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# NEW CYCLIC KAUTZ DIGRAPHS WITH OPTIMAL DIAMETER

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Abstract. We obtain a new family of digraphs with minimal diameter, that is, given the number of vertices and out-degree, there is no other digraph with a smaller diameter. This new family of digraphs are called 'modified cyclic digraphs' $MCK(d,\ell)$, and it is derived from the Kautz digraphs $K(d,\ell)$ and from the so-called cyclic Kautz digraphs $CK(d,\ell)$.

The cyclic Kautz digraphs $CK(d,\ell)$ were defined as the digraphs whose vertices are labeled by all possible sequences $a_1\dots a_\ell$ of length $\ell$, such that each character $a_i$ is chosen from an alphabet of $d+1$ distinct symbols, where the consecutive characters in the sequence are different (as in Kautz digraphs), and also requiring that $a_1\neq a_\ell$. Their arcs are between vertices $a_1a_2\dots a_\ell$ and $a_2\dots a_\ell a_{\ell+1}$, with $a_1\neq a_\ell$ and $a_2\neq a_{\ell+1}$. Since $CK(d,\ell)$ do not have minimal diameter for their number of vertices, we construct the modified cyclic Kautz digraphs to obtain the same diameter as in the Kautz digraphs, and we also show that $MCK(d,\ell)$ are $d$-out-regular. Moreover, for $t\geq 1$, we compute the number of vertices of the iterated line digraphs $L^t(CK(d,\ell))$.

## 1. Introduction

Searching for graphs or digraphs with maximum number of vertices given maximum degree $\Delta$ and diameter $D$, or with minimum diameter given maximum degree $\Delta$ and number of vertices $N$ are two very prominent problems in graph theory. These problems are called the $(\Delta,D)$ problem and the

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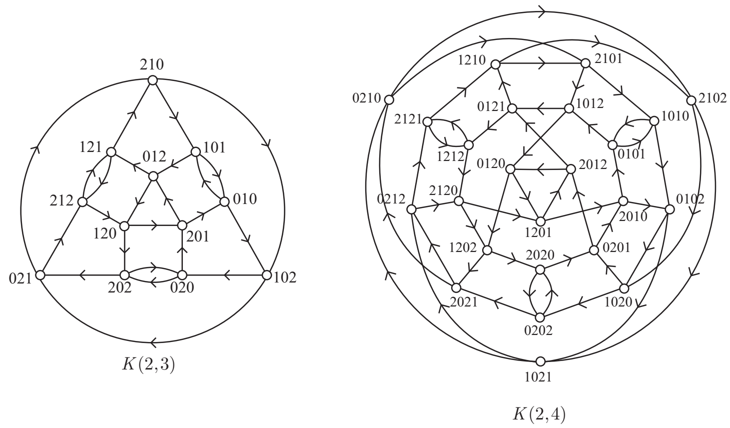


FIGURE 1. The Kautz digraphs $K(2,3)$ and $K(2,4)$.

$(\Delta, N)$ problem, respectively. See the comprehensive survey by Miller and Širáň [6] for more information. In this paper, we obtain a new family of digraphs with minimal diameter in the sense that, given the number of vertices and out-degree, there is no other digraph with a smaller diameter. This new family is called modified cyclic Kautz digraphs $MCK(d,\ell)$, and it is derived from the Kautz digraphs $K(d,\ell)$ and from the so-called cyclic Kautz digraphs $CK(d,\ell)$.

It is well-known that, for some integers $d \geq 2$ and $\ell \geq 1$, the Kautz digraphs $K(d,\ell)$ have vertices labeled by all possible sequences $a_1 \dots a_\ell$ of length $\ell$ with different consecutive symbols, $a_i \neq a_{i+1}$ for $i = 1, \dots, \ell-1$, from an alphabet $\Sigma$ of $d+1$ distinct symbols. The Kautz digraphs $K(d,\ell)$ have arcs between vertices $a_1 a_2 \dots a_\ell$ and $a_2 \dots a_\ell a_{\ell+1}$. See Figure 1 for a pair of examples.

The cyclic Kautz digraphs $CK(d,\ell)$ (see Figure 2) were defined in [2], as the digraphs whose labels of their vertices are as the ones of the Kautz digraphs, with the additional requirement that the first and last symbols must also be different ($a_1 \neq a_\ell$). The cyclic Kautz digraphs $CK(d,\ell)$ have arcs between vertices $a_1 a_2 \dots a_\ell$ and $a_2 \dots a_\ell a_{\ell+1}$, with $a_i \neq a_{i+1}$, $a_1 \neq a_\ell$, and $a_2 \neq a_{\ell+1}$.

Note that $CK(d,\ell)$ are subdigraphs of $K(d,\ell)$. Unlike in $K(d,\ell)$, any label of a vertex of $CK(d,\ell)$ can be cyclically shifted to form a label of another of its vertices. Note that, in contrast to the Kautz digraphs, $CK(d,\ell)$ are not $d$-regular (neither $d$-out-regular). Therefore, for $CK(d,\ell)$ the meaning of $d$ is the size of the alphabet minus one and, for $\ell > 3$, $d$ also corresponds to the maximum out-degree of $CK(d,\ell)$.

In [1, 2] the authors showed that the cyclic Kautz digraphs $CK(d,\ell)$ have number of vertices

$$n_{d,\ell} = (-1)^\ell d + d^\ell, \tag{1.1}$$

number of arcs

$$m_{d,\ell} = (d+1)d^{\ell} - (2d-1)n_{d,\ell-1} = (d+1)d^{\ell} - (2d-1)((-1)^{\ell-1}d + d^{\ell-1}), \tag{1.2}$$

and diameter

$$D = \begin{cases} 1 & \text{for } d = 1, \ell \geq 2 \text{ if } \ell \text{ is even; for } \ell = 1; \\ 2 & \text{for } d > 1, \ell = 2; \\ 2\ell - 1 & \text{for } d = 2, \ell = 4;\ \text{for } d = 3, \ell \geq 3;\ \text{for } d \geq 4, \ell = 3; \\ 2\ell - 2 & \text{for } d \geq 4, \ell \geq 4. \end{cases}$$

Note that, for $d = 1$ and $\ell \geq 2$ when $\ell$ is odd, the cyclic Kautz digraphs do not exist. Moreover, for $d = 2$ and $\ell \geq 3$ when $\ell \neq 4$, the cyclic Kautz digraphs are not connected.

It is known that the diameter of a line digraph $L(G)$ of a strongly connected digraph $G$ is $D(L(G)) = D(G) + 1$, even if $G$ is a nonregular digraph with the exception of directed cycles (see Fiol, Yebra, and Alegre [5]). With the line digraph technique, we obtain digraphs whose diameter is only one unit larger than the one of the original digraph. Applying this technique iteratively, we get digraphs whose diameter is asymptotically minimal for a large number of iterations. For this reason, we calculate the number of vertices of digraphs obtained with this technique for the cyclic Kautz digraphs. Computing the number of vertices of a $t$-iterated line digraph is easy for regular digraphs. Still, in the case of nonregular digraphs, it becomes an interesting combinatorial problem, which can be quite difficult to solve. Fiol and Lladó defined in [4] the partial line digraph $PL(G)$ of a digraph $G$, where some (but not necessarily all, as in the line digraph $L(G)$) of the vertices in $G$ become arcs in $PL(G)$. For a comparison between the partial digraph technique and other construction techniques to obtain digraphs with minimum diameter, see Miller, Slamin, Ryan, and Baskoro [7]. We use the partial line digraph technique to obtain the modified cyclic Kautz digraphs $MCK(d, \ell)$.

In this paper, in Section 2, we construct the modified cyclic Kautz digraphs $MCK(d, \ell)$ to obtain digraphs with the same diameter as the Kautz digraphs, and we show that $MCK(d, \ell)$ are $d$-out-regular. Moreover, in Section 3, we obtain the number of vertices of the $t$-iterated line digraph of $CK(d, \ell)$ for $1 \leq t \leq \ell - 2$, and for the case of $CK(d, 4)$ for all values of $t$. For the particular case of $CK(2, 4)$, these numbers of vertices follow a Fibonacci sequence with starting values 18 and 30.

We use the habitual notation for digraphs, that is, a *digraph* $G = (V, E)$ consists of a (finite) set $V = V(G)$ of vertices and a set $E = E(G)$ of arcs (directed edges) between vertices of $G$. There are no *multiple arcs*, that is, there is at most one arc from each vertex to any other. If $a = (u, v)$ is an arc between vertices $u$ and $v$, then vertex $u$ (and arc $a$) is adjacent to vertex $v$, and vertex $v$ (and arc $a$) is adjacent from $u$. Let $\Gamma^+_G(v)$ and $\Gamma^-_G(v)$ denote the set of vertices adjacent from and to vertex $v$, respectively. Their cardinalities are the *out-degree* $\delta^+_G(v) = |\Gamma^+_G(v)|$ of vertex $v$, and the

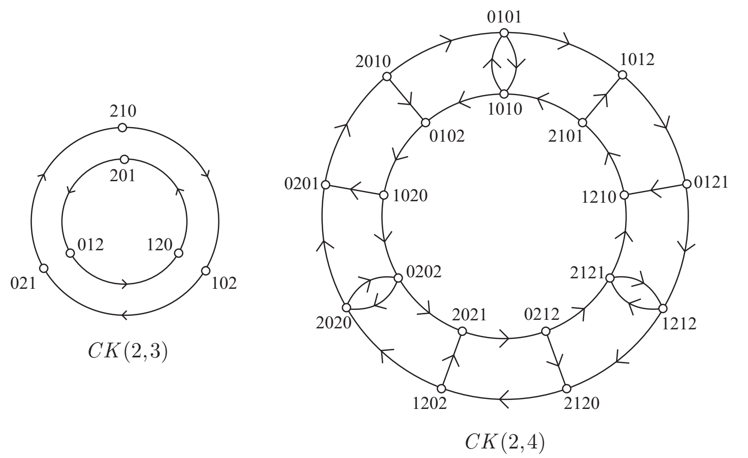


FIGURE 2. The cyclic Kautz digraphs $CK(2,3)$ and $CK(2,4)$.

*in-degree* $\delta^-_G(v) = |\Gamma^-_G(v)|$ of vertex $v$. A digraph $G$ is called *$d$-out-regular* if $\delta^+_G(v) = d$, *$d$-in-regular* if $\delta^-_G(v) = d$, and *$d$-regular* if $\delta^+_G(v) = \delta^-_G(v) = d$, for all $v \in V$.

## 2. The modified cyclic Kautz digraphs

Recall that the diameter of the Kautz digraphs is optimal, that is, for a fixed out-degree $d$ and number of vertices $(d+1)d^{\ell-1}$, the Kautz digraph $K(d,\ell)$ has the smallest diameter ($D = \ell$) among all digraphs with $(d+1)d^{\ell-1}$ vertices and degree $d$ (see, for example, Miller and Širáň [6]). Since the diameter of the cyclic Kautz digraphs $CK(d,\ell)$ is greater than the diameter of the Kautz digraphs $K(d,\ell)$, we construct the *modified cyclic Kautz digraphs* $MCK(d,\ell)$ by adding some arcs to $CK(d,\ell)$ to obtain the same diameter as $K(d,\ell)$, without increasing the maximum degree.

In a cyclic Kautz digraph $CK(d,\ell)$, a vertex labeled with $a_2 \dots a_{\ell+1}$ is forbidden if $a_2 = a_{\ell+1}$. For each such label, we replace the first symbol $a_2$ by one of the possible symbols $a'_2$ such that now $a'_2 \neq a_3, a_{\ell+1}$ (so $a'_2 \dots a_{\ell+1}$ represents a vertex). Then, we add arcs from vertex $a_1 \dots a_\ell$ to vertex $a'_2 \dots a_{\ell+1}$, with $a_1 \neq a_\ell$ and $a'_2 \neq a_3, a_{\ell+1}$. Note that $CK(d,\ell)$ and $MCK(d,\ell)$ have the same vertices, because we only add arcs to $CK(d,\ell)$ to obtain $MCK(d,\ell)$. See a pair of examples of modified cyclic Kautz digraphs in Figure 3.

The Kautz digraphs $K(d,\ell)$ can also be defined as iterated line digraphs of the complete symmetric digraphs $K_{d+1}$ (see Fiol, Yebra, and Alegre [5]). This means that the Kautz digraphs $K(d,\ell)$ can be obtained as the line digraph of $K(d,\ell-1)$. Namely,

$$K(d,\ell) = L^{\ell-1}(K_{d+1}),$$
$$K(d,\ell) = L(K(d,\ell-1)),$$

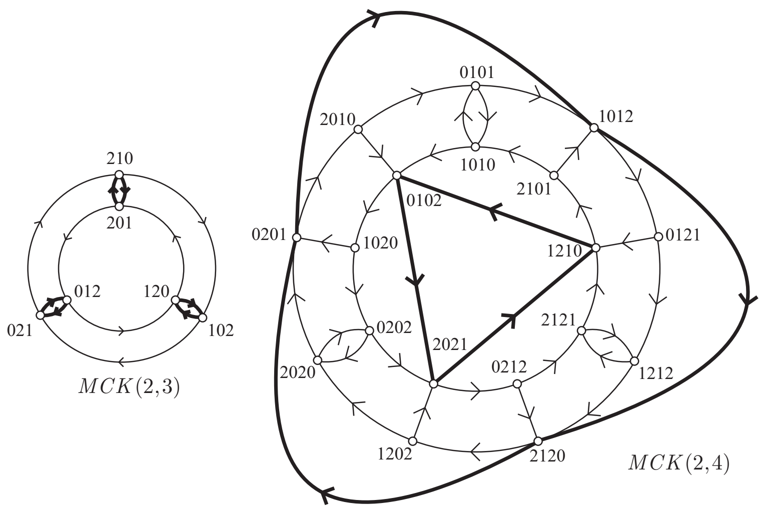


FIGURE 3. Modified cyclic Kautz digraphs $MCK(2,3)$ and $MCK(2,4)$ (the thick lines are the arcs added with respect to the corresponding cyclic Kautz digraphs).

where $L$ is the (1-iterated) line digraph of a digraph, and $L^t$ is the $t$-iterated line digraph.

Fiol and Lladó [4] defined the partial line digraph as follows. Let $E' \subseteq E$ be a subset of arcs that are adjacent to all vertices of $G$, that is, $\{v; (u,v) \in E'\} = V$. A digraph $PL(G)$ is said to be a *partial line digraph* of $G$ if its vertices represent the arcs of $E'$, that is, $V(PL(G)) = \{uv; (u,v) \in E'\}$, and a vertex $uv$ is adjacent to vertices $v'w$, for each $w \in \Gamma^+_G(v)$, where

$$v' = \begin{cases} v & \\ & \text{if } vw \in V(PL(G)), \\ \text{any other element of } \Gamma^-_G(w) \text{ such that } v'w \in V(PL(G)) & \\ & \text{otherwise.} \end{cases}$$

See an example of this definition in Figure 4.

**Theorem 2.1** ([4])**.** *Let $G = (V,E)$ be a $d$-out-regular digraph (with $d > 1$) of order $N$ and diameter $D$. Then, the order $N_{PL}$ and diameter $D_{PL}$ of a partial line digraph $PL(G)$ satisfy $N \le N_{PL} \le dN$, $D \le D_{PL} \le D+1$. Moreover, $PL(G)$ is also $d$-out-regular.*

Note that $N_{PL} = N$ if and only if $PL(G)$ is isomorphic to $G$. Moreover, $N_{PL} = |E|$ if and only if $PL(G)$ is isomorphic to $L(G)$.

**Theorem 2.2.** *The modified cyclic Kautz digraph $MCK(d,\ell)$ has the following properties:*

(*a*) *It is $d$-out-regular.*

(*b*) *Its diameter is $D = \ell$, which is the same as the diameter of the Kautz digraph $K(d, \ell)$.*

*Proof.* First, we show that the modified cyclic Kautz digraph $MCK(d, \ell)$ can be obtained as a partial line digraph of Kautz digraph $K(d, \ell - 1)$: $MCK(d, \ell) = PL(K(d, \ell - 1))$. See a scheme of the process for obtaining $MCK(d, \ell)$ from $K(d, \ell - 1)$ in Figure 4. The vertices of $MCK(d, \ell)$ are the same vertices as the ones of $CK(d, \ell)$, that is, sequences $a_1 \dots a_\ell$ with $a_i \neq a_{i+1}$ and $a_1 \neq a_\ell$, for $i = 1, \dots, \ell - 1$. To obtain a $MCK(d, \ell)$, we add some arcs to $CK(d, \ell)$. Since label $a_2 a_3 \dots a_\ell a_2$ is forbidden, it does not belong to $MCK(d, \ell)$. The added arcs are the following:

$$a_1 a_2 a_3 \dots a_\ell \longrightarrow a'_2 a_3 \dots a_\ell a_2.$$

Moreover, the vertices of $K(d, \ell - 1)$ are $a_1 \dots a_{\ell-1}$ with $a_i \neq a_{i+1}$ for $i = 1, \dots, \ell - 2$, and have arcs between vertices $a_1 \dots a_{\ell-1}$ and $a_2 \dots a_\ell$. Then, the arcs of $K(d, \ell-1)$ are $a_1 \dots a_\ell$, with $a_i \neq a_{i+1}$ for $i = 1, \dots, \ell-1$. Let $E$ be the set of arcs of $K(d, \ell - 1)$, and let $E' \subseteq E$ be the subset of arcs that satisfies $a_1 \neq a_\ell$. Now we apply the partial digraph technique to $E'$. Replacing the arcs of $E'$ by vertices, we obtain that the vertices of $PL(K(d, \ell-1))$ are $a_1 \dots a_\ell$, with $a_i \neq a_{i+1}$ and $a_1 \neq a_\ell$, for $i = 1, \dots, \ell-1$. According to the definition of partial line digraph, there are two kinds of arcs in $PL(K(d, \ell-1))$. The first kind of arcs goes from vertex $a_1 \dots a_\ell$ to vertex $a_2 \dots a_{\ell+1}$, both vertices belonging to the set of vertices of $PL(K(d, \ell - 1))$. The second kind of arcs goes from vertex $a_1 \dots a_\ell$ to vertex $a'_2 a_3 \dots a_\ell a_2$, where we replaced the forbidden label $a_2 a_3 \dots a_\ell a_2$ in $PL(K(d, \ell - 1))$ for $a'_2 a_3 \dots a_\ell a_2$, for a value of $a'_2$ such that $a'_2 \neq a_2, a_3$. Since we obtain the same vertices and arcs in $MCK(d, \ell)$ as in $PL(K(d, \ell - 1))$, they are the same digraph.

(*a*) Since $K(d, \ell - 1)$ is $d$-out-regular (indeed, it is $d$-regular), then by Theorem 2.1 its partial line digraph $PL(K(d, \ell - 1)) = MCK(d, \ell)$ is also $d$-out-regular.

(*b*) Since the diameter of $K(d, \ell-1)$ is $D = \ell-1$, then by Theorem 2.1 the diameter of its partial line digraph $PL(K(d, \ell - 1)) = MCK(d, \ell)$ is $\ell$. The diameter of $PL(K(d, \ell - 1))$ cannot be $\ell - 1$, because $PL(K(d, \ell - 1)) \neq K(d, \ell - 1)$. Then, the diameter of $MCK(d, \ell)$ is $\ell$, which is the same as the diameter of $K(d, \ell)$.

□

## 3. Line digraphs iterations of the cyclic Kautz digraphs

As done with the Kautz digraphs $K(d, \ell)$, here we compute the number of vertices of the $t$-iterated line digraphs of the cyclic Kautz digraphs $CK(d, \ell)$, which are nonregular digraphs. In contrast with the case of regular digraphs, the resolution of the nonregular case is not immediate.

As said in the introduction, the diameter of a line digraph $L(G)$ of a digraph $G$ is $D(L(G)) = D(G) + 1$, even if $G$ is a nonregular digraph with

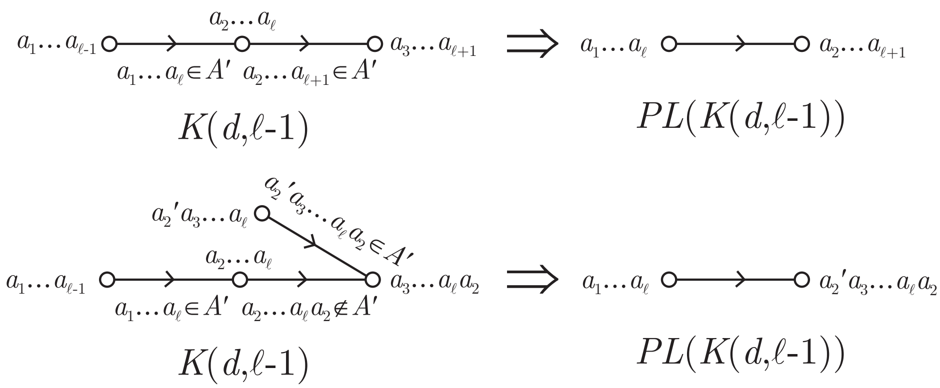


FIGURE 4. Both cases in the partial line digraph of $K(d, \ell - 1)$.

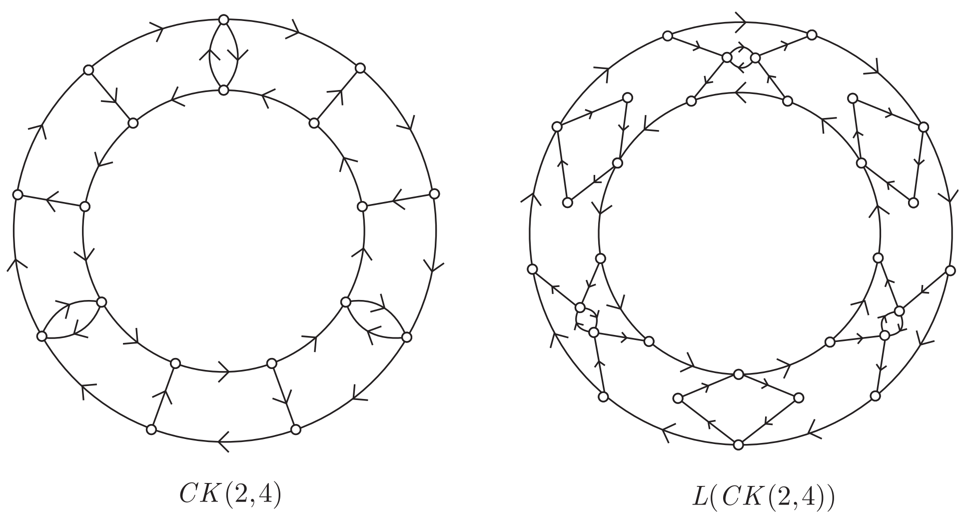


FIGURE 5. $CK(2, 4)$ and its line digraph at iteration $t = 1$.

the exception of directed cycles. Then, with the line digraph technique, we obtain digraphs whose diameter is asymptotically minimal for a large number of iterations (see Fiol, Yebra, and Alegre [5], and Fàbrega and Fiol [3]). For this reason, we calculate the number of vertices of digraphs obtained with this technique. In Figure 5, there is an example of a $CK(d, \ell)$ and its line digraph.

**Theorem 3.1.** *Let $\ell(\geq 3)$ and $d(\geq 1)$ be integers. Then, the number of vertices of the $t$-iterated line digraph $L^t(CK(d, \ell))$, for $1 \leq t \leq \ell - 2$, is*

$$(d^2 - d + 1)^t d^{\ell - t} + \frac{1}{2}(-1)^\ell d\Big[d^t(d+1) - (d-2)^t(d-1)\Big].$$

Recall that two vertices of $CK(d,\ell)$ are adjacent when they have the form $a_1a_2\dots a_\ell$ and $a_2\dots a_\ell a_{\ell+1}$ (with $a_1 \neq a_\ell$ and $a_2 \neq a_{\ell+1}$). This suggests representing an arc of $CK(d,\ell)$ as a sequence of $\ell+1$ characters $a_1a_2\dots a_\ell a_{\ell+1}$ satisfying $a_i \neq a_{i+1}$ for $1 \leq i \leq \ell$, $a_1 \neq a_\ell$, and $a_2 \neq a_{\ell+1}$. Note that $a_1$ can be equal to $a_{\ell+1}$. See Figure 6 for an example. The arcs of $CK(d,\ell)$ are the vertices of the iterated line digraph of $CK(d,\ell)$ at the first iteration $t=1$. Two such vertices are adjacent when they have the form $a_1a_2\dots a_\ell a_{\ell+1}$ and $a_2a_3\dots a_{\ell+1}a_{\ell+2}$, with $a_1 \neq a_\ell$, $a_2 \neq a_{\ell+1}$, and $a_3 \neq a_{\ell+2}$. Therefore, a vertex of $L^2(CK(d,\ell))$ can be represented by a sequence of $\ell+2$ characters satisfying $a_i \neq a_{i+1}$, $a_1 \neq a_\ell$, $a_2 \neq a_{\ell+1}$, and $a_3 \neq a_{\ell+2}$. In general, for $0 \leq t \leq \ell-2$, the vertices of $L^t(CK(d,\ell))$ are represented by sequences $a_1a_2\dots a_{\ell+t}$ satisfying $a_i \neq a_{i+1}$ for $1 \leq i \leq \ell+t-1$ and $a_i \neq a_{i+\ell-1}$ for $1 \leq i \leq t+1$. We denote the set of these sequences by $\mathcal{S}$ for any $t$ and $\ell$. See Figure 7.

For the case $d=1$, if $\ell$ is even, $CK(d,\ell)$ is a digon (that is, two vertices with two opposite arcs), namely the two sequences of alternating characters, and $CK(d,\ell)$ has no vertices if $\ell$ is odd. It is easy to verify that this also holds for the set $\mathcal{S}$. Thus, $L^t(CK(d,\ell))$ has the claimed number of vertices.

Let us now prove the case $d \geq 2$. To count the number of elements of $\mathcal{S}$, we first only count sequences of the form $a_1a_2\dots a_\ell$ for $\ell$ even, with $a_i \neq a_{i+1}$ for $1 \leq i \leq \ell-1$ and $a_i \neq a_{i+\ell/2}$ for $1 \leq i \leq \ell/2$. We denote this set of sequences of even length by $\mathcal{S}' \subset \mathcal{S}$.

We partition $\mathcal{S}'$ into two classes $\mathcal{C}_\ell$ and $\mathcal{D}_\ell$, where $\mathcal{C}_\ell$ is the set of those sequences of $\mathcal{S}'$ that have $a_{\ell/2+1} = a_\ell$, and $\mathcal{D}_\ell$ is the set of the remaining ones. We also introduce an auxiliary class of sequences $\mathcal{B}_\ell$, which is defined as $\mathcal{D}_\ell$ with the further restriction that $a_{\ell/2+1} = a_{\ell/2}$; hence, the elements of $\mathcal{B}_\ell$ are not sequences of $\mathcal{S}$. See Figure 8. We denote the cardinalities of $\mathcal{C}_\ell, \mathcal{D}_\ell$, and $\mathcal{B}_\ell$ with $C_\ell, D_\ell$, and $B_\ell$, respectively.

For the first values of $\ell$, it is easy to calculate

$$
\begin{aligned}
&B_4 = (d+1)d^2, && B_6 = (d+1)d(d-1)^3,\\
&C_4 = 0, && C_6 = (d+1)d(d^3-2d^2+3d-1),\\
&D_4 = (d+1)d(d-1)^2, && D_6 = (d+1)d(d-1)^2(d^2-2d+3).
\end{aligned}
$$

For $\ell > 6$, we generate all sequences of $\mathcal{B}_\ell$, $\mathcal{C}_\ell$, and $\mathcal{D}_\ell$ from all sequences of $\mathcal{B}_{\ell-2}$, $\mathcal{C}_{\ell-2}$, and $\mathcal{D}_{\ell-2}$. This is done by inserting a new character between $a_{\ell/2}$ and $a_{\ell/2+1}$, and another new character after $a_\ell$. See Figure 9. Let us now describe how to generate sequences from the class $\mathcal{B}_\ell$ using only the classes $\mathcal{C}_{\ell-2}$ and $\mathcal{D}_{\ell-2}$. There is only one possibility to insert a new character between $a_{\ell/2}$ and $a_{\ell/2+1}$, that is, $a_{\text{new}} = a_{\ell/2+1}$. The other new character $a_{\ell+2}$ has to be different from $a_\ell$ and from $a_{\text{new}} = a_{\ell/2+1}$. If we start with a sequence from $\mathcal{D}_{\ell-2}$, then $a_{\text{new}} \neq a_\ell$, and there are $d-1$ possible ways to insert character $a_{\ell+2}$. If we start with a sequence from $\mathcal{C}_{\ell-2}$, then $a_{\text{new}} = a_\ell$, and there are $d$ possible ways to insert character $a_{\ell+1}$. We therefore obtain

$$B_\ell = (d-1)D_{\ell-2} + d\,C_{\ell-2}.$$

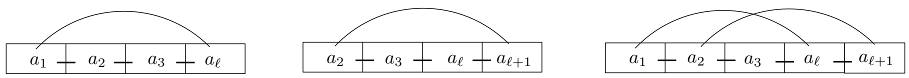


FIGURE 6. Sequences representing two vertices of $CK(d,4)$ and the arc between them. The lines between characters indicate that these characters must be different.

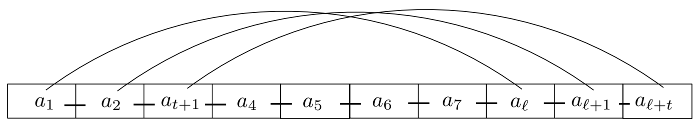


FIGURE 7. A sequence of $\ell+t$ characters that represents a vertex of the iterated line digraph of a cyclic Kautz digraph $CK(d,\ell=8)$ at iteration $t=2$.

Similar arguments show

$$C_\ell = (d-1)D_{\ell-2} + d\,B_{\ell-2},$$

$$D_\ell = (d^2-3d+3)D_{\ell-2} + (d-1)^2C_{\ell-2} + (d-1)^2B_{\ell-2}.$$

Note that every sequence is generated exactly once.

**Lemma 3.2.** *The system*

$$\begin{aligned} B_\ell &= (d-1)D_{\ell-2} + d\,C_{\ell-2} \\ C_\ell &= (d-1)D_{\ell-2} + d\,B_{\ell-2} \\ D_\ell &= (d^2-3d+3)D_{\ell-2} + (d-1)^2C_{\ell-2} + (d-1)^2B_{\ell-2} \end{aligned}$$

*with initial values*

$$\begin{aligned} B_6 &= (d+1)d(d-1)^3 \\ C_6 &= (d+1)d(d^3-2d^2+3d-1) \\ D_6 &= (d+1)d(d-1)^2(d^2-2d+3) \end{aligned}$$

*has solution*

$$\begin{aligned} B_\ell &= d(d^2-d+1)^{\ell/2-1} + \frac{1}{2}(-1)^{\ell/2-1}d(d-1)(d-2)^{\ell/2-1} \\ &\quad + \frac{1}{2}(-1)^{\ell/2}(d+1)d^{\ell/2}, \\ C_\ell &= \frac{1}{2}(-1)^{\ell/2-1}(d-1)d(d-2)^{\ell/2-1} + d(d^2-d+1)^{\ell/2-1} \\ &\quad - \frac{1}{2}(-1)^{\ell/2}d^{\ell/2}(d+1), \\ D_\ell &= (d-1)d\left((d^2-d+1)^{\ell/2-1} - (-1)^{\ell/2-1}(d-2)^{\ell/2-1}\right). \end{aligned}$$

This lemma can be proven by using simple computations.

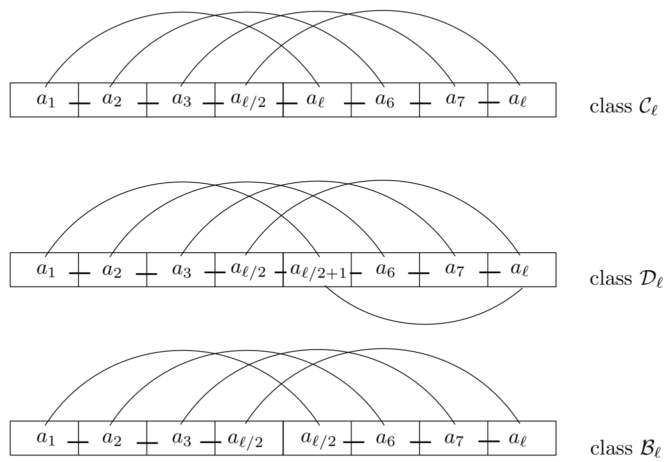


FIGURE 8. The classes $\mathcal{B}_\ell, \mathcal{C}_\ell$, and $\mathcal{D}_\ell$.

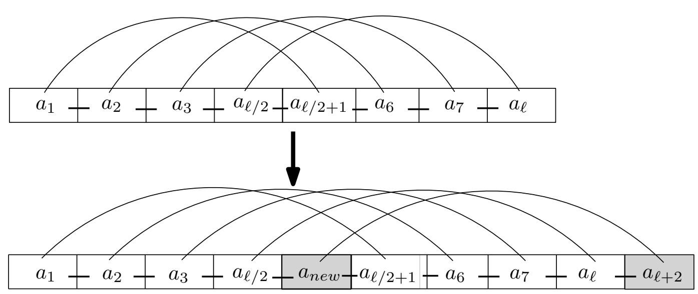


FIGURE 9. A sequence of length $\ell+2$ is obtained from one of length $\ell$ by inserting a new character between $a_{\ell/2}$ and $a_{\ell/2+1}$ and another one, $a_{\ell+2}$, after $a_\ell$.

*Remark:* From Lemma 3.2, we immediately obtain the number of vertices of the $t = (\ell-2)$-iteration of the line digraph of a cyclic Kautz digraph $CK(d,\ell)$ for $d \geq 2$ and $\ell \geq 3$, which is $C_{2\ell-2} + D_{2\ell-2}$.

We now count the number of sequences from the set $\mathcal{S}$ in general. Recall that an element of $\mathcal{S}$ has the form $a_1a_2\dots a_{\ell+t}$ satisfying $a_i \neq a_{i+1}$ for $1 \leq i \leq \ell+t-1$ and $a_i \neq a_{i+\ell-1}$ for $1 \leq i \leq t+1$. In the following, we set $r = t+1$ (we assume $r$ and $d$ are fixed integers) and represent the set of sequences $\mathcal{S}$ by $\mathcal{E}_j$, where $j = 0, 1, 2, \dots$ An element of $\mathcal{E}_j$ has the form $a_1 \dots a_{2r+j}$ satisfying $a_i \neq a_{i+1}$ for $1 \leq i \leq 2r+j-1$, and $a_i \neq a_{i+r+j}$ for $1 \leq i \leq r$. $E_j$ is the cardinality of $\mathcal{E}_j$. Observe that $E_0 = C_{2r} + D_{2r}$. To determine $\mathcal{E}_1$, we insert a new character between characters $a_r$ and $a_{r+1}$ in each sequence of $\mathcal{B}_{2r}, \mathcal{C}_{2r}$, and $\mathcal{D}_{2r}$. This gives

$$E_1 = d\,B_{2r} + (d-1)(C_{2r} + D_{2r}).$$

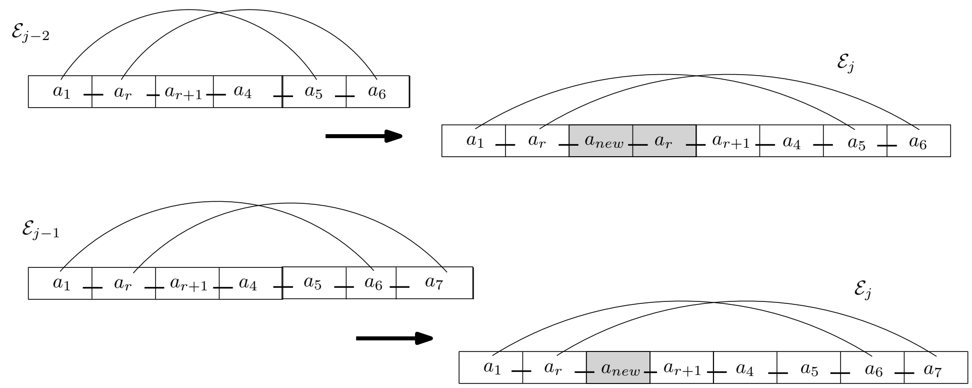


FIGURE 10. The sequences of $\mathcal{E}_j$ are obtained from the sequences of $\mathcal{E}_{j-1}$ and $\mathcal{E}_{j-2}$. Here, $r = 2$ and $j = 4$.

For $j > 1$, $\mathcal{E}_j$ is obtained from $\mathcal{E}_{j-1}$ and $\mathcal{E}_{j-2}$. In each sequence of the set $\mathcal{E}_{j-1}$, we insert a new character between the characters $a_r$ and $a_{r+1}$, which can be done in $d-1$ ways. In each sequence of the set $\mathcal{E}_{j-2}$, we first duplicate the character $a_r$, and then insert a new character between these two characters $a_r$. This can be done in $d-1$ ways. Figure 10 depicts the insertion procedure.

Note that each sequence of $\mathcal{E}_j$ is generated exactly once. Thus, we only need to solve the recursion given in the following result.

**Lemma 3.4.** *For fixed integers $r$ and $d$, the recursion*

$$E_j = (d-1)E_{j-1} + d\,E_{j-2}$$

*with initial values $E_0 = C_{2r} + D_{2r}$ and $E_1 = d\,B_{2r} + (d-1)(C_{2r} + D_{2r})$ has solution*

$$E_j = (-1)^j (B_{2r} + C_{2r} + D_{2r}) \frac{1 - (-d)^{j+1}}{d+1} - B_{2r}(-1)^j.$$

Again this lemma can be proven with simple computations.

We now use $B_{2r} + C_{2r} + D_{2r} = (d+1)d(d^2-d+1)^{r-1}$ and

$$B_{2r} = d(d^2-d+1)^{r-1} + \frac{1}{2}(-1)^{r-1} d(d-1)(d-2)^{r-1} + \frac{1}{2}(-1)^r (d+1)d^r.$$

Then,

$$E_j = (d^2-d+1)^{r-1} d^{j+2} + \frac{1}{2}(-1)^{r+j}(d-2)^{r-1}(d-1)d + \frac{1}{2}(-1)^{r+j+1} d^r (d+1).$$

From $E_j$ we now obtain the number of vertices of the iterated line graph of $CK(d,\ell)$ at iteration $t \geq 1$. Since $t = r-1$ and $j = \ell - t - 2$, we arrive at the claimed formula of Theorem 3.1:

$$(d^2-d+1)^t d^{\ell-t} + \frac{1}{2}(-1)^{\ell+1}(d-2)^t(d-1)d + \frac{1}{2}(-1)^\ell d^{t+1}(d+1).$$

*Remark:* Theorem 3.1 also holds for $t = 0$ if, for the particular case $d = 2$, the indeterminate form $(d-2)^t$ is defined as 1.

Note that, in general, the $t$-iterated line digraph of a cyclic Kautz digraph is neither a Kautz digraph, nor a cyclic Kautz digraph. But if the length is $\ell = 2$, then it is clear that $CK(d, 2)$ (and all its iterated line digraphs) are Kautz digraphs.

Theorem 3.1 gives the number of vertices at the $t$-iteration of the line digraph of $CK(d, \ell)$, with $1 \le t \le \ell - 2$. Now we compute the number of vertices of $L^t(CK(d, \ell))$ without restriction on the value of $t$, for the particular case $\ell = 4$. Let $N_t$ denote the number of vertices of $L^t(CK(d, 4))$.

**Proposition 3.6.** *The number of vertices $N_t$ of the iterated line digraph of $CK(d, 4)$ at iteration $t \ge 0$ is*

$$N_t = \alpha \left( \frac{d - 1 + \sqrt{d^2 - 2d + 5}}{2} \right)^t + \beta \left( \frac{d - 1 - \sqrt{d^2 - 2d + 5}}{2} \right)^t,$$

*where*

$$\alpha = \frac{1}{2} d(d+1) \left( d^2 - d + 1 + \frac{d^3 - 2d^2 + 4d - 1}{\sqrt{d^2 - 2d + 5}} \right)$$
$$\beta = \frac{1}{2} d(d+1) \left( d^2 - d + 1 - \frac{d^3 - 2d^2 + 4d - 1}{\sqrt{d^2 - 2d + 5}} \right).$$

*Moreover, if $d = 2$, $N_t$ follows a Fibonacci sequence with initial values* 18 *and* 30*.*

*Proof.* In the following, we construct $L^t(CK(d, \ell))$ by adding a new symbol $a_{4+t}$ to every vertex $a_1 \dots a_{3+t}$ of $L^{t-1}(CK(d, \ell))$. We make a partition of $L^{t-1}(CK(d, 4))$ into two sets, one with sequences such that $a_{t+1} = a_{t+3}$, and the other one with sequences that satisfy $a_{t+1} \neq a_{t+3}$. In the first case, the cardinality of the number of vertices is called $N^+_{t-1}$ and, in the second one, it is called $N^-_{t-1}$. Hence, $N_{t-1} = N^+_{t-1} + N^-_{t-1}$. See a scheme for $\ell = 4$ in Figure 11, where the lines drawn between symbols indicate that they must be different.

With $d + 1$ symbols, given the characters $a_{t+1}$ and $a_{t+3}$, for $a_{t+4}$, there are $d$ possible symbols corresponding to $N^+_{t-1}$ (that is, with $a_{t+1} = a_{t+3}$), and $d - 1$ corresponding to $N^-_{t-1}$ (that is, with $a_{t+1} \neq a_{t+3}$). Then, $N_t = d\, N^+_{t-1} + (d-1)N^-_{t-1}$. Moreover, $N_{t-2} = N^+_{t-1}$, because from a vertex of $L^{t-2}(CK(d, 4))$ there is only one possible symbol for the character $a_{t+3}$ to obtain a vertex of $L^{t-1}(CK(d, 4))$, such that $a_{t+1} = a_{t+3}$. See again Figure 11. Thus, $N^-_{t-1} = N_{t-1} - N_{t-2}$, and $N_t$ satisfies the recurrence equation $N_t = (d-1)N_{t-1} + N_{t-2}$, with initial conditions $N_0 = n_{d,4} = d^4 + d$ (see eq. (1.1)) and $N_1 = m_{d,4} = d(d+1)(d^3 - 2d^2 + 3d - 1)$ (see eq. (1.2)).

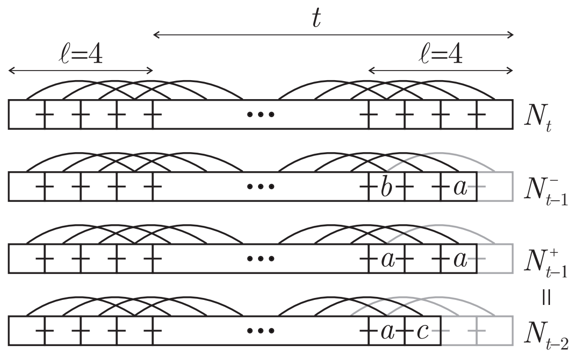


FIGURE 11. Scheme to obtain the number of vertices in the $t$-iterated line digraph of $CK(d,4)$. We indicate in grey the characters that have to be added to obtain a sequence with $t+4$ characters. The lines between two symbols represent that they must be different.

Solving this recurrence equation, we have

$$N_t = \alpha\left(\frac{d-1+\sqrt{d^2-2d+5}}{2}\right)^t + \beta\left(\frac{d-1-\sqrt{d^2-2d+5}}{2}\right)^t.$$

With the initial conditions $N_0$ and $N_1$, we get

$$\begin{aligned}\alpha &= \frac{1}{2}d(d+1)\left(d^2-d+1+\frac{d^3-2d^2+4d-1}{\sqrt{d^2-2d+5}}\right),\\ \beta &= \frac{1}{2}d(d+1)\left(d^2-d+1-\frac{d^3-2d^2+4d-1}{\sqrt{d^2-2d+5}}\right).\end{aligned}$$

As a particular case, if $d=2$, then the recurrence equation is $N_t = N_{t-1} + N_{t-2}$, with initial values $N_0 = 18$ and $N_1 = 30$. Then,

$$N_t = \left(9+\frac{21}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^t + \left(9-\frac{21}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^t,$$

and we obtain the values $N_t = 18, 30, 48, 78, \dots$ for $t = 0, 1, 2, 3, \dots$, which is a Fibonacci sequence with initial values 18 and 30. □

We leave as an open problem to find the number of vertices of the $t$-iterated cyclic Kautz digraph $CK(d,\ell)$ for the remaining values of $d$, $t$, and $\ell$.

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