{s_1}\ \overline{s_0})^2\ (\overline{s_2}\ \overline{s_1})^2\ \overline{s_0}\ \overline{s_1}$

Table 2. Additional relations $R_1(\overline{s_0},\overline{s_1},\overline{s_2}) = e$, $R_2(\overline{s_0},\overline{s_1},\overline{s_2}) = e$ defining string C-groups $\overline{\Gamma}$ of order 512, rank 3, and type $\{4,8\}$.

$\overline{\Gamma}$	$\tau = R_1(s_0,s_1,s_2)$, $R_2(s_0,s_1,s_2) = e$	$\tau = R_2(s_0,s_1,s_2)$, $R_1(s_0,s_1,s_2) = e$	$\tau = R_1(s_0,s_1,s_2)$, $R_2(s_0,s_1,s_2) = \tau$
$[4,8]^*512_a$	$[4,8]^*1024_c$	$[4,8]^*1024_d$	$[4,8]^*1024_e$
$[4,8]^*512_b$	$[4,8]^*1024_a$	$[4,8]^*1024_c$	$[4,8]^*1024_f$
$[4,8]^*512_c$	$[4,8]^*1024_c$	$[4,8]^*1024_g$	$[4,8]^*1024_h$
$[4,8]^*512_d$	$[4,8]^*512_d$	$[4,8]^*1024_a$	$[4,8]^*512_d$

Table 3. String C-groups of order 1024, rank 3, and type $\{4,8\}$ in Class III.

Γ	Class of Γ	Additional Relations for Γ
$[4,8]^*1024_a$	II	$(s_0s_1s_2s_1)^8 = e$
$[4,8]^*1024_b$	II	$(s_0s_1s_2s_1)^8 = \tau$
$[4,8]^*1024_c$	III	$\tau = (s_0s_1s_2s_1)^4$, $(s_2s_0(s_1s_2)^3s_0s_1)^2 = e$
$[4,8]^*1024_d$	III	$\tau = (s_2s_0(s_1s_2)^3s_0s_1)^2$, $(s_0s_1s_2s_1)^4 = e$
$[4,8]^*1024_e$	III	$\tau = (s_0s_1s_2s_1)^4$, $(s_2s_0(s_1s_2)^3s_0s_1)^2 = e$
$[4,8]^*1024_f$	III	$\tau = (s_2s_0s_1)^8$, $(s_0(s_1s_2)^3s_1)^2 = e$
$[4,8]^*1024_g$	III	$\tau = (s_2s_0(s_1s_2)^3s_0s_1)^2$, $(s_2(s_0s_1s_2s_1)^2)^2 = e$
$[4,8]^*1024_h$	III	$\tau = (s_2(s_0s_1s_2s_1)^2)^2$, $(s_2s_0(s_1s_2)^3s_0s_1)^2 = e$

Table 4. String C-groups of order 1024, rank 3, and type $\{4,8\}$.

Enumeration for other types can be done analogously. We summarize the results and provide in **Table 5** the number of nondegenerate string C-groups of rank 3 for each type, up to isomorphism.

Type	Count
$\{4,4\}$	1
$\{4,8\}$	8
$\{4,16\}$	14
$\{4,32\}$	4
$\{4,64\}$	2
$\{4,128\}$	2
$\{8,8\}$	22
$\{8,16\}$	40
$\{8,32\}$	8
$\{8,64\}$	4
$\{16,16\}$	10
$\{16,32\}$	10

Table 5. The number of nondegenerate string C-groups of order 1024 and rank 3.

4. Rank at least four

The enumeration of non-degenerate string C-groups of order 1024 and rank at least 4 in this section will follow an approach that is different from the method of *central extension* employed in the previous section. This method, which is adopted from Chapter 4 of [12], computes for smooth quotients of *universal string C-groups*.

Definition 4.1. *Let $\mathcal{C}(\Gamma',\Gamma'')$ be the class or collection of all string C-groups $\langle t_0,t_1,\dots,t_{n-1}\rangle$ of rank n and type $\{p_{0,1},p_{1,2},\dots,p_{n-2,n-1}\}$ such that $\langle t_0,t_1,\dots,t_{n-2}\rangle \simeq \Gamma'$ and $\langle t_1,t_2,\dots,t_{n-1}\rangle \simeq \Gamma''$. We define $\mathbf{U}(\Gamma',\Gamma'')$ to be the group generated by $s_0,s_1,\dots,s_{n-1}$ with set of defining relations formed by combining the following:*

(i) the Coxeter relations of a group of type $\{p_{0,1},p_{1,2},\dots,p_{n-2,n-1}\}$;
(ii) $R'_1(s_0,s_1,\dots,s_{n-2}) = R'_2(s_0,s_1,\dots,s_{n-2}) = \dots = R'_{b'}(s_0,s_1,\dots,s_{n-2}) = e$; and
(iii) $R''_1(s_1,s_2,\dots,s_{n-1}) = R''_2(s_1,s_2,\dots,s_{n-1}) = \dots = R''_{b''}(s_1,s_2,\dots,s_{n-1}) = e$,

where

$$R'_1(t_0,t_1,\dots,t_{n-2}) = R'_2(t_0,t_1,\dots,t_{n-2}) = \dots = R'_{b'}(t_0,t_1,\dots,t_{n-2}) = e$$

and

$$R''_1(t_1,t_2,\dots,t_{n-1}) = R''_2(t_1,t_2,\dots,t_{n-1}) = \dots = R''_{b''}(t_1,t_2,\dots,t_{n-1}) = e$$

are the additional relations defining Γ' and Γ'', respectively. If $\mathbf{U}(\Gamma',\Gamma'')$ is a string C-group, we call it the universal string C-group in the class $\mathcal{C}(\Gamma',\Gamma'')$.

When Γ' and Γ'' are clear from context, we use the shorthand $\mathbf{U}$ *to denote the universal string C-group.*

The next theorem characterizes string C-groups in a class.

Theorem 4.2 ([12, p. 97])**.** *Suppose Γ' and Γ'' are string C-groups of rank $n-1$. If the class $\mathcal{C}(\Gamma', \Gamma'')$ is nonempty, then every string C-group in this class is a quotient of $\mathbf{U}(\Gamma', \Gamma'')$.*

It then follows from the above discussion that to enumerate nondegenerate string C-groups Γ of order 1024 and rank at least 4, we need to analyze the universal string C-group in each class to which Γ belongs. To identify these classes, we consider the restrictions provided in **Lemma 2.5**. If Γ has rank 4, then it has one of the following types (up to duality): $\{4, 4, 2^a\}$ for $2 \leq a \leq 5$; $\{4, 8, 2^a\}$ for $2 \leq a \leq 4$; $\{4, 16, 2^a\}$ for $2 \leq a \leq 3$; $\{4, 32, 4\}$; $\{8, 4, 2^e\}$ for $3 \leq a \leq 4$; and $\{8, 8, 8\}$. On the other hand, if Γ has rank greater than 4, then it must have rank 5 and must have one of the following types (up to duality): $\{4, 4, 4, 2^a\}$ where $a = 2, 3$ or $\{4, 4, 8, 4\}$. Further, if Γ belongs to class $\mathcal{C}(\Gamma', \Gamma'')$, then $|\Gamma'|$, $|\Gamma''| \leq 256$.

To significantly reduce the classes to be considered in the enumeration, we make use of the fact that if Γ'_d and Γ''_d are the duals of Γ' and Γ'', respectively, then $\mathbf{U}(\Gamma', \Gamma'')$ and $\mathbf{U}(\Gamma''_d, \Gamma'_d)$ are duals of each other. Thus, once the class $\mathcal{C}(\Gamma', \Gamma'')$ has been considered, its dual class $\mathcal{C}(\Gamma''_d, \Gamma'_d)$ can be disregarded.

From each class, we determine whether the group $\mathbf{U} := \mathbf{U}(\Gamma', \Gamma'')$ described in **Definition 4.1** is a universal string C-group of order at least 1024. Since, by definition, $\mathbf{U}$ has a string Coxeter diagram, it is sufficient to check only the intersection condition. As a consequence of the main result in [2], the only required statements needed to be verified to satisfy the intersection condition for groups of rank 4 or rank 5 are provided in **Lemma 2.3** and **Lemma 2.4**.

If $\mathbf{U}$ is a string C-group, then based on $|\mathbf{U}|$, we proceed with the identification of all smooth quotients of this group which are string C-groups of order 1024. If $|\mathbf{U}| < 1024$, we do not obtain such a string C-group in the class. If $1024 \leq |\mathbf{U}| < \infty$, on the other hand, we use GAP to enumerate these quotients. We do this by listing down all normal subgroups $N \trianglelefteq \mathbf{U}$ of index 1024, forming the quotient groups $\mathbf{U}/N$, and checking each quotient for smoothness and satisfaction of the intersection condition. In particular, if $|\mathbf{U}| = 1024$, then we take $N = 1$. As we shall see later, there is only a single class in which $|\mathbf{U}| = \infty$. This class will be considered special and will be dealt with separately in **Section 5**. We illustrate the method in more detail through specific examples below.

Example 4.3. *Consider the string C-groups $\Gamma' = [4, 4]^*32 = \langle t_0, t_1, t_2 \rangle$ of order 32 with additional relation $(t_0t_1t_2t_1)^2 = e$ and $\Gamma'' = [4, 4]^*128 = \langle t_1, t_2, t_3 \rangle$ of order 128 with additional relation $(t_1t_2t_3t_2)^4 = e$. This pair of groups defines the class $\mathcal{C}([4, 4]^*32, [4, 4]^*128)$. Using GAP, it can be verified* *that the group* $\mathbf{U} = \langle s_0, s_1, s_2, s_3\rangle$ *of type* $\{4,4,4\}$ *defined by the relations*

$$s_0^2 = s_1^2 = s_2^2 = s_3^2 = e,$$
$$(s_0s_1)^4 = (s_0s_2)^2 = (s_0s_3)^2 = (s_1s_2)^4 = (s_1s_3)^2 = (s_2s_3)^4 = e,$$
$$(s_0s_1s_2s_1)^2 = (s_1s_2s_3s_2)^4 = e$$

is the universal string C-group in the class and has order 512. Consequently, we do not obtain a string C-group of order 1024 in the class.

Example 4.4. *Let us take the string C-groups* $\Gamma' = [4,4]^*32 = \langle t_0, t_1, t_2\rangle$ *in* ***Example 4.3*** *and* $\Gamma'' = [4,4]^*256 = \langle t_1, t_2, t_3\rangle$ *of order 256 with additional relation* $(t_3t_1t_2)^8 = e$*. This pair of groups defines the class* $\mathcal{C}([4,4]^*32,\ [4,4]^*256)$*. Using GAP, it can be verified that the group* $\mathbf{U} = \langle s_0, s_1, s_2, s_3\rangle$ *of type* $\{4,4,4\}$ *defined by the relations*

$$s_0^2 = s_1^2 = s_2^2 = s_3^2 = e,$$
$$(s_0s_1)^4 = (s_0s_2)^2 = (s_0s_3)^2 = (s_1s_2)^4 = (s_1s_3)^2 = (s_2s_3)^4 = e,$$
$$(s_0s_1s_2s_1)^2 = (s_3s_1s_2)^8 = e$$

is the universal string C-group in the class and has order 1024. We denote this group by $[4,4,4]^*1024_a$*.*

Example 4.5. *Let us now consider the string C-groups* $\Gamma' = [4,4]^*64 = \langle t_0, t_1, t_2\rangle$ *of order 64 with additional relation* $(t_2t_0t_1)^4 = e$ *and* $\Gamma'' = [4,8]^*128_a = \langle t_1, t_2, t_3\rangle$ *of order 128 with additional relation* $t_3t_1t_2t_3(t_2t_1)^2(t_3t_2)^2t_1t_2 = e$*. This pair of groups defines the class* $\mathcal{C}([4,4]^*64,\ [4,8]^*128_a)$*. Using GAP, it can be verified that the group* $\mathbf{U} = \langle s_0, s_1, s_2, s_3\rangle$ *of type* $\{4,4,8\}$ *defined by the relations*

$$s_0^2 = s_1^2 = s_2^2 = s_3^2 = e,$$
$$(s_0s_1)^4 = (s_0s_2)^2 = (s_0s_3)^2 = (s_1s_2)^4 = (s_1s_3)^2 = (s_2s_3)^8 = e,$$
$$(s_2s_0s_1)^4 = s_3s_1s_2s_3(s_2s_1)^2(s_3s_2)^2s_1s_2 = e$$

is the universal string C-group in the class and has order 2048.

To determine the string C-groups of type $\{4,4,8\}$ *which are quotients of the universal group, we list down all normal subgroups* N *of index 1024, take the quotient* $\mathbf{U}/N$ *for each* N*, and check which of the resulting quotients are smooth and satisfy the intersection condition. Note in this case that* N *must be generated by a central involution* $\tau \in \mathbf{U}$*, since* $|\mathbf{U}| = 2048$*. We accomplish these tasks with the help of GAP, which gave us three values for* τ*:*

(i) *If* $\tau = s_1s_2s_3s_2s_1s_0s_1(s_3s_2)^2s_3s_1s_0$*, then* $\mathbf{U}/N$ *is a smooth quotient and satisfies the intersection condition. We denote the resulting string C-group by* $[4,4,8]^*1024_e$*.*

(ii) *If* $\tau = (s_1 s_2 s_3 s_2 s_1 s_0)^2$, *then* $\mathbf{U}/N$ *is likewise a smooth quotient and satisfies the intersection condition. We denote the resulting string C-group by* $[4,4,8]^*1024_f$.

(iii) *If* $\tau = (s_2 s_3)^4$, *then* $\mathbf{U}/N$ *satisfies the intersection condition. However, it is not a smooth quotient since it has type* $\{4,4,4\}$.

We summarize the results for each type and provide in **Table 6** the number of nondegenerate string C-groups of rank at least 4, up to isomorphism. Notice in this table that only when Γ has type $\{4,4,4\}$ and belongs to the class defined by $\Gamma' = [4,4]^*128$ and $\Gamma'' = [4,4]^*64$ do we obtain an infinite universal string C-group $\mathbf{U}$. We thus concentrate on this class in the next section and enumerate the finite string C-groups belonging to it. The enumeration of these groups entails an analysis of $\mathbf{U}$ and its subgroup structure. Since $\mathbf{U}$ is directly linked to the string Coxeter group $[4,4]$, we begin the next section by stating results for smooth quotients of this Coxeter group.

Type	Γ'	Γ''	$\lvert\mathbf{U}(\Gamma',\Gamma'')\rvert$	Count
$\{4,4,4\}$	$[4,4]^*32$	$[4,4]^*256$	1024	1
$\{4,4,4\}$	$[4,4]^*64$	$[4,4]^*64$	1024	1
$\{4,4,4\}$	$[4,4]^*128$	$[4,4]^*64$	∞	2
$\{4,4,8\}$	$[4,4]^*32$	$[4,8]^*256_a$	1024	1
$\{4,4,8\}$	$[4,4]^*32$	$[4,8]^*256_b$	1024	1
$\{4,4,8\}$	$[4,4]^*32$	$[4,8]^*256_c$	1024	1
$\{4,4,8\}$	$[4,4]^*32$	$[4,8]^*256_d$	1024	1
$\{4,4,8\}$	$[4,4]^*64$	$[4,8]^*128_a$	2048	2
$\{4,4,8\}$	$[4,4]^*64$	$[4,8]^*128_b$	2048	2
$\{4,4,8\}$	$[4,4]^*128$	$[4,8]^*64_a$	1024	1
$\{4,4,8\}$	$[4,4]^*128$	$[4,8]^*64_b$	1024	1
$\{4,4,16\}$	$[4,4]^*32$	$[4,16]^*256_a$	1024	1
$\{4,4,16\}$	$[4,4]^*32$	$[4,16]^*256_b$	1024	1
$\{4,4,16\}$	$[4,4]^*64$	$[4,16]^*128_a$	1024	1
$\{4,4,16\}$	$[4,4]^*64$	$[4,16]^*128_b$	1024	1
$\{4,4,32\}$	$[4,4]^*32$	$[4,32]^*256_a$	1024	1
$\{4,4,32\}$	$[4,4]^*32$	$[4,32]^*256_b$	1024	1
$\{4,8,4\}$	$[4,8]^*64_a$	$[8,4]^*256_a$	1024	1
$\{4,8,4\}$	$[4,8]^*64_a$	$[8,4]^*256_b$	1024	1
$\{4,8,4\}$	$[4,8]^*64_a$	$[8,4]^*256_c$	1024	1
$\{4,8,4\}$	$[4,8]^*64_a$	$[8,4]^*256_d$	1024	1
$\{4,8,4\}$	$[4,8]^*64_b$	$[8,4]^*256_a$	1024	1
$\{4,8,4\}$	$[4,8]^*64_b$	$[8,4]^*256_b$	1024	1

Type	Γ'	Γ''	$\lvert\mathbf{U}(\Gamma',\Gamma'')\rvert$	Count
$\{4,8,4\}$	$[4,8]^*64_b$	$[8,4]^*256_c$	1024	1
$\{4,8,4\}$	$[4,8]^*64_b$	$[8,4]^*256_d$	1024	1
$\{4,8,4\}$	$[4,8]^*128_a$	$[8,4]^*128_a$	2048	2
$\{4,8,4\}$	$[4,8]^*128_a$	$[8,4]^*128_b$	2048	2
$\{4,8,4\}$	$[4,8]^*128_b$	$[8,4]^*128_b$	2048	2
$\{4,8,8\}$	$[4,8]^*64_a$	$[8,8]^*256_a$	1024	1
$\{4,8,8\}$	$[4,8]^*64_a$	$[8,8]^*256_b$	1024	1
$\{4,8,8\}$	$[4,8]^*64_a$	$[8,8]^*256_c$	1024	1
$\{4,8,8\}$	$[4,8]^*64_a$	$[8,8]^*256_d$	1024	1
$\{4,8,8\}$	$[4,8]^*64_a$	$[8,8]^*256_e$	1024	1
$\{4,8,8\}$	$[4,8]^*64_a$	$[8,8]^*256_f$	1024	1
$\{4,8,8\}$	$[4,8]^*64_a$	$[8,8]^*256_g$	1024	1
$\{4,8,8\}$	$[4,8]^*64_a$	$[8,8]^*256_h$	1024	1
$\{4,8,8\}$	$[4,8]^*64_b$	$[8,8]^*256_a$	1024	1
$\{4,8,8\}$	$[4,8]^*64_b$	$[8,8]^*256_b$	1024	1
$\{4,8,8\}$	$[4,8]^*64_b$	$[8,8]^*256_c$	1024	1
$\{4,8,8\}$	$[4,8]^*64_b$	$[8,8]^*256_d$	1024	1
$\{4,8,8\}$	$[4,8]^*64_b$	$[8,8]^*256_e$	1024	1
$\{4,8,8\}$	$[4,8]^*64_b$	$[8,8]^*256_f$	1024	1
$\{4,8,8\}$	$[4,8]^*64_b$	$[8,8]^*256_g$	1024	1
$\{4,8,8\}$	$[4,8]^*64_b$	$[8,8]^*256_h$	1024	1
$\{4,8,8\}$	$[4,8]^*128_a$	$[8,8]^*128_a$	1024	1
$\{4,8,8\}$	$[4,8]^*128_a$	$[8,8]^*128_b$	1024	1
$\{4,8,8\}$	$[4,8]^*128_a$	$[8,8]^*128_c$	1024	1
$\{4,8,8\}$	$[4,8]^*128_a$	$[8,8]^*128_d$	1024	1
$\{4,8,8\}$	$[4,8]^*128_b$	$[8,8]^*128_a$	1024	1
$\{4,8,8\}$	$[4,8]^*128_b$	$[8,8]^*128_b$	1024	1
$\{4,8,8\}$	$[4,8]^*128_b$	$[8,8]^*128_c$	1024	1
$\{4,8,8\}$	$[4,8]^*128_b$	$[8,8]^*128_d$	1024	1
$\{4,8,16\}$	$[4,8]^*64_a$	$[8,16]^*256_a$	1024	1
$\{4,8,16\}$	$[4,8]^*64_a$	$[8,16]^*256_b$	1024	1
$\{4,8,16\}$	$[4,8]^*64_a$	$[8,16]^*256_c$	1024	1
$\{4,8,16\}$	$[4,8]^*64_a$	$[8,16]^*256_d$	1024	1
$\{4,8,16\}$	$[4,8]^*64_a$	$[8,16]^*256_e$	1024	1
$\{4,8,16\}$	$[4,8]^*64_a$	$[8,16]^*256_f$	1024	1
$\{4,8,16\}$	$[4,8]^*64_b$	$[8,16]^*256_c$	1024	1

Type	Γ'	Γ''	$\lvert\mathbf{U}(\Gamma',\Gamma'')\rvert$	Count
$\{4,8,16\}$	$[4,8]^*64_b$	$[8,16]^*256_d$	1024	1
$\{4,8,16\}$	$[4,8]^*64_b$	$[8,16]^*256_e$	1024	1
$\{4,8,16\}$	$[4,8]^*64_b$	$[8,16]^*256_f$	1024	1
$\{4,16,4\}$	$[4,16]^*128_a$	$[4,16]^*256_a$	1024	1
$\{4,16,4\}$	$[4,16]^*128_a$	$[4,16]^*256_b$	1024	1
$\{4,16,4\}$	$[4,16]^*128_b$	$[4,16]^*256_a$	1024	1
$\{4,16,4\}$	$[4,16]^*128_b$	$[4,16]^*256_b$	1024	1
$\{4,16,8\}$	$[4,16]^*128_a$	$[16,8]^*256_a$	1024	1
$\{4,16,8\}$	$[4,16]^*128_a$	$[16,8]^*256_b$	1024	1
$\{4,16,8\}$	$[4,16]^*128_a$	$[16,8]^*256_c$	1024	1
$\{4,16,8\}$	$[4,16]^*128_a$	$[16,8]^*256_d$	1024	1
$\{4,16,8\}$	$[4,16]^*128_a$	$[16,8]^*256_e$	1024	1
$\{4,16,8\}$	$[4,16]^*128_a$	$[16,8]^*256_f$	1024	1
$\{4,16,8\}$	$[4,16]^*128_b$	$[16,8]^*256_a$	1024	1
$\{4,16,8\}$	$[4,16]^*128_b$	$[16,8]^*256_b$	1024	1
$\{4,16,8\}$	$[4,16]^*128_b$	$[16,8]^*256_c$	1024	1
$\{4,16,8\}$	$[4,16]^*128_b$	$[16,8]^*256_d$	1024	1
$\{4,16,8\}$	$[4,16]^*128_b$	$[16,8]^*256_e$	1024	1
$\{4,16,8\}$	$[4,16]^*128_b$	$[16,8]^*256_f$	1024	1
$\{4,32,4\}$	$[4,32]^*256_a$	$[32,4]^*256_a$	1024	1
$\{4,32,4\}$	$[4,32]^*256_a$	$[32,4]^*256_b$	1024	1
$\{4,32,4\}$	$[4,32]^*256_b$	$[32,4]^*256_b$	1024	1
$\{8,4,8\}$	$[8,4]^*64_a$	$[4,8]^*128_a$	1024	1
$\{8,4,8\}$	$[8,4]^*64_a$	$[4,8]^*128_b$	1024	1
$\{8,4,8\}$	$[8,4]^*64_b$	$[4,8]^*128_a$	1024	1
$\{8,4,8\}$	$[8,4]^*64_b$	$[4,8]^*128_b$	1024	1
$\{8,4,16\}$	$[8,4]^*64_a$	$[4,16]^*128_a$	1024	1
$\{8,4,16\}$	$[8,4]^*64_a$	$[4,16]^*128_b$	1024	1
$\{8,4,16\}$	$[8,4]^*64_b$	$[4,16]^*128_a$	1024	1
$\{8,4,16\}$	$[8,4]^*64_b$	$[4,16]^*128_b$	1024	1
$\{8,8,8\}$	$[8,8]^*128_a$	$[8,8]^*128_a$	1024	1
$\{8,8,8\}$	$[8,8]^*128_a$	$[8,8]^*128_b$	1024	1
$\{8,8,8\}$	$[8,8]^*128_a$	$[8,8]^*128_c$	1024	1
$\{8,8,8\}$	$[8,8]^*128_a$	$[8,8]^*128_d$	1024	1
$\{8,8,8\}$	$[8,8]^*128_b$	$[8,8]^*128_a$	1024	1
$\{8,8,8\}$	$[8,8]^*128_b$	$[8,8]^*128_b$	1024	1

Type	Γ'	Γ''	$\lvert\mathbf{U}(\Gamma', \Gamma'')\rvert$	Count
$\{8, 8, 8\}$	$[8, 8]^*128_b$	$[8, 8]^*128_d$	1024	1
$\{8, 8, 8\}$	$[8, 8]^*128_c$	$[8, 8]^*128_a$	1024	1
$\{8, 8, 8\}$	$[8, 8]^*128_c$	$[8, 8]^*128_d$	1024	1
$\{8, 8, 8\}$	$[8, 8]^*128_d$	$[8, 8]^*128_d$	1024	1
$\{4, 4, 4, 4\}$	$[4, 4, 4]^*128$	$[4, 4, 4]^*256_a$	1024	1
$\{4, 4, 4, 4\}$	$[4, 4, 4]^*256_a$	$[4, 4, 4]^*256_b$	1024	1
$\{4, 4, 4, 8\}$	$[4, 4, 4]^*128$	$[4, 4, 8]^*256_a$	1024	1
$\{4, 4, 4, 8\}$	$[4, 4, 4]^*128$	$[4, 4, 8]^*256_b$	1024	1
$\{4, 4, 8, 4\}$	$[4, 4, 8]^*256_a$	$[4, 8, 4]^*256_b$	1024	1
$\{4, 4, 8, 4\}$	$[4, 4, 8]^*256_a$	$[4, 8, 4]^*256_d$	1024	1
$\{4, 4, 8, 4\}$	$[4, 4, 8]^*256_b$	$[4, 8, 4]^*256_a$	1024	1
$\{4, 4, 8, 4\}$	$[4, 4, 8]^*256_b$	$[4, 8, 4]^*256_c$	1024	1

Table 6: The number of nondegenerate string C-groups of order 1024 and rank at least 4.

5. Class $\mathcal{C}([4, 4]^*128, [4, 4]^*64)$

We begin with the following lemma.

Lemma 5.1 ([12, pp. 364-365,368]). *Let $G = \langle t_0, t_1, t_2\rangle$ be a smooth quotient of the string Coxeter group $[4, 4]$. Then for some positive integer s,* $\mathrm{ord}(t_0t_1t_2) = 2s$ *and either*

(i) $\mathrm{ord}(t_0t_1t_2t_1) = s$,

$$G = (\langle t_0, t_2^{t_1}\rangle \times \langle t_2, t_0^{t_1}\rangle) \rtimes \langle t_1\rangle \simeq (\mathbf{D}_s \times \mathbf{D}_s) \rtimes \mathbf{Z}_2$$

with additional relation $(t_0t_1t_2t_1)^s = e$ *and order* $8s^2$*; or*

(ii) $\mathrm{ord}(t_0t_1t_2t_1) = 2s$,

$$G = (\langle t_1, t_1^{t_0}\rangle \times \langle t_1^{t_2}, t_1^{t_0t_2}\rangle) \rtimes \langle t_0, t_2\rangle \simeq (\mathbf{D}_s \times \mathbf{D}_s) \rtimes \mathbf{Z}_2^2$$

with additional relation $(t_0t_1t_2)^{2s} = e$ *and order* $16s^2$.

It follows that every element in G can be written uniquely as a product of the form $d_0d_1d_2$, where $d_0, d_1 \in \mathbf{D}_s$ and $d_2 \in \mathbf{Z}_2$ or $\mathbf{Z}_2^2$. Using this property we can compute for the center of G when s is a power of 2:

Lemma 5.2. *Let $G = \langle t_0, t_1, t_2\rangle$ be a smooth quotient of $[4, 4]$ and let $s = 2^m$ for some non-negative integer m.*

(i) If $|G| = 8s^2$*, then* $Z(G) = \begin{cases} \langle (t_0t_1)^2, (t_1t_2)^2\rangle & \text{if } m = 1, \\ \langle (t_0t_1t_2)^s\rangle & \text{if } m \geq 2. \end{cases}$

(ii) If $|G| = 16s^2$*, then* $Z(G) = \begin{cases} \langle (t_0t_2), (t_0t_1)^2, (t_1t_2)^2\rangle & \text{if } m = 0, \\ \langle (t_0t_1t_2t_1)^s\rangle & \text{if } m \geq 1. \end{cases}$

Consider the class $\mathcal{C}([4,4]^*128, [4,4]^*64)$ and let $\mathbf{U} = \langle s_0, s_1, s_2, s_3\rangle$ be the universal string C-group in this class. Then $[4,4]^*128 \simeq \langle s_0, s_1, s_2\rangle$ and $[4,4]^*64 \simeq \langle s_1, s_2, s_3\rangle$. Since $128 = 8\cdot 4^2$ and $64 = 16\cdot 2^2$, then $\mathbf{U}$ is defined by the additional relations $(s_0s_1s_2s_1)^4 = e$ and $(s_1s_2s_3)^4 = e$. The next result gives a decomposition of $\mathbf{U}$ as a product of proper subgroups of $\mathbf{U}$.

Theorem 5.3 ([11])**.** *The universal string C-group $\mathbf{U} = \langle s_0, s_1, s_2, s_3\rangle$ in the class $\mathcal{C}([4,4]^*128, [4,4]^*64)$ is isomorphic to $(W_1\times W_2)\rtimes \mathbf{Z}_2^2$, where $W_1 = \langle s_2^{s_1s_3}, s_0, s_2^{s_1}\rangle$, $W_2 = \langle s_2, s_0^{s_1}, s_2^{s_3}\rangle$, and $\mathbf{Z}_2^2 = \langle s_1, s_3\rangle$. Moreover, $|\mathbf{U}| = \infty$ and $W_2 = W_1^{s_1}$ is isomorphic to the string Coxeter group $[4,4]$.*

It follows from above that if Γ is a string C-group of order 1024 in the class $\mathcal{C}([4,4]^*128, [4,4]^*64)$, then $\Gamma \simeq [(W_1\times W_2)\rtimes \mathbf{Z}_2^2]/N$ for some $N \trianglelefteq (W_1\times W_2)\rtimes\mathbf{Z}_2^2$. Moreover, Γ is generated by $\overline{s_0} = s_0N$, $\overline{s_1} = s_1N$, $\overline{s_2} = s_2N$, $\overline{s_3} = s_3N$. We claim that $N \le W_1\times W_2$. To prove this claim, consider the homomorphism $\varphi_i : \Gamma \to \{-1,1\}$ in **Lemma 2.8** which sends $\overline{s_i}$ to -1 and $\overline{s_j}$ to 1 if $j\neq i$. Let $\rho : (W_1\times W_2)\rtimes\mathbf{Z}_2^2 \to \Gamma$ be the canonical homomorphism sending s_i to $\overline{s_i}$ and take the composition $\psi_i = (\varphi_i\circ\rho) : (W_1\times W_2)\rtimes {\mathbf{Z_2}}^2 \to \{-1,1\}$. Suppose there exists an element $g\in N-(W_1\times W_2)$. Then for $i = 0,1,2,3$, we have

$$\psi_i(g) = \varphi_i(\rho(g)) = \varphi_i(gN) = \varphi_i(N) = 1.$$

In the case when $g = w_1w_2s_1$, where $w_1\in W_1$ and $w_2\in W_2$, we obtain

$$\begin{aligned}\psi_1(g) &= \psi_1(w_1w_2s_1) = \psi_1(w_1)\psi_1(w_2)\psi_1(s_1)\\ &= \varphi_1(\overline{w_1})\varphi_1(\overline{w_2})\varphi_1(\overline{s_1}) = (1)(1)(-1).\end{aligned}$$

The equalities $\varphi_1(\overline{w_1}) = 1$ and $\varphi_1(\overline{w_2}) = 1$ follows from the fact that each generator of W_1 or W_2 is sent to 1 by φ_1. Since it was established $\psi_i(g) = 1$ for any i, we arrived at a contradiction. We likewise arrive at the same contradiction by using the homomorphism φ_3 in the case when $g = w_1w_2s_3$ or $g = w_1w_2s_1s_3$. Thus, we have shown that $N\le W_1\times W_2$, whence $\Gamma \simeq (W_1\times W_2)/N\rtimes\mathbf{Z}_2^2$. We summarize the result in **Lemma 5.4**.

Lemma 5.4. *Let $\mathbf{U} = \langle s_0, s_1, s_2, s_3\rangle$ be the universal string C-group in the class $\mathcal{C}([4,4]^*128, [4,4]^*64)$ and let Γ be a string C-group of order 1024 in this class. Then $\Gamma\simeq\mathbf{U}/N$ for some $N\le W_1\times W_2$. Furthermore, $\Gamma\simeq(W_1\times W_2)/N\rtimes\mathbf{Z}_2^2$.*

It follows immediately from **Lemma 5.4** that $|(W_1\times W_2)/N| = 256$. Let $N_1 = W_1\cap N$ and $N_2 = W_2\cap N_2$ and, hence, $N_2 = N_1^{s_1}$. It can be shown, by computing the orders of the pairwise products of its generating elements, that W_1/N_1 (and, hence, also W_2/N_2) is a smooth quotient of $[4,4]$. Now let t_0, t_1, t_2 (resp. u_0, u_1, u_2) be the generators $s_2^{s_1s_3}$, s_0, $s_2^{s_1}$ (resp. s_2, $s_0^{s_1}$, $s_2^{s_3}$) of W_1 (resp. W_2) and $\overline{t_0}$, $\overline{t_1}$, $\overline{t_2}$ (resp. $\overline{u_0}$, $\overline{u_1}$, $\overline{u_2}$) be their canonical images in $\Gamma\simeq\mathbf{U}/N$. Since $\psi(s_2^{s_1s_3}\cdot s_0\cdot s_2^{s_1}) = -1$, we must have $\mathrm{ord}(\overline{t_0}\,\overline{t_1}\,\overline{t_2})\ge 2$. We consider two cases below. In each case, we seek group relations involving these generators or their canonical images in certain quotient groups.

5.1. **ord**$(\overline{t_0}\ \overline{t_1}\ \overline{t_2}) = 2$**.** If $\mathrm{ord}(\overline{t_0}\ \overline{t_1}\ \overline{t_2}) = 2$, then by **Lemma 5.1** with $s = 1$, we must have $|W_1/N_1| = |W_2/N_2| = 8$ or 16 implying that

$$|(W_1 \times W_2)/(N_1 \times N_2)| = |(W_1/N_1) \times (W_2/N_2)| = 8^2 \text{ or } 16^2.$$

Since $N_1 \times N_2 \leq N$, we have

$$\begin{aligned} 16^2 &\geq [W_1 \times W_2 : N_1 \times N_2] \\ &= [W_1 \times W_2 : N][N : N_1 \times N_2] \\ &= 256 \cdot [N : N_1 \times N_2] \end{aligned}$$

which gives us $[N : N_1 \times N_2] = 1$ or, equivalently, $N = N_1 \times N_2$. Hence, $|W_1/N_1| = |W_2/N_2| = 16$.

Therefore, aside from the relations $(s_0s_1s_2s_1)^4 = (s_1s_2s_3)^4 = e$ determined by the class $\mathcal{C}([4,4]^*128, [4,4]^*64)$, the additional relations for Γ contains the additional relations defining W_1/N_1 and W_2/N_2. Since these subgroups are conjugate, it is enough to consider the relation coming from W_1/N_1, which is

$$(\overline{t_0}\ \overline{t_1}\ \overline{t_2})^2 = (\overline{s_3}\ \overline{s_2}\ \overline{s_1}\ \overline{s_0}\ \overline{s_1}\ \overline{s_3}\ \overline{s_2})^2 = e$$

by **Lemma 5.1**.

It can be checked using GAP that Γ defined by the relations

$$s_0^2 = s_1^2 = s_2^2 = s_3^2 = e,$$

$$(s_0s_1)^4 = (s_0s_2)^2 = (s_0s_3)^2 = (s_1s_2)^4 = (s_1s_3)^2 = (s_2s_3)^4 = e,$$

$$(s_0s_1s_2s_1)^4 = (s_1s_2s_3)^4 = (s_3s_2s_1s_0s_1s_3s_2)^2 = e$$

is indeed a string C-group of order 1024 that belongs to class $\mathcal{C}([4,4]^*128, [4,4]^*64)$.

5.2. $\mathrm{ord}(\overline{t_0}\ \overline{t_1}\ \overline{t_2}) \geq 4$**.** Take a nonidentity element of $N/(N_1 \times N_2)$. It can be written in the form $w_1w_2(N_1 \times N_2)$, where $w_1w_2 \in N$ such that $w_1 \in W_1 - N_1$ and $w_2 \in W_2 - N_2$. For any $\omega_1 \in W_1$, we have $(w_1w_2)^{-1}(w_1w_2)^{\omega_1} = (w_1w_2)^{-1}w_1^{\omega_1}w_2 \in N$ and, thus,

$$(w_1w_2)^{-1}(w_1w_2)^{\omega_1} = w_2^{-1}w_1^{-1}\omega_1^{-1}w_1\omega_1w_2 = w_1^{-1}\omega_1^{-1}w_1\omega_1 \in W_1 \cap N = N_1.$$

Consequently, $w_1\omega_1N_1 = \omega_1w_1N_1$ and, hence, $w_1N_1 \in Z(W_1/N_1)$. Similarly, for any $\omega_2 \in W_2$, we have $w_2\omega_2N_2 = \omega_2w_2N_2$ and, hence, $w_2N_2 \in Z(W_2/N_2)$. Thus,

$$N/(N_1 \times N_2) \leq Z((W_1 \times W_2)/(N_1 \times N_2)) \simeq Z(W_1/N_1) \times Z(W_2/N_2).$$

If $\mathrm{ord}(\overline{t_0}\ \overline{t_1}\ \overline{t_2}) \geq 4$, then by **Lemma 5.1** with $s \geq 2$, we must have $|W_1/N_1| = |W_2/N_2| \geq 32$. To show that we have equality, we suppose, for the sake of contradiction, that $|W_1/N_1| = |W_2/N_2| \geq 64$. By **Lemma 5.2**, $|Z(W_1/N_1)| = |Z(W_2/N_2)| = 2$. Since $w_1(N_1 \times N_2) \in Z((W_1 \times W_2)/(N_1 \times$

N_2)) but $w_1(N_1 \times N_2) \notin N/(N_1 \times N_2)$, then $|N/(N_1 \times N_2)| \leq 2$. It follows that

$$\begin{aligned} 64^2 &\leq [W_1 \times W_2 : N_1 \times N_2] \\ &= [W_1 \times W_2 : N][N : N_1 \times N_2] \\ &\leq [W_1 \times W_2 : N] \cdot 2, \end{aligned}$$

which gives us $[W_1 \times W_2 : N] \geq 64^2/2 = 2048$ contradicting $|(W_1 \times W_2)/N| = 256$. Thus, $|W_1/N_1| = |W_2/N_2| = 32$.

Now observe from **Lemma 5.2** that $|Z(W_1/N_1)| = |Z(W_2/N_2)| = 4$ and $Z(W_1/N_1) \times Z(W_2/N_2) \simeq \mathbf{Z}_2^4$. Since

$$32^2 = [W_1 \times W_2 : N_1 \times N_2] = [W_1 \times W_2 : N][N : N_1 \times N_2] = 256 \cdot [N : N_1 \times N_2],$$

we have $[N : N_1 \times N_2] = 4$ and $N/(N_1 \times N_2) \simeq \mathbf{Z}_2^2$. Given the restrictions for the form of a nonidentity element in $N/(N_1 \times N_2)$, we obtain six possible candidates for this group. Out of six, however, only one will provide additional relations defining a group in the class $\mathcal{C}([4,4]^*128, [4,4]^*64)$. This is

$$N/(N_1 \times N_2) = \langle (\tilde{t_0}\tilde{t_1})^2(\tilde{u_0}\tilde{u_1})^2, (\tilde{t_1}\tilde{t_2})^2(\tilde{u_1}\tilde{u_2})^2 \rangle,$$

where $\tilde{t_0}$, $\tilde{t_1}$, $\tilde{t_2}$ (resp. $\tilde{u_0}$, $\tilde{u_1}$, $\tilde{u_2}$) are the canonical images of t_0, t_1, t_2 (resp. u_0, u_1, u_2) in $(W_1 \times W_2)/(N_1 \times N_2)$. Hence, we have the relations

$$(\tilde{s}_1\tilde{s}_3\tilde{s}_2\tilde{s}_3\tilde{s}_1\tilde{s}_0)^2(\tilde{s}_2\tilde{s}_1\tilde{s}_0\tilde{s}_1)^2 = (\tilde{s}_0\tilde{s}_1\tilde{s}_2\tilde{s}_1)^2(\tilde{s}_1\tilde{s}_0\tilde{s}_1\tilde{s}_3\tilde{s}_2\tilde{s}_3)^2 = e.$$

It can be checked using GAP that Γ defined by the relations

$$s_0^2 = s_1^2 = s_2^2 = s_3^2 = e,$$

$$(s_0s_1)^4 = (s_0s_2)^2 = (s_0s_3)^2 = (s_1s_2)^4 = (s_1s_3)^2 = (s_2s_3)^4 = e,$$

$$(s_0s_1s_2s_1)^4 = (s_1s_2s_3)^4 = e,$$

$$(s_1s_3s_2s_3s_1s_0)^2(s_2s_1s_0s_1)^2 = (s_0s_1s_2s_1)^2(s_1s_0s_1s_3s_2s_3)^2 = e$$

is another string C-group of order 1024 that belongs to class $\mathcal{C}([4,4]^*128, [4,4]^*64)$.

We summarize the results of this section in the next theorem.

Theorem 5.5. *There are two string C-groups* $\Gamma = \langle s_0, s_1, s_2, s_3 \rangle$ *of order 1024 in the class* $\mathcal{C}([4,4]^*128, [4,4]^*64)$. *One is defined by the additional relations*

$$(s_0s_1s_2s_1)^4 = (s_1s_2s_3)^4 = (s_3s_2s_1s_0s_1s_3s_2)^2 = e$$

while the other is defined by the additional relations

$$(s_0s_1s_2s_1)^4 = (s_1s_2s_3)^4 = e,$$

$$(s_1s_3s_2s_3s_1s_0)^2(s_2s_1s_0s_1)^2 = (s_0s_1s_2s_1)^2(s_1s_0s_1s_3s_2s_3)^2 = e.$$

6. Concluding remarks

In this paper, we determined the nondegenerate string C-groups of order 1024. The first part characterized the string C-groups of rank 3, which were obtained using the technique of central extension of string C-groups of order 512 by $\mathbf{Z}_2$. This approach can be employed in an inductive manner to determine string C-groups of order 2^m, for $m \geq 11$, that are generated by three involutions. The second part of the work characterized the string C-groups of rank at least 4 by computing smooth quotients of universal string C-groups. The case when the string C-group Γ has type $\{4, 4, 4\}$ and belongs to the class $\mathcal{C}([4,4]^*128, [4,4]^*64)$ gives rise to an infinite universal string C-group $\mathbf{U}$. This was dealt with separately by an analysis of the structure of $\mathbf{U}$. For future work, we intend to study the different geometric properties and realizations of the abstract regular polytopes associated with these groups.

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Department of Information and Communication Sciences, Sophia University, Ichigaya Campus, 4 Yonban-cho, Chiyoda-ku, Tokyo 102-0081, Japan
E-mail address: y-gomi@sophia.ac.jp

Department of Mathematics, Ateneo de Manila University, Katipunan Avenue, Loyola Heights, Quezon City 1108, Philippines
E-mail address: mloyola@ateneo.edu

Department of Mathematics, Ateneo de Manila University, Katipunan Avenue, Loyola Heights, Quezon City 1108, Philippines
E-mail address: mdelaspenas@ateneo.edu