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# TRIPLE PRODUCT SUMS OF CATALAN TRIANGLE NUMBERS

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ABSTRACT. By means of quadratic transformations for the well-poised hypergeometric series, several reduction and transformation formulae are derived for the triple product sums of Catalan triangle numbers. One of them confirms a conjecture made recently by Miana, Ohtsuka and Romero [18, 2017].

## 1. INTRODUCTION AND MOTIVATION

In classical combinatorics, the Catalan numbers (cf. [10, §1.15] and [11, §5.4])

$$C_n = \frac{1}{n+1}\binom{2n}{n} \text{ with } n \in \mathbb{N}_0,$$

are one of the most fascinating sequences and have numerous interpretations in enumerative problems (cf. Chu [8], Hilton et al. [14], and Stanley [20, Exercise 6.19]). There are also several amazing arithmetic properties, for example, the nonlinear recurrence

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k},$$

the Touchard formula [23] (see also [17, p. 319])

$$C_{n+1} = \sum_{0 \le k \le n/2} 2^{n-2k}\binom{n}{2k} C_k,$$

as well as the formula due to Jonah [17, p. 325] (see also [25])

$$\binom{m+1}{n} = \sum_{k=0}^{n} \binom{m-2k}{n-k} C_k.$$

As an extension of Catalan numbers, Shapiro [19, 1976] introduced and investigated Catalan triangles (see also Sun and Ma [22]) with the entries

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given by

$$B_{n,k}=\frac{k}{n}\binom{2n}{n-k} \text{ where } k,\ n\in\mathbb{N} \text{ with } k\le n. \tag{1}$$

Recently, Gutiérrez et al. [13] found the following two summation formulae

$$\sum_{k=1}^{n} kB_{n,k}^2 \;=\; \binom{n+1}{2}C_nC_{n-1}, \tag{2}$$

$$\sum_{k=1}^{n} k^2B_{n,k}^2 \;=\; (3n-2)C_{2n-2}. \tag{3}$$

The more general problem of evaluating the binomial moments

$$\Theta_\gamma(n)=\sum_{k=1}^{n}k^\gamma\binom{2n}{n-k}^2 \tag{4}$$

was resolved by Chen and Chu [2] and extended by Chu [7], Slavik [21], and Guo and Zeng [12]. When the weight factor "$k^\gamma$" is replaced by falling (or rising) factorials, the related binomial sums have been evaluated by Kilic [15] and Kilic and Prodinger [16]. Further binomial sums have been examined by the author in [3–6, 9].

The purpose of the present paper is to investigate the following triple product sum of binomial coefficients

$$\Omega_\gamma^\delta(m,n,p)=\sum_{k\ge\delta}\big(k-\tfrac{\delta}{2}\big)^\gamma\binom{2m+\delta}{m+k}\binom{2n+\delta}{n+k}\binom{2p+\delta}{p+k}, \tag{5}$$

where $\delta=0$ or $1$ and $m,\ n,\ p,\ \gamma\in\mathbb{N}_0$. As preliminaries, we shall review in the next section a few identities for terminating well-poised series and partial sums related to quadratic transformations. They will be utilized in Section 3 to derive several reduction and transformation formulae for the triple product sums of Catalan triangles. One particular example for $\Omega_3^0(m,n,n)$ confirms a conjecture made by Miana, Ohtsuka, and Romero [18, Conjecture 4.2], which has been the primary motivation for the present research. Finally in Section 4, we shall briefly discuss how to evaluate the binomial moments $\Omega_\gamma^\delta(m,n,p)$ for $\gamma\in\mathbb{N}$ in accordance with $\delta=0$ and 1.

Throughout the paper, we shall adopt the notation of Bailey [1, §2.1] for the classical hypergeometric series

$${}_{1+p}F_p\left[\begin{matrix}a_0,\,a_1,\,a_2,\,\cdots,\,a_p\\ b_1,\,b_2,\,\cdots,\,b_p\end{matrix}\,\Big|\,z\right]=\sum_{k=0}^{\infty}\frac{(a_0)_k(a_1)_k(a_2)_k\cdots(a_p)_k}{k!(b_1)_k(b_2)_k\cdots(b_p)_k}\,z^k,$$

where the rising and falling factorials are given by the $\Gamma$-function ratios

$$(x)_n=\frac{\Gamma(x+n)}{\Gamma(x)} \quad\text{and}\quad \langle x\rangle_n=\frac{\Gamma(1+x)}{\Gamma(1+x-n)}.$$

The multiparameter forms of the shifted factorial will be abbreviated as

$$\begin{bmatrix}\alpha,\ \beta,\ \cdots,\ \gamma\\ A,\ B,\ \cdots,\ C\end{bmatrix}_n = \frac{(\alpha)_n(\beta)_n\cdots(\gamma)_n}{(A)_n(B)_n\cdots(C)_n}.$$

## 2. Quadratic Transformation and Partial Sums

As a preliminary, a quadratic transformation due to Bailey [1] for the very well-poised ${}_4F_3$-series will be examined. From its special case at $x=-1$, we derive several equivalent expressions for the partial ${}_4F_3$-sums, that will be utilized in the next section to examine triple product sums of Catalan triangle numbers.

In the theory of classical hypergeometric series, the following quadratic transformation formula is well-known

$$\begin{aligned}(6)\qquad &{}_3F_2\left[\begin{matrix}a, & b, & c\\ & 1+a-b, & 1+a-c\end{matrix}\,\middle|\,x\right]\\ &= (1-x)^{-a}\times {}_3F_2\left[\begin{matrix}\frac{a}{2},\ \frac{1+a}{2},\ 1+a-b-c\\ 1+a-b,\ 1+a-c\end{matrix}\,\middle|\,\frac{-4x}{(1-x)^2}\right].\end{aligned}$$

Whipple [24, 1927] discovered it by making the replacement $w\to -w/x$ and then letting $n\to\infty$ in the following transformation between a well-poised ${}_4F_3$-series and the Saalschützian ${}_5F_4$-series (see also Bailey [1, §4.5])

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}a, & b, & c, & -n\\ & 1+a-b, & 1+a-c, & w\end{matrix}\,\middle|\,1\right] = \frac{(w-a)_n}{(w)_n}\\ &\times {}_5F_4\left[\begin{matrix}1+a-w,\ \frac{a}{2},\ \frac{1+a}{2},\ 1+a-b-c,\ -n\\ 1+a-b,\ 1+a-c,\ \frac{1+a-w-n}{2},\ \frac{2+a-w-n}{2}\end{matrix}\,\middle|\,1\right].\end{aligned}$$

There is another quadratic formula due to Bailey [1, p. 97]

$$\begin{aligned}(7)\qquad &{}_4F_3\left[\begin{matrix}a, & 1+\frac{a}{2}, & b, & c\\ & \frac{a}{2}, & 1+a-b, & 1+a-c\end{matrix}\,\middle|\,x\right]\\ &= \frac{1+x}{(1-x)^{a+1}}\times {}_3F_2\left[\begin{matrix}\frac{1+a}{2},\ \frac{2+a}{2},\ 1+a-b-c\\ 1+a-b,\ 1+a-c\end{matrix}\,\middle|\,\frac{-4x}{(1-x)^2}\right].\end{aligned}$$

For a formal power series $f(x)$, denote by $[x^n]f(x)$ the coefficient of $x^n$ in $f(x)$. Then $f(x)/(1-x)$ results in the generating function of the partial sums of the coefficients of $f(x)$. This can be explicitly expressed as

$$f(x)=\sum_{n\geq 0}c_nx^n \rightleftharpoons [x^n]\frac{f(x)}{1-x}=\sum_{k=0}^{n}c_k.$$

Given a hypergeometric ${}_pF_q$-series, let ${}_pF_q^{(n)}$ be the partial sum of ${}_pF_q$ to $n+1$ terms with its summation index running from 0 to $n$. Now replacing

$x$ by $-x$, we can reformulate the quadratic relation (7) as

$$\frac{1}{1-x}\,{}_4F_3\left[\begin{matrix}a,\ 1+\frac{a}{2},\ \ b,\ \ c\\ \frac{a}{2},\ 1+a-b,\ 1+a-c\end{matrix}\,\middle|\,-x\right]$$
$$=\ (1+x)^{-a-1}\times{}_3F_2\left[\begin{matrix}\frac{1+a}{2},\ \frac{2+a}{2},\ 1+a-b-c\\ 1+a-b,\ 1+a-c\end{matrix}\,\middle|\,\frac{4x}{(1+x)^2}\right].$$

Extracting the coefficient of $[x^n]$, we can proceed

$$\begin{aligned}
&{}_4F_3^{(n)}\left[\begin{matrix}a,\ 1+\frac{a}{2},\ \ b,\ \ c\\ \frac{a}{2},\ 1+a-b,\ 1+a-c\end{matrix}\,\middle|\,-1\right]\\
&=[x^n](1+x)^{-a-1}\times{}_3F_2\left[\begin{matrix}\frac{1+a}{2},\ \frac{2+a}{2},\ 1+a-b-c\\ 1+a-b,\ 1+a-c\end{matrix}\,\middle|\,\frac{4x}{(1+x)^2}\right]\\
&=\sum_{k=0}^{n}\left[\begin{matrix}\frac{1+a}{2},\ \frac{2+a}{2},\ 1+a-b-c\\ 1,\ 1+a-b,\ 1+a-c\end{matrix}\right]_k[x^{n-k}]\frac{4^k}{(1+x)^{1+a+2k}}\\
&=\sum_{k=0}^{n}\binom{-1-a-2k}{n-k}\left[\begin{matrix}\frac{1+a}{2},\ \frac{2+a}{2},\ 1+a-b-c\\ 1,\ 1+a-b,\ 1+a-c\end{matrix}\right]_k 4^k.
\end{aligned}$$

Taking into account the relation

$$\binom{-1-a-2k}{n-k}4^k=\binom{-1-a}{n}\left[\begin{matrix}-n,\ 1+a+n\\ \frac{1+a}{2},\ \frac{1+a}{2}\end{matrix}\right]_k$$

we get the following interesting expression for partial sums.

**Lemma 1** (Partial sum expression)**.**

$$\begin{aligned}
&{}_4F_3^{(n)}\left[\begin{matrix}a,\ 1+\frac{a}{2},\ \ b,\ \ c\\ \frac{a}{2},\ 1+a-b,\ 1+a-c\end{matrix}\,\middle|\,-1\right]\\
&=\ \binom{-1-a}{n}\times{}_3F_2\left[\begin{matrix}-n,\ 1+a+n,\ 1+a-b-c\\ 1+a-b,\ \ 1+a-c\end{matrix}\,\middle|\,1\right].
\end{aligned}$$

Instead, we do not have such fortune to evaluate partial sums for the ${}_3F_2$-series in Whipple's transformation (6). We are content to write down its special case $x=-1$ for the subsequent application.

**Lemma 2** (Transformation formula)**.**

$${}_3F_2\left[\begin{matrix}a,\ \ b,\ \ c\\ 1+a-b,\ 1+a-c\end{matrix}\,\middle|\,-1\right]=2^{-a}{}_3F_2\left[\begin{matrix}\frac{a}{2},\ \frac{1+a}{2},\ 1+a-b-c\\ 1+a-b,\ 1+a-c\end{matrix}\,\middle|\,1\right].$$

The formula stated in Lemma 1 is remarkable because it transforms a very well-poised partial sum into another terminating, but "zero" balanced series (with the sum of its numerator parameters equal to that of the denominator parameters). It may be of interest to look for further equivalent expressions of the partial sum ${}_4F_3^{(n)}(-1)$ stated in Lemma 1. This can be carried out by using the following transformations for terminating ${}_3F_2$-series.

**Lemma 3** (Transformation formulae)**.**

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}-n,a,b\\ c,d\end{matrix}\,\middle|\,1\right]\\
&=\frac{(c-a)_n}{(c)_n}{}_3F_2\left[\begin{matrix}-n, & a, & d-b\\ d, & 1+a-c-n\end{matrix}\,\middle|\,1\right] &(8)\\
&=\frac{(c+d-a-b)_n}{(c)_n}{}_3F_2\left[\begin{matrix}-n,\ d-a,\ d-b\\ d,\ c+d-a-b\end{matrix}\,\middle|\,1\right] &(9)\\
&=\left[\begin{matrix}a,c+d-a-b\\ c,\quad d\end{matrix}\right]_n{}_3F_2\left[\begin{matrix}-n, & c-a, & d-a\\ 1-a-n, c+d-a-b\end{matrix}\,\middle|\,1\right] &(10)\\
&=\left[\begin{matrix}c-a,d-a\\ c,\quad d\end{matrix}\right]_n{}_3F_2\left[\begin{matrix}-n,\ a,\ 1+a+b-c-d-n\\ 1+a-c-n,1+a-d-n\end{matrix}\,\middle|\,1\right]. &(11)
\end{aligned}
$$

Among these formulae, the first one results from a limiting case of the transformation between two terminating balanced ${}_4F_3$-series due to Bailey [1, §7.2], which can also be verified easily by combining the series rearrangement with the Chu–Vandermonde formula (cf. Bailey [1, §1.3] as follows

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}-n,a,b\\ c,d\end{matrix}\,\middle|\,1\right]&=\sum_{k=0}^{n}\left[\begin{matrix}-n,a\\ 1,c\end{matrix}\right]_k{}_2F_1\left[\begin{matrix}-k,d-b\\ d\end{matrix}\,\middle|\,1\right]\\
&=\sum_{i=0}^{n}(-1)^i\left[\begin{matrix}-n,a,d-b\\ 1,\ c,\ \ d\end{matrix}\right]_i{}_2F_1\left[\begin{matrix}i-n,a+i\\ c+i\end{matrix}\,\middle|\,1\right].
\end{aligned}
$$

Other three formulae in Lemma 3 follow by iterating (8) appropriately, where (11) has been given explicitly by Bailey [1, §10.1].

By applying Lemma 3 to the ${}_3F_2$-series displayed in Lemma 1, we get further four equivalent expressions for the well-poised partial ${}_4F_3^{(n)}$-sum.

**Lemma 4** (Partial sum expressions)**.**

$$
\begin{aligned}
&{}_4F_3^{(n)}\left[\begin{matrix}a, 1+\frac{a}{2}, & b, & c\\ \frac{a}{2}, & 1+a-b, & 1+a-c\end{matrix}\,\middle|\,-1\right]\\
&=\frac{(1+a)_n}{(1+a-c)_n}\times{}_2F_1^{(n)}\left[\begin{matrix}c,\ -b-n\\ 1+a-b\end{matrix}\,\middle|\,1\right] &(12)\\
&=\left[\begin{matrix}1+a,1+b\\ 1,1+a-b\end{matrix}\right]_n\times{}_3F_2\left[\begin{matrix}-n, & b, & 1+a+n\\ & b+1, & 1+a-c\end{matrix}\,\middle|\,1\right] &(13)\\
&=\left[\begin{matrix}1+a,1+a-b-c\\ 1+a-b,1+a-c\end{matrix}\right]_n\times{}_2F_1^{(n)}\left[\begin{matrix}b,\quad c\\ b+c-a-n\end{matrix}\,\middle|\,1\right] &(14)\\
&=\left[\begin{matrix}1+a,\ 1+b,\ 1+c\\ 1+a-b,1+a-c,-n\end{matrix}\right]_n\times{}_3F_2\left[\begin{matrix}1,\ -n,\ 1+a+n\\ 1+b,\ 1+c\end{matrix}\,\middle|\,1\right]. &(15)
\end{aligned}
$$

Among these transformations, we point out that the ${}_3F_2$-series in (13) is "exotic".

## 3. Triple Product Sums in Catalan Triangles

Based on the hypergeometric series results prepared in the previous section, we shall examine four initial cases of $\Omega_\gamma^\delta(m,n,p)$ with $\gamma,\ \delta=0,\ 1$. When $\delta=1$, we have the fortune to establish reduction and transformation formulae even for the partial sums of triple products of Catalan triangle numbers. In addition, we shall prove a reduction formula for $\Omega_3^0(m,n,p)$ that confirms, in particular for $p=n$, a conjecture made recently by Miana, Ohtsuka, and Romero [18, Conjecture 4.2].

3.1. $\Omega_0^0(m,n,p)$**.** Suppose that $m=\min\{m,n,p\}$ for three nonnegative numbers $m$, $n$, $p$. For the terminating bilateral series below, it is not difficult to show, by inverting the summation index $k\to -k$, the following relation

$$\begin{aligned}
&{}_3H_3\left[\begin{matrix}-m, & -n, & -p\\ 1+m, & 1+n, & 1+p\end{matrix}\,\Big|\,-1\right]\\
&=\sum_{k=-m}^{m}(-1)^k\left[\begin{matrix}-m, & -n, & -p\\ 1+m, & 1+n, & 1+p\end{matrix}\right]_k\\
&=1+2\sum_{k\geq 1}(-1)^k\left[\begin{matrix}-m, & -n, & -p\\ 1+m, & 1+n, & 1+p\end{matrix}\right]_k\\
&=1+2\sum_{k\geq 1}\frac{\binom{2m}{m+k}\binom{2n}{n+k}\binom{2p}{p+k}}{\binom{2m}{m}\binom{2n}{n}\binom{2p}{p}}.
\end{aligned}$$

On the other hand, by shifting the summation index $k\to k-m$, we have

$$\begin{aligned}
{}_3H_3\left[\begin{matrix}-m, & -n, & -p\\ 1+m, & 1+n, & 1+p\end{matrix}\,\Big|\,-1\right]=&\frac{\binom{2n}{m+n}\binom{2p}{m+p}}{\binom{2m}{m}\binom{2n}{n}\binom{2p}{p}}\\
&\times{}_3F_2\left[\begin{matrix}-2m, & -m-n, & -m-p\\ & 1-m+n, & 1-m+p\end{matrix}\,\Big|\,-1\right].
\end{aligned}$$

By combining the last two equations, we get the expression

$$\begin{aligned}
&\sum_{k\geq 0}\binom{2m}{m+k}\binom{2n}{n+k}\binom{2p}{p+k}\\
&=\frac{1}{2}\binom{2m}{m}\binom{2n}{n}\binom{2p}{p}+\frac{1}{2}\binom{2n}{m+n}\binom{2p}{m+p}\\
&\times{}_3F_2\left[\begin{matrix}-2m, & -m-n, & -m-p\\ & 1-m+n, & 1-m+p\end{matrix}\,\Big|\,-1\right].
\end{aligned}$$

For the last ${}_3F_2$-series, applying the transformation displayed in Lemma 2

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix}-2m, & -m-n, & -m-p\\ & 1-m+n, & 1-m+p\end{matrix}\,\Big|\,-1\right]\\
&=4^m\times{}_3F_2\left[\begin{matrix}-m,\frac{1}{2}-m,1+n+p\\ 1-m+n,1-m+p\end{matrix}\,\Big|\,1\right]
\end{aligned}\tag{16}$$

and then replacing the resulting ${}_3F_2$-series by its reversal

$$
{}_3F_2\left[\begin{matrix}-m,\frac12-m,1+n+p\\1-m+n,1-m+p\end{matrix}\,\Big|\,1\right]=\left[\begin{matrix}\frac12,\ 1+n+p\\1-m+n,1-m+p\end{matrix}\right]_m
\times{}_3F_2\left[\begin{matrix}-m,\ -n,\ -p\\\frac12,-m-n-p\end{matrix}\,\Big|\,1\right]
$$

we derive the following expression with the afore-assumed parameter restriction $m=\min\{m,n,p\}$ being removed by symmetry.

**Theorem 5.** *For three nonnegative integers $m$, $n$, $p$, there holds*

$$
\sum_{k\ge0}\binom{2m}{m+k}\binom{2n}{n+k}\binom{2p}{p+k}=\frac12\binom{2m}{m}\binom{2n}{n}\binom{2p}{p}
+{}_3F_2\left[\begin{matrix}-m,\ -n,\ -p\\\frac12,-m-n-p\end{matrix}\,\Big|\,1\right]\frac{(m+n+p)!(2m)!(2n)!(2p)!}{2(m+n)!(m+p)!(n+p)!m!n!p!}.
$$

Applying (9) to the last ${}_3F_2$-series

$$
{}_3F_2\left[\begin{matrix}-m,\ -n,\ -p\\\frac12,-m-n-p\end{matrix}\,\Big|\,1\right]=\frac{(\frac12)_m}{(1+n+p)_m}{}_3F_2\left[\begin{matrix}-m,\ \frac12+n,\frac12+p\\\frac12,\ \frac12-m\end{matrix}\,\Big|\,1\right]
$$

leads us to the following equivalent expression.

**Proposition 6.** *For three nonnegative integers $m$, $n$, $p$, there holds*

$$
\sum_{k\ge0}\binom{2m}{m+k}\binom{2n}{n+k}\binom{2p}{p+k}=\frac12\binom{2m}{m}\binom{2n}{n}\binom{2p}{p}
+\frac{(\frac12)_m(2m)!(2n)!(2p)!}{2(m+n)!(m+p)!m!n!p!}{}_3F_2\left[\begin{matrix}-m,\ \frac12+n,\frac12+p\\\frac12,\ \frac12-m\end{matrix}\,\Big|\,1\right].
$$

If we apply (11) to the ${}_3F_2$-series displayed on the right hand side of (16)

$$
{}_3F_2\left[\begin{matrix}-m,\frac12-m,1+n+p\\1-m+n,1-m+p\end{matrix}\,\Big|\,1\right]=\left[\begin{matrix}1+n,\quad 1+p\\1-m+n,1-m+p\end{matrix}\right]_m
\times{}_3F_2\left[\begin{matrix}-m,\frac12,1+n+p\\1+n,1+p\end{matrix}\,\Big|\,1\right]
$$

we would get another equivalent expression.

**Corollary 7.** *For three nonnegative integers $m$, $n$, $p$, there holds*

$$
\sum_{k\ge0}\binom{2m}{m+k}\binom{2n}{n+k}\binom{2p}{p+k}=\frac12\binom{2m}{m}\binom{2n}{n}\binom{2p}{p}
+2^{2m-1}\binom{2n}{n}\binom{2p}{p}{}_3F_2\left[\begin{matrix}\frac12,\ -m,1+n+p\\1+n,\quad 1+p\end{matrix}\,\Big|\,1\right].
$$

3.2. $\Omega_1^0(m,n,p)$. Let $\lambda$ be a natural number subject to $0<\lambda\le m$. Suppose that $m=\min\{m,n,p\}$. By inverting the summation order through $k\to m-k$, it is not difficult to reformulate the binomial sum in terms of hypergeometric series

$$\begin{aligned}\sum_{k=\lambda}^{m}k\binom{2m}{m+k}\binom{2n}{n+k}\binom{2p}{p+k} &= m\binom{2n}{n-m}\binom{2p}{p-m}\\ &\times{}_4F_3^{(m-\lambda)}\left[\begin{matrix}-2m, 1-m, & -m-n, & -m-p\\ -m, & 1-m+n, & 1-m+p\end{matrix}\,\Big|\,-1\right].\end{aligned}\tag{17}$$

Applying Lemma 1 to the last ${}_4F_3$-sum, we have

$$\begin{aligned}&{}_4F_3^{(m-\lambda)}\left[\begin{matrix}-2m, 1-m, & -m-n, & -m-p\\ -m, & 1-m+n, & 1-m+p\end{matrix}\,\Big|\,-1\right]\\ &=\binom{2m-1}{m-\lambda}{}_3F_2\left[\begin{matrix}\lambda-m, 1-m-\lambda, 1+n+p\\ 1-m+n, 1-m+p\end{matrix}\,\Big|\,1\right].\end{aligned}$$

For the above ${}_3F_2$-series, its summation index runs from 0 to $m-\lambda$ in view of $\lambda>0$. By taking its reversal

$$\begin{aligned}{}_3F_2\left[\begin{matrix}\lambda-m, 1-m-\lambda, 1+n+p\\ 1-m+n, 1-m+p\end{matrix}\,\Big|\,1\right] &= \left[\begin{matrix}2\lambda, 1+n+p\\ 1-m+n, 1-m+p\end{matrix}\right]_{m-\lambda}\\ &\times{}_3F_2\left[\begin{matrix}\lambda-m, \lambda-n, & \lambda-p\\ 2\lambda, & \lambda-m-n-p\end{matrix}\,\Big|\,1\right]\end{aligned}$$

we obtain the following symmetric expression with the afore-assumed restriction $m=\min\{m,n,p\}$ being removed by symmetry.

**Theorem 8.** *For four nonnegative integers $m$, $n$, $p$, and $\lambda$ subject to the condition $0<\lambda\le\min\{m,n,p\}$, there holds*

$$\begin{aligned}\sum_{k=\lambda}^{m}k\binom{2m}{m+k}\binom{2n}{n+k}\binom{2p}{p+k} &= {}_3F_2\left[\begin{matrix}\lambda-m, \lambda-n, \lambda-p\\ 2\lambda, \lambda-m-n-p\end{matrix}\,\Big|\,1\right]\\ \times\frac{(m+n+p-\lambda)!(2m)!(2n)!(2p)!}{2(2\lambda-1)!(m+n)!(m+p)!(n+p)!(m-\lambda)!(n-\lambda)!(p-\lambda)!}&.\end{aligned}$$

According to (9), the last ${}_3F_2$-series can further be reduced to

$$\begin{aligned}{}_3F_2\left[\begin{matrix}\lambda-m, \lambda-n, \lambda-p\\ 2\lambda, \lambda-m-n-p\end{matrix}\,\Big|\,1\right] &= \frac{(m-\lambda)!}{(n+p+1)_{m-\lambda}}\\ &\times{}_2F_1^{(m-\lambda)}\left[\begin{matrix}\lambda+n, \lambda+p\\ 2\lambda\end{matrix}\,\Big|\,1\right].\end{aligned}$$

We therefore establish the following equivalent expression.

**Proposition 9.** *Let $m$, $n$, $p$, and $\lambda$ be four nonnegative integers that satisfy the condition $0 < \lambda \leq \min\{m, n, p\}$. We have*

$$\sum_{k=\lambda}^{m} k\binom{2m}{m+k}\binom{2n}{n+k}\binom{2p}{p+k} = {}_2F_1{}^{(m-\lambda)}\left[\begin{matrix}\lambda+n,\ \lambda+p\\ 2\lambda\end{matrix}\,\middle|\,1\right] \\ \times \frac{(2m)!(2n)!(2p)!}{2(2\lambda-1)!(m+n)!(m+p)!(n-\lambda)!(p-\lambda)!}.$$

Instead, applying (14) to (17) and then inverting the summation order (valid also for $\lambda = 0$) for the resulting ${}_2F_1$-sum, we have

$$\begin{aligned}&{}_4F_3^{(m-\lambda)}\left[\begin{matrix}-2m,\ 1-m,\ \ -m-n,\ \ \ -m-p\\ -m,\ \ 1-m+n,\ 1-m+p\end{matrix}\,\middle|\,-1\right]\\ &=\left[\begin{matrix}1-2m, 1+n+p\\ 1-m+n, 1-m+p\end{matrix}\right]_{m-\lambda}{}_2F_1{}^{(m-\lambda)}\left[\begin{matrix}-m-n, -m-p\\ \lambda-m-n-p\end{matrix}\,\middle|\,1\right]\\ &=\left[\begin{matrix}\lambda+m, \lambda+n+1, \lambda+p+1\\ 1, 1-m+n, 1-m+p\end{matrix}\right]_{m-\lambda}{}_3F_2\left[\begin{matrix}1,\ \lambda-m, 1+n+p\\ 1+\lambda+n, 1+\lambda+p\end{matrix}\,\middle|\,1\right]\end{aligned}$$

which yields another equivalent expression.

**Corollary 10.** *Let $m$, $n$, $p$, and $\lambda$ be four nonnegative integers satisfying the condition $0 \leq \lambda \leq \min\{m, n, p\}$. We have*

$$\sum_{k=\lambda}^{m} k\binom{2m}{m+k}\binom{2n}{n+k}\binom{2p}{p+k} = {}_3F_2\left[\begin{matrix}1,\ \lambda-m, 1+n+p\\ 1+\lambda+n, 1+\lambda+p\end{matrix}\,\middle|\,1\right] \\ \times \frac{m+\lambda}{2}\binom{2m}{m+\lambda}\binom{2n}{n+\lambda}\binom{2p}{p+\lambda}.$$

3.3. $\Omega_0^1(m, n, p)$**.** Suppose that with $m = \min\{m, n, p\}$ for three natural numbers $m, n, p$. For the terminating bilateral series below, it is not difficult to show, by inverting the summation index $k \to -k-1$, the following relation

$$\begin{aligned}&{}_3H_3\left[\begin{matrix}-1-m, -1-n, -1-p\\ 1+m,\ \ \ 1+n,\ \ \ 1+p\end{matrix}\,\middle|\,-1\right]\\ &=\sum_{k=-m}^{m+1}(-1)^k\left[\begin{matrix}-1-m, -1-n, -1-p\\ 1+m,\ \ \ 1+n,\ \ \ 1+p\end{matrix}\right]_k\\ &=2\sum_{k\geq 1}(-1)^k\left[\begin{matrix}-1-m, -1-n, -1-p\\ 1+m,\ \ \ 1+n,\ \ \ 1+p\end{matrix}\right]_k\\ &=2\sum_{k\geq 1}\frac{\binom{2m+1}{m+k}\binom{2n+1}{n+k}\binom{2p+1}{p+k}}{\binom{2m+1}{m}\binom{2n+1}{n}\binom{2p+1}{p}}.\end{aligned}$$

On the other hand, by shifting the summation index $k \to k - m$, we have

$$ {}_3H_3\left[\begin{matrix} -1-m, & -1-n, & -1-p \\ 1+m, & 1+n, & 1+p \end{matrix} \,\Big|\, -1\right] = \frac{\binom{2n+1}{n-m}\binom{2p+1}{p-m}}{\binom{2m+1}{m}\binom{2n+1}{n}\binom{2p+1}{p}} \times {}_3F_2\left[\begin{matrix} -1-2m, & -1-m-n, & -1-m-p \\ & 1-m+n, & 1-m+p \end{matrix} \,\Big|\, -1\right]. $$

By combining the last two equations, we get the expression

$$ \sum_{k\geq 1}\binom{2m+1}{m+k}\binom{2n+1}{n+k}\binom{2p+1}{p+k} = \frac{1}{2}\binom{2n+1}{n-m}\binom{2p+1}{p-m} \times {}_3F_2\left[\begin{matrix} -1-2m, & -1-m-n, & -1-m-p \\ & 1-m+n, & 1-m+p \end{matrix} \,\Big|\, -1\right]. $$

For the last ${}_3F_2$-series, applying the transformation displayed in Lemma 2

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix} -1-2m, & -1-m-n, & -1-m-p \\ & 1-m+n, & 1-m+p \end{matrix} \,\Big|\, -1\right] \\ &= 2^{1+2m}{}_3F_2\left[\begin{matrix} -m, -\frac{1}{2}-m, 2+n+p \\ 1-m+n, 1-m+p \end{matrix} \,\Big|\, 1\right] \end{aligned} \tag{18} $$

and then replacing the resulting ${}_3F_2$-series by its reversal

$$ {}_3F_2\left[\begin{matrix} -m, -\frac{1}{2}-m, 2+n+p \\ 1-m+n, 1-m+p \end{matrix} \,\Big|\, 1\right] = \left[\begin{matrix} \frac{3}{2}, \ 2+n+p \\ 1-m+n, 1-m+p \end{matrix}\right]_m \times {}_3F_2\left[\begin{matrix} -m, \ -n, \ -p \\ \frac{3}{2}, -1-m-n-p \end{matrix} \,\Big|\, 1\right] $$

we derive the following expression with the afore-assumed restricted condition $m = \min\{m, n, p\}$ being removed by symmetry.

**Theorem 11.** *For three nonnegative integers $m$, $n$, $p$, there holds*

$$ \sum_{k\geq 1}\binom{2m+1}{m+k}\binom{2n+1}{n+k}\binom{2p+1}{p+k} = {}_3F_2\left[\begin{matrix} -m, \ -n, \ -p \\ \frac{3}{2}, -1-m-n-p \end{matrix} \,\Big|\, 1\right] \times \frac{(m+n+p+1)!(2m+1)!(2n+1)!(2p+1)!}{(m+n+1)!(m+p+1)!(n+p+1)!m!n!p!}. $$

Applying (9) to the last ${}_3F_2$-series

$$ {}_3F_2\left[\begin{matrix} -m, \ -n, \ -p \\ \frac{3}{2}, -1-m-n-p \end{matrix} \,\Big|\, 1\right] = \frac{(\frac{1}{2})_m}{(2+n+p)_m}{}_3F_2\left[\begin{matrix} -m, \ \frac{3}{2}+n, \frac{3}{2}+p \\ \frac{3}{2}, \ \frac{1}{2}-m \end{matrix} \,\Big|\, 1\right] $$

leads us to the following equivalent expression.

**Proposition 12.** *For three nonnegative integers $m$, $n$, $p$, there holds*

$$\sum_{k\ge1}\binom{2m+1}{m+k}\binom{2n+1}{n+k}\binom{2p+1}{p+k} = {}_3F_2\left[\begin{matrix}-m,\ \frac32+n,\ \frac32+p\\ \frac32,\ \frac12-m\end{matrix}\,\Big|\,1\right]$$
$$\times\frac{(\frac12)_m(2m+1)!(2n+1)!(2p+1)!}{(m+n+1)!(m+p+1)!m!n!p!}.$$

If we apply (11) to the ${}_3F_2$-series displayed in (18)

$${}_3F_2\left[\begin{matrix}-m,-\frac12-m,2+n+p\\ 1-m+n,1-m+p\end{matrix}\,\Big|\,1\right]=\left[\begin{matrix}2+n,\quad 2+p\\ 1-m+n,1-m+p\end{matrix}\right]_m$$
$$\times\,{}_3F_2\left[\begin{matrix}-m,\ \frac12,\ 2+n+p\\ 2+n,\ 2+p\end{matrix}\,\Big|\,1\right]$$

we would get another equivalent expression.

**Corollary 13.** *For three nonnegative integers $m$, $n$, $p$, there holds*

$$\sum_{k\ge1}\binom{2m+1}{m+k}\binom{2n+1}{n+k}\binom{2p+1}{p+k}=4^m\binom{2n+1}{n}\binom{2p+1}{p}$$
$$\times\,{}_3F_2\left[\begin{matrix}-m,\ \frac12,2+n+p\\ 2+n,2+p\end{matrix}\,\Big|\,1\right].$$

3.4. $\Omega_1^1(m,n,p)$. Suppose that $m=\min\{m,n,p\}$. For a natural number $\lambda$ satisfying the condition $0<\lambda\le m+1$, we may reformulate the following binomial sum, by inverting the summation order through $k\to 1+m-k$, in terms of hypergeometric series

$$\sum_{k=\lambda}^{m+1}\left(k-\tfrac12\right)\binom{2m+1}{m+k}\binom{2n+1}{n+k}\binom{2p+1}{p+k}=\left(m+\tfrac12\right)\binom{2n+1}{n-m}\binom{2p+1}{p-m}$$
$$\times{}_4F_3^{(1+m-\lambda)}\left[\begin{matrix}-1-2m,\ \frac12-m,\ -1-m-n,-1-m-p\\ -\frac12-m,\ 1-m+n,\ 1-m+p\end{matrix}\,\Big|\,-1\right].\tag{19}$$

First transforming the last ${}_4F_3$-sum by Lemma 1

$${}_4F_3^{(1+m-\lambda)}\left[\begin{matrix}-1-2m,\ \frac12-m,\ -1-m-n,-1-m-p\\ -\frac12-m,\ 1-m+n,\ 1-m+p\end{matrix}\,\Big|\,-1\right]$$
$$=\binom{2m}{1+m-\lambda}{}_3F_2\left[\begin{matrix}\lambda-m-1,1-m-\lambda,\ 2+n+p\\ 1-m+n,1-m+p\end{matrix}\,\Big|\,1\right]$$

and then considering the reversal of the resulting ${}_3F_2$-series

$${}_3F_2\left[\begin{matrix}\lambda-m-1,1-m-\lambda,\ 2+n+p\\ 1-m+n,1-m+p\end{matrix}\,\Big|\,1\right]=\left[\begin{matrix}2\lambda-1,2+n+p\\ 1-m+n,1-m+p\end{matrix}\right]_{1+m-\lambda}$$
$$\times\,{}_3F_2\left[\begin{matrix}\lambda-m-1,\lambda-n-1,\quad \lambda-p-1\\ 2\lambda-1,\quad \lambda-2-m-n-p\end{matrix}\,\Big|\,1\right]$$

we derive the following symmetric expression with the afore-assumed restriction $m=\min\{m,n,p\}$ being removed by symmetry.

**Theorem 14.** *For four nonnegative integers $m$, $n$, $p$, and $\lambda$ subject to the condition $0 < \lambda \le \min\{m,n,p\}$, there holds*

$$\begin{aligned}&\sum_{k=\lambda}^{m+1}\big(k-\tfrac{1}{2}\big)\binom{2m+1}{m+k}\binom{2n+1}{n+k}\binom{2p+1}{p+k}\\&={}_3F_2\left[\begin{matrix}\lambda-m-1,\lambda-n-1,\lambda-p-1\\2\lambda-1,\lambda-m-n-p-2\end{matrix}\,\Big|\,1\right]\\&\times\frac{(2+m+n+p-\lambda)!}{(1+m-\lambda)!(1+n-\lambda)!(1+p-\lambda)!}\\&\times\frac{(2m+1)!(2n+1)!(2p+1)!}{2(2\lambda-2)!(1+m+n)!(1+m+p)!(1+n+p)!}.\end{aligned}$$

Rewriting further the last ${}_3F_2$-series, through (9), as the ${}_2F_1$-sum below

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}\lambda-m-1,\lambda-n-1,\lambda-p-1\\2\lambda-1,\lambda-m-n-p-2\end{matrix}\,\Big|\,1\right]\\&=\frac{(1+m-\lambda)!}{(n+p+2)_{1+m-\lambda}}\,{}_2F_1{}^{(1+m-\lambda)}\left[\begin{matrix}\lambda+n,\ \lambda+p\\2\lambda-1\end{matrix}\,\Big|\,1\right]\end{aligned}$$

leads us to the following equivalent expression.

**Proposition 15.** *Let $m, n, p$, and $\lambda$ be four nonnegative integers that satisfy the condition $0 < \lambda \le 1+\min\{m,n,p\}$. We have*

$$\begin{aligned}\sum_{k=\lambda}^{m+1}\big(k-\tfrac{1}{2}\big)\binom{2m+1}{m+k}\binom{2n+1}{n+k}\binom{2p+1}{p+k}&={}_2F_1{}^{(1+m-\lambda)}\left[\begin{matrix}\lambda+n,\lambda+p\\2\lambda-1\end{matrix}\,\Big|\,1\right]\\&\times\frac{(2m+1)!(2n+1)!(2p+1)!}{2(2\lambda-2)!(m+n+1)!(m+p+1)!(n-\lambda+1)!(p-\lambda+1)!}.\end{aligned}$$

Furthermore, applying (14) to the ${}_4F_3$-sum displayed in (19) and then inverting the summation order (valid for $\lambda=0$ too) for the resulting ${}_2F_1$-sum, we get

$$\begin{aligned}&{}_4F_3^{(1+m-\lambda)}\left[\begin{matrix}-1-2m,\ \tfrac{1}{2}-m,\ -1-m-n,-1-m-p\\-\tfrac{1}{2}-m,\ 1-m+n,\ \ 1-m+p\end{matrix}\,\Big|\,-1\right]\\&=\left[\begin{matrix}-2m,2+n+p\\1-m+n,1-m+p\end{matrix}\right]_{1+m-\lambda}{}_2F_1{}^{(1+m-\lambda)}\left[\begin{matrix}-1-m-n,-1-m-p\\\lambda-2-m-n-p\end{matrix}\,\Big|\,1\right]\\&=\left[\begin{matrix}\lambda+m,\lambda+n+1,\lambda+p+1\\1,1-m+n,1-m+p\end{matrix}\right]_{1+m-\lambda}\times{}_3F_2\left[\begin{matrix}1,\ \lambda-m-1,2+n+p\\1+\lambda+n,1+\lambda+p\end{matrix}\,\Big|\,1\right]\end{aligned}$$

which gives rise to another equivalent expression.

**Corollary 16.** *Let $m$, $n$, $p$, and $\lambda$ be four nonnegative integers that satisfy the condition $0 \leq \lambda \leq 1+\min\{m,n,p\}$. We have*

$$\begin{aligned}&\sum_{k=\lambda}^{m+1}(k-\tfrac{1}{2})\binom{2m+1}{m+k}\binom{2n+1}{n+k}\binom{2p+1}{p+k}\\&= {}_3F_2\left[\begin{matrix}1,\ \lambda-m-1, 2+n+p\\ 1+\lambda+n, 1+\lambda+p\end{matrix}\ \middle|\ 1\right]\\&\times\ \frac{m+\lambda}{2}\binom{2m+1}{m+\lambda}\binom{2n+1}{n+\lambda}\binom{2p+1}{p+\lambda}.\end{aligned}$$

3.5. $\Omega_3^0(m,n,p)$**.** In accordance with the equation

$$2k^3 = (m+k)(n+k)(p+k) - (m-k)(n-k)(p-k) - 2k(mn+mp+np)$$

we can derive the following relation

$$\begin{aligned}\sum_{k\geq 1} B_{m,k}B_{n,k}B_{p,k} =& \frac{1}{2}\binom{2m}{m}\binom{2n}{n}\binom{2p}{p} - \frac{mn+mp+np}{mnp}\\ &\times\sum_{k\geq 1}k\binom{2m}{m+k}\binom{2n}{n+k}\binom{2p}{p+k},\end{aligned}\tag{20}$$

where we have made, by the telescoping, the following evaluation

$$\begin{aligned}\binom{2m}{m}\binom{2n}{n}\binom{2p}{p} &= 8\sum_{k\geq 1}\binom{2m-1}{m+k-1}\binom{2n-1}{n+k-1}\binom{2p-1}{p+k-1}\\ &- 8\sum_{k\geq 1}\binom{2m-1}{m+k}\binom{2n-1}{n+k}\binom{2p-1}{p+k}.\end{aligned}$$

Specifying $\lambda = 1$ in Proposition 9, we can employ this particular case to reformulate the binomial sum (20) as

$$\begin{aligned}&\sum_{k\geq 1}k\binom{2m}{m+k}\binom{2n}{n+k}\binom{2p}{p+k}\\ &= \frac{(2m)!(2n)!(2p)!}{2(m+n)!(m+p)!(n-1)!(p-1)!}\,{}_2F_1{}^{(m-1)}\left[\begin{matrix}n+1, p+1\\ 2\end{matrix}\ \middle|\ 1\right]\\ &= \frac{(2m)!(2n)!(2p)!}{2(m+n)!(m+p)!n!(p-1)!}\sum_{k=1}^{m}\binom{n+k-1}{n-1}\binom{p+k-1}{p}.\end{aligned}$$

We therefore find the following reduction formula.

**Theorem 17.** *Let $m$, $n$, $p$ be three natural numbers. We have*

$$\begin{aligned}\sum_{k\geq 1} B_{m,k}B_{n,k}B_{p,k} &= \frac{1}{2}\binom{2m}{m}\binom{2n}{n}\binom{2p}{p} - \frac{mn+mp+np}{2mn}\\ &\times \frac{(2m)!(2n)!(2p)!}{(m+n)!(m+p)!n!p!}\sum_{k=1}^{m}\binom{n+k-1}{n-1}\binom{p+k-1}{p}.\end{aligned}$$

Now we are in position to confirm the following conjectured formula.

**Proposition 18** (Miana–Ohtsuka–Romero [18, Conjecture 4.2])**.** *Let $m$ and $n$ be two natural numbers. We have*

$$\sum_{k\geq 1} B_{m,k}B_{n,k}^2 = \frac{1}{2}\binom{2m}{m}\binom{2n}{n}^2 - \frac{2m+n}{2\nu} \times \frac{\binom{2m}{m}\binom{2n}{n}^2}{\binom{m+n}{n}\binom{n+\nu}{n}} \sum_{k=1}^{\nu}\binom{\mu+k-1}{\mu}\binom{n+k-1}{n-1},$$

*where $\mu = \max\{m,n\}$ and $\nu = \min\{m,n\}$.*

In fact, it is routine to check that when $m \leq n$, the last formula corresponds to the $p = n$ case of Theorem 17. Instead, when $m > n$, we can get it from Theorem 17 by making the exchange $m \rightleftharpoons n$ and then letting $p = n$.

## 4. Moments on Triple Product Sums

In evaluating the moments of Catalan numbers, Chen–Chu [2, Equations 9,10] find the following identity

$$x^{2\gamma} = \sum_{j=0}^{\gamma}(-1)^j\langle y+x\rangle_j\langle y-x\rangle_j\sigma_{j,\gamma-j}(y), \tag{21}$$

where $\sigma_{j,\ell}$ is given by

$$\sigma_{j,\ell}(y) = \frac{2(-1)^j}{\langle 2y\rangle_{2j+1}}\sum_{i=0}^{j}\binom{2y}{i}\binom{2j-2y}{j-i}(y-i)^{2j+2\ell+1}. \tag{22}$$

We shall show that the same identity can be utilized to treat the moments of triple product of Catalan triangle numbers. The next theorem reduces the evaluation of $\Omega_\gamma^\delta(m,n,p)$ to those of $\Omega_0^\delta(m,n,p)$ and $\Omega_1^\delta(m,n,p)$, that have already been done in the previous section.

**Theorem 19.** *Let $m$, $n$, $p$, $\gamma$ be four nonnegative numbers and $\varepsilon$, $\delta = 0$ or $1$. We have*

$$\Omega_{\varepsilon+2\gamma}^\delta(m,n,p) = \sum_{j=0}^{\gamma}(-1)^j\langle 2m+\delta\rangle_{2j}\sigma_{j,\gamma-j}\big(m+\tfrac{\delta}{2}\big)\Omega_\varepsilon^\delta(m-j,n,p).$$

*Proof.* Recall the binomial sum by definition (5)

$$\Omega_\gamma^0(m,n,p) = \sum_{k\geq 0}k^\gamma\binom{2m}{m+k}\binom{2n}{n+k}\binom{2p}{p+k}.$$

We can manipulate it, by utilizing the case $x = k$ and $y = m$ of (21–22), as follows

$$\begin{aligned}
\Omega^0_{\varepsilon+2\gamma}(m,n,p) &= \sum_{k\geq 0} k^{\varepsilon}\binom{2m}{m+k}\binom{2n}{n+k}\binom{2p}{p+k} \\
&\times \sum_{j=0}^{\gamma}(-1)^j\langle m+k\rangle_j\langle m-k\rangle_j\sigma_{j,\gamma-j}(m) \\
&= \sum_{j=0}^{\gamma}(-1)^j\langle 2m\rangle_{2j}\sigma_{j,\gamma-j}(m) \\
&\times \sum_{k\geq 0} k^{\varepsilon}\binom{2m-2j}{m+k-j}\binom{2n}{n+k}\binom{2p}{p+k}.
\end{aligned}$$

We have therefore derived the following equality

$$\Omega^0_{\varepsilon+2\gamma}(m,n,p) = \sum_{j=0}^{\gamma}(-1)^j\langle 2m\rangle_{2j}\sigma_{j,\gamma-j}(m)\Omega^0_{\varepsilon}(m-j,n,p)$$

which corresponds to the case $\delta = 0$ of the theorem.

Analogously, another binomial sum defined by (5)

$$\Omega^1_{\gamma}(m,n,p) = \sum_{k\geq 1}\left(k-\tfrac{1}{2}\right)^{\gamma}\binom{2m+1}{m+k}\binom{2n+1}{n+k}\binom{2p+1}{p+k}$$

can be reformulated, with the help of the equations (21–22) under the replacements $x = k - 1/2$ and $y = m + 1/2$, as follows

$$\begin{aligned}
\Omega^1_{\varepsilon+2\gamma}(m,n,p) &= \sum_{k\geq 1}\left(k-\tfrac{1}{2}\right)^{\varepsilon}\binom{2m+1}{m+k}\binom{2n+1}{n+k}\binom{2p+1}{p+k} \\
&\times \sum_{j=0}^{\gamma}(-1)^j\langle m+k\rangle_j\langle m-k+1\rangle_j\sigma_{j,\gamma-j}\left(m+\tfrac{1}{2}\right) \\
&= \sum_{j=0}^{\gamma}(-1)^j\langle 2m+1\rangle_{2j}\sigma_{j,\gamma-j}\left(m+\tfrac{1}{2}\right) \\
&\times \sum_{k\geq 1}\left(k-\tfrac{1}{2}\right)^{\varepsilon}\binom{2m-2j+1}{m+k-j}\binom{2n+1}{n+k}\binom{2p+1}{p+k}.
\end{aligned}$$

This results in the following reduction formula

$$\Omega^1_{\varepsilon+2\gamma}(m,n,p) = \sum_{j=0}^{\gamma}(-1)^j\langle 2m+1\rangle_{2j}\sigma_{j,\gamma-j}\left(m+\tfrac{1}{2}\right)\Omega^1_{\varepsilon}(m-j,n,p)$$

which confirms the case $\delta = 1$ of the theorem. □

By carrying out the same procedure, we can prove the following reduction formula for the more general multiple product sums. The details will not be produced.

**Theorem 20.** *Let $\gamma$, $m$ and $\tilde{n} = \{n_1, n_2, \cdots, n_\ell\}$ be $\ell + 2$ nonnegative numbers. For $\varepsilon$, $\delta = 0$ or $1$, define the multiple product sums by*

$$\Lambda_\gamma^\delta(m, \tilde{n}) \;=\; \sum_{k \geq \delta} \big(k - \tfrac{\delta}{2}\big)^\gamma \binom{2m+\delta}{m+k} \prod_{i=1}^{p} \binom{2n_i+\delta}{n_i+k}.$$

*Then the following reduction formula holds*

$$\Lambda_{\varepsilon+2\gamma}^\delta(m, \tilde{n}) \;=\; \sum_{j=0}^{\gamma} (-1)^j \langle 2m+\delta\rangle_{2j} \sigma_{j,\gamma-j}\big(m + \tfrac{\delta}{2}\big) \Lambda_\varepsilon^\delta(m-j, \tilde{n}).$$

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