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REGULARITY IN WEIGHTED GRAPHS A SYMMETRIC FUNCTION APPROACH

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Abstract. This work describes how the class of k-regular multigraphs with edge multiplicities from a finite set can be expressed using symmetric species results of Méndez. Consequently, the generating functions can be computed systematically using the scalar product of symmetric functions. This gives conditions on when the classes are D-finite using criteria of Gessel, and a potential route to asymptotic enumeration formulas.

1. Introduction

The asymptotic enumeration of regular graphs is a compelling topic that has appeared in many forms in combinatorics over the past half century. There are several approaches, and each has its own conditions and results. Here, we revisit the symmetric function approach, introduced by Goulden, Jackson, and Reilly [10], generalized by Gessel [9], and automated by Chyzak, Mishna, and Salvy [4]. The goal of this work is to give insight on the following problem posed by McKay in a recent Oberwolfach problem session:

Problem 1.1 (McKay [16])**.** *Let $\mathcal{J}$ and $\mathcal{J}^*$ be subsets of the nonnegative integers, and let $\mathbf{d} = (d_1, \dots, d_n)$ be a vector of nonnegative integers. Let $M(n, \mathcal{J}, \mathcal{J}^*)$ be the number of symmetric matrices whose diagonal entries are drawn from $\mathcal{J}^*$, off-diagonal entries from $\mathcal{J}$, and whose row sums are $d_1, \dots, d_n$. As usual in graph theory, entries on the diagonal are counted twice. We are interested in the asymptotic value of $M(n, \mathcal{J}, \mathcal{J}^*)$ in the sparse case, where the row sums do not grow very quickly with n.*

We consider this problem in the case that the d_i take on a finite number of values. (We call this set of possible values $\mathcal{D}$.) We show that the generating function of the sequence $M(n, \mathcal{J}, \mathcal{J}^*)$ is D-finite under certain conditions. (We recall the definitions in Section 3.) This gives a template for the form

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of the asymptotic expansion requested by McKay, and identifies them as an example of the multiassembly species construction of Méndez [17].

A univariate generating function is D-finite if it satisfies a linear differential equation with polynomial coefficients. This property has considerable implications on asymptotic enumeration: D-finite functions have restrictions on their asymptotic form; asymptotic information is encoded in the differential equation that it satisfies; they can be treated with a number of automated tools. Although we do not compute asymptotic formulas here, the fact that the generating functions are D-finite can be useful for precisely such a computation. For example, there are restrictions on the constant and the sub-exponential growth. See [6, Theorem 2] for more details.

We cast this problem in graph theoretic language as follows in order to state our results precisely. Let $\mathcal{J}$ and $\mathcal{D}$ be sets of positive integers, and suppose additionally that $\mathcal{D}$ is finite. Let $\mathcal{G}_{\mathcal{J},\mathcal{D},n}$ be the set of well-labelled graphs on n vertices where edge weights are from $\mathcal{J} \cup \{0\}$ and the sum of weights of the edges incident to any given vertex is an element of $\mathcal{D}$. The case when all $d_i = k$ is the case of k-regular graphs. Here, a graph is *well-labelled* if the label set is $\{1, 2, \ldots, n\}$, where n is the number of vertices. Initially, we only consider graphs without loops and hence $\mathcal{J}^* = \{0\}$. We later describe how to generalize to other finite $\mathcal{J}^*$.

The following is our main result. It appears below as Theorem 4.3.

Theorem. *Let $\mathcal{J}$ and $\mathcal{D}$ be finite sets of positive integers. Let $G_{\mathcal{J},\mathcal{D}}(z)$ be the generating function for the class $\mathcal{G}_{\mathcal{J},\mathcal{D}}$ of well-labelled simple graphs where edge weights are from $\mathcal{J} \cup \{0\}$ and the sum of weights of the edges incident to any given vertex is an element of $\mathcal{D}$. Then,*

$$G_{\mathcal{J},\mathcal{D}}(z) = \sum_n |\mathcal{G}_{\mathcal{J},\mathcal{D},n}| \, z^n$$

is D-finite. Furthermore, the differential equation satisfied by $G_{\mathcal{J},\mathcal{D}}(z)$ is theoretically computable.

The condition of finiteness on $\mathcal{J}$ is not necessary, but is a consequence of the finiteness of $\mathcal{D}$. We show how to relax the condition of finiteness, once there is more notation developed.

To illustrate the notation, remark $G_{\{1\},\{k\}}(z)$ is the generating function for simple, labelled, k-regular graphs, and $G_{\{1,2,\ldots,k\},\{k\}}(z)$ is the generating function for labelled k-regular multigraphs. The D-finiteness of these generating functions for generic k was conjectured by Goulden, Jackson and Reilly [10], and was proved by Gessel [9]. Our strategy proves the more general result by using the work of Méndez [17] to create the same framework as Gessel [9], to which the work of Chyzak et al. [4] then applies. This amounts to building a symmetric function encoding of the full class of graphs, and then performing a subseries extraction to realize the degree restriction. These are all theoretically effective, and hence the differential equations are potentially computable.

As Gessel noted, the class of all regular graphs is *not* D-finite. Consequently, these classes are likely to constitute the largest classes of regular graphs with D-finite generating functions.

1.1. **Contribution: Structure of graphs with controlled edge multiplicities.** McKay remarked in his problem "The simplest non-trivial case is $\mathcal{J}^* = \{0\}$ and $\mathcal{J} = \{0, 2, 3\}$". In our notation, $\mathcal{J}^* = \{0\}$ corresponds to the criterion that the graphs have no loops, and our $\mathcal{J}$ is the same as McKay's. We first consider $d_i = k$ for all i, and loosen this to consider then $d_i \in \mathcal{D}$ for finite $\mathcal{D}$ and conclude with some more general comments. We also address $\mathcal{J}^* = \{0, 1\}$.

Our principal contribution is a new formulation of McKay's problem, and the resulting proof of D-finiteness. We find our approach to be of interest as it gives a new way to view his problem, and also a new example for the class of symmetric species. In this context, generalizations to hypergraph variants are very natural.

In Section 2 we describe how to write the graph generating functions using symmetric functions. The construction generalizes our previous work [18] in a straightforward way using the multiassemblies studied by Méndez [17]. The construction uses the species theory formalism [2], but we leave the category theoretic details to previous sources to avoid a rather substantial detour that is well described elsewhere. We remark, however, this appears to be different than the description of graphs as a species recently developed by Gainer-Dewar and Gessel [8]. That said, our Proposition 3.3 bears some resemblance to the formulas in Section 6 of Henderson's species generalization [12], which is the formalism upon which their work is based. Perhaps this problem is a good entry point for that theory.

The D-finiteness result follows quickly once we adapt Waring's formula. This is explained in Section 4. We conclude with some directions on how to weaken the conditions as stated.

2. Labelled graph generating functions

We start with a systematic encoding of graph classes using symmetric functions.

2.1. **Simple graphs and X-generating functions.** Let G be a simple graph with vertex set $V(G) \subset \mathbb{Z}^+$, and edge set $E(G)$. To each vertex i we associate the weight x_i and to the graph G we associate the monomial $\pi(G)$ defined

$$\pi(G) = \prod_{\{i,j\} \in E(G)} x_i x_j = x_1^{d_1} x_2^{d_2} \dots x_k^{d_k},$$

where d_i is the degree of the vertex i. Let $\mathbf{G}(X)$ be the generating function of the set of all labelled simple graphs $\mathcal{G}$, where the vertex weights are a subset of $X = \{x_1, x_2, \dots, \}$:

$$\mathbf{G}(X) = \sum_{G \in \mathcal{G}} \pi(G) = \prod_{i<j} (1 + x_i x_j) .$$

(We remark that the well-labelled graphs is a subset of $\mathcal{G}$). To see the second equality, note that every edge is either present once, or not at all. Similarly, if $\overline{\mathcal{G}}$ is the set of graphs that permit multiple edges (but not loops),

$$\overline{\mathbf{G}}(X) = \sum_{G \in \overline{\mathcal{G}}} \pi(G) = \prod_{i<j} \left(\frac{1}{1 - x_i x_j} \right) ,$$

as every edge exists some nonnegative integer number of times.

Under this description, the set $\mathcal{G}$ is a symmetric species [17], and the series encoding $\mathbf{G}(X)$ is what Méndez calls the associated X-generating function. Our strategy is to determine the X-generating function for the class of graphs of $\mathcal{G}_{\mathcal{J}}$ in which edge weights are incorporated, and treated as multiplicities. (Hence, it is essential that the edge weights be positive integers.) It is straightforward to get an expression for this, and then it is a mechanical manipulation to get a form we desire.

The X-generating function for the class of labelled graphs with graphs edge weights from the set of positive integers $\mathcal{J}$ is

$$\mathbf{G}_{\mathcal{J}}(X) = \sum_{G \in \mathcal{G}_{\mathcal{J}}} \pi(G) = \prod_{i<j} \left(1 + \sum_{s \in \mathcal{J}} (x_i x_j)^s \right) . \tag{2.1}$$

For example, $\mathbf{G}_{\{1,2,3,\dots\}}(X) = \overline{\mathbf{G}}(X)$. Frequently we simplify expressions by omitting X.

To determine the series for graph classes with loops it is sufficient to change the product index from $i < j$ to $i \le j$. If, rather, the loops have weights from a different set, say $\mathcal{J}^*$, we multiply Eq. (2.1) by the product $\prod_i \big(1 + \sum_{s \in \mathcal{J}^*} (x_i x_i)^s\big)$.

The number of well-labelled graphs in this class with given degree sequence $d_1, \dots, d_n$ is the coefficient of the monomial $x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}$. For graph classes defined in our main theorem, and the specified degree vectors, this is the value under investigation by McKay. In standard generating function notation we write the coefficient as

$$[x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}] \mathbf{G}_{\mathcal{J}} .$$

As is always the case for symmetric species, the X-generating functions are symmetric functions. We access the coefficient using classic symmetric function operations.

2.2. **Expressing $\mathbf{G}_{\mathcal{J}}(X)$ using symmetric functions.** Since we can relabel any graph with a different set of labels and it remains in the class, $\mathcal{G}_{\mathcal{J}}$ is a symmetric class with respect to the graph labels. We leverage this underlying symmetry to rewrite the generating function in terms of symmetric functions.

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be an integer partition of n, a fact which we denote by $\lambda \vdash n$. Let $X = x_1, x_2, \dots$ be an infinite, but countable, variable set. Then the symmetric function $m_\lambda(X)$ (or simply m_λ) is defined

$$m_\lambda(X) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \dots x_{i_k}^{\lambda_k}$$

where the sum is over all k-tuples of distinct positive integers $(i_1, i_2, \dots, i_k)$. This is the monomial symmetric function indexed by λ. The set of the monomial symmetric functions form a basis for a vector space of symmetric functions over $\mathbb{Q}$.

We express the classic elementary, complete, and power sum symmetric functions in the monomial basis as follows:

$$e_n = m_{(1,1,\dots,1)}, \quad h_n = \sum_{\lambda \vdash n} m_\lambda, \quad p_n = m_{(n)}.$$

Recall the definition $e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_k}$. The set of the elementary symmetric functions indexed by partitions also forms a basis for a vector space of symmetric functions. This is also true for the sets of h_λ and p_λ respectively, which are similarly defined. We work in the ring of series in the power sum symmetric functions, $\mathbb{Q}[[p_1, p_2, p_3, \dots]]$. In particular, we are interested in elements $\mathbf{R}(X)$ of the form

$$\mathbf{R}(X) = \sum_{0<n} \sum_{\lambda \vdash n} c_\lambda p_\lambda(X).$$

The symmetric function operation known as *plethysm* is essential to our solution. Given two symmetric functions u and v, the inner product defines the quantity $u[v]$ by defining the following rules, with symmetric functions u, v, w and α, β from the underlying field:

$$(\alpha u + \beta v)[w] = \alpha u[w] + \beta v[w], \quad (uv)[w] = u[w]v[w],$$

and, most importantly, if $w = \sum_\lambda c_\lambda p_\lambda$ then $p_n[w] = \sum_\lambda c_\lambda p_{(n\lambda_1)} p_{(n\lambda_2)} \dots$. For example, we can deduce that $w[p_n] = p_n[w]$, and in particular that $p_n[p_m] = p_{nm}$. In a mnemonic way:

(2.2)

$$w[p_n] = w(p_{1n}, p_{2n}, \dots, p_{kn}, \dots) \qquad \text{whenever} \qquad w = w(p_1, p_2, \dots, p_k, \dots).$$

Let $\mathbf{H} = \sum_n h_n$ and $\mathbf{E} = \sum_n e_n$. Gessel noted that $\mathbf{G}$ and $\overline{\mathbf{G}}$ can both be expressed using plethysm:

$$\mathbf{G} = \prod_{i<j} (1 + x_i x_j) = \sum_n e_n[e_2] = \mathbf{E}[e_2], \tag{2.3}$$

$$\overline{\mathbf{G}} = \prod_{i<j} \left(\frac{1}{1 - x_i x_j} \right) = \sum_n h_n[e_n] = \mathbf{H}[e_2]. \tag{2.4}$$

Given Eq. (2.2), often plethysm expressions are easier to manipulate when the symmetric functions are written in the power sum basis. We do this next in Section 2.3, and this is followed by a discussion on how to derive the plethym expressions for $\mathbf{G}_\mathcal{J}$ in Section 3.

2.3. **Expressions in the power sum basis.** We recall the following classic lemma as it guides our work. It shows how to express an infinite sum of h_n as a function of power sum symmetric functions.

Lemma 2.1 (Waring formula)**.** *The following equations are true:*

$$\mathbf{H} = \sum_n h_n = \prod_{i<j} \frac{1}{1-x_i} = \exp\left(\sum_{0<k} p_k/k\right),$$

$$\mathbf{E} = \sum_n e_n = \prod_{i<j} (1+x_i) = \exp\left(\sum_{0<k} (-1)^{k+1} p_k/k\right).$$

Proof. The proof is an elementary series manipulation:

$$\log \prod_{0<i} \frac{1}{1-x_i} = \sum_{0<i} \log \frac{1}{1-x_i} = \sum_{0<i}\sum_{0<k} x_i^k/k = \sum_{0<k}\sum_{0<i} x_i^k/k = \sum_{0<k} p_k/k.$$

$\square$

Indeed, the plethysms are easier to analyse given this form:

$$\begin{aligned} \overline{\mathbf{G}} &= \mathbf{H}[e_2] = \exp\left(\sum_{0<k} \frac{1}{k} p_k\right)[e_2] \\ &= \exp(\sum p_k[e_2]/k) \\ &= \exp(\sum \frac{1}{k} p_k[p_1^2/2 - p_2/2]) \\ &= \exp(\sum \frac{1}{2k}(p_k^2 - p_{2k})). \end{aligned}$$

We can similarly express $\mathbf{E}[e_2]$ (and indeed $\mathbf{H}[h_2]$, $\mathbf{E}[h_2]$).

3. Labelled graphs as a symmetric species

One of the innovations of species theory [2, 13], is a rigorous combinatorial interpretation of the plethysm operation in terms of natural compositions of combinatorial structures [1]. Plethysm as a symbolic analog to composition has been well studied since Pólya's composition theorem. Asymmetric series of Labelle [14] is also an important relative to the X-generating functions that we seek.

The combinatorial understanding of the composition in the particular case of $\mathbf{H}$ and $\mathbf{E}$ is formally developed by Méndez [17], and we gave a direct interpretation in the case of graphs, and other variants in [18]. In particular, [18] contains a description of graphs as multisort, and ultimately symmetric, species. The interpretation is as follows: a simple labelled graph is a set of edges. Edges are sets of atomic structures. Each atom is coloured a colour from the infinite set $X = \{x_1, x_2 \ldots, \}$, and atoms of the same colour are identified to form a vertex. The combinatorial composition of a set of edges is reflected in X-generating function by a plethysm.

In the notation of classic species, E is the species of sets, and E_2 is the species of a set of cardinality two. Then, the class of all labelled multigraphs with loops allowed is given by the multisort species $E[E_2](X)$, and the associated cycle index series is $\mathbf{H}[h_2]$. A graph is viewed as a set of edges, but in the original theory, repetitions are not handled directly via multisort species. The innovation of the multiassembly construction of Méndez does permit repetition of elements. A *multiassembly of type* λ, denoted M_λ, is a multiset where the multiplicities of the elements are prescribed by the parts of the partition λ. For example, a multiassembly of type $\lambda = (1, 1, \ldots, 1)$ is a usual set without repetitions. Example 3.7 and Proposition 3.9 of [17] describe how to get the X-generating function associated a composition of a multiassembly and a some object. In particular, the X-generating function of a multiassembly $M_\lambda(X)$ is $m_\lambda(X)$, and the composition is realized by plethysm.

Example 3.1. *We view graphs as multiassemblies of edges, where edge multiplicities are given by λ. The edges themselves are multiassemblies of vertices. For example, the species $M_{(3,2)}[M_{(1,1)}(X)]$ is the set of graphs with one edge of multiplicity two, and one edge of multiplicity three, under all possible positive integer labellings. There are two underlying graphs that could be described this way, if we consider all possible labellings: a set of two disconnected edges $E(G) = \{\{i, j\}, \{k, \ell\}\}$; and the path graph on three vertices with edge set $E(G) = \{\{i, j\}, \{i, k\}\}$. In each case, suppose the first edge has weight 2, and the second weight 3 , and hence the monomial encodings are respectively $x_i^2 x_J^2 x_k^3 x_\ell^3$ and $x_i^5 x_j^2 x_k^3$. The sum over all labellings is thus $m_{5,3,2} + m_{3,3,2,2}$. One easily verifies that $m_{5,3,2} + m_{3,3,2,2} = m_{(3,2)}[m_{(1,1)}]$.*

The X-generating function $\mathbf{G}_\mathcal{J}$ of $\mathcal{G}_\mathcal{J}$ requires a sum over all possible integer partitions where the parts are elements of $\mathcal{J}$. Towards a more compact notation, define the symmetric function $f_{\mathcal{J},n}$ as follows:

$$f_{\mathcal{J},n} = \sum_{\lambda \vdash n; \lambda_i \in \mathcal{J}} m_\lambda. \tag{3.1}$$

To unravel the definition, note that $f_{\{2,3\},6} = m_{2,2,2} + m_{3,3}$, $f_{\{1\},n} = e_n$, and $f_{\{1,\ldots,n\},n} = h_n$. We gather these observations together as a simple proposition for reference.

Proposition 3.2. *Fix $\mathcal{J}$, a nonempty set of positive integers, and $X = x_1, x_2, \ldots,$ an infinite, but countable, set of labels.*

(1) *The symmetric species of simple graphs $\mathcal{G}_\mathcal{J}$ is isomorphic to*

$$\cup_{\lambda \vdash n; \lambda_i \in \mathcal{J}} M_\lambda[M_{(1,1)}(X)];$$

(2) *For any degree sequence $\mathbf{d} = (d_1, \ldots, d_n)$ with sum $|\mathbf{d}| = d_1 + \cdots + d_n$, the coefficient*

$$[x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n}] \sum_{\lambda \vdash |\mathbf{d}|; \lambda_i \in \mathcal{J}} m_\lambda[e_2]$$

is precisely the number of graphs in $\mathcal{G}_{\mathcal{J},\mathbb{N},n}$ with degree sequence $\mathbf{d}$. Equivalently, this is the coefficient of $m_{\mathbf{d}}$ when expanded in the multinomial basis of symmetric functions.[1]

The analogous class of graphs permitting loops, denoted $\mathcal{G}^{\circ}_{\mathcal{J}}$, is isomorphic as species to

$$\bigcup_{\lambda\vdash n;\lambda_i\in\mathcal{J}} M_\lambda[M_{(1,1)}(X)+M_{(2)}(X)],$$

and the coefficient

$$[x_1^{d_1}x_2^{d_2}\dots x_n^{d_n}]\sum_{\lambda\vdash D;\lambda_i\in\mathcal{J}} m_\lambda[h_2]$$

is precisely the number of graphs in $\mathcal{G}^{\circ}_{\mathcal{J},\mathbb{N},n}$ with degree sequence $\mathbf{d}$.

3.1. **Series expressions.** To prove the D-finiteness results, we need the X-generating functions in a different format. We define $\mathbf{F}_{\mathcal{J}} = \sum_n f_{\mathcal{J},n}$ and hence $\mathbf{F}_{\{1\}} = \mathbf{E}$ and $\mathbf{F}_{\{1,2,\dots\}} = \mathbf{H}$. Now, $\mathbf{F}_{\mathcal{J}} = \prod_{0<i}(1+\sum_{s\in\mathcal{J}} x_i^s)$. We can generalize Lemma 2.1 to express $\mathbf{F}_{\mathcal{J}}$ in the power sum symmetric function basis. The proof follows from very basic manipulations.

Proposition 3.3. *Let $\mathcal{J} = \{j_1,\dots,j_\ell\}$ be a set of distinct positive integers. Then,*

$$\mathbf{F}_{\mathcal{J}} = \sum_{0<n} f_{\mathcal{J},n} = \exp\left(\sum_{0<n} a_n p_n\right)$$

where a_n is the following sum taken over all compositions $\alpha = (\alpha_1,\alpha_2,\dots, a_k)$ of n such that each part α_i is contained in $\mathcal{J}$, and the number of parts, k, is denoted $\ell(\alpha)$:

$$a_n = -\sum_{\alpha}\frac{(-1)^{\ell(\alpha)}}{\ell(\alpha)}. \tag{3.2}$$

Proof. First we note that as the elements of $\mathcal{J}$ are all positive, a_n is well defined since the number of such compositions is finite. Next we apply the

[1]With $\mathbf{d}$ normalized to be decreasing if necessary.

same log-exp expansion, and some very basic coefficient extraction formulas:

$$\begin{aligned}
\log \sum_{0<n} f_{\mathcal{J},n} &= \log \prod_{0<i} (1 + \sum_{s \in \mathcal{J}} x_i^s) \\
&= -\sum_{0<i} \log \frac{1}{1 - \sum_{s\in\mathcal{J}}(-x_i^s)} \\
&= \sum_{0<i}\sum_{0<k} \frac{(-1)^{k+1}}{k} (\sum_{s\in\mathcal{J}} x_i^s)^k \\
&= \sum_{0<i}\sum_{0<n} a_n x_i^n \\
&= \sum_{0<n} a_n p_n.
\end{aligned}$$

□

There are a few simplifications to note. If $\mathcal{J} = \{1\}$, then $a_n = (-1)^n/n$, since there is only one term in the summation in Equation (3.2). When $\mathcal{J} = \{1, 2, \dots\}$, $a_n = 1/n$ as the sum is over all compositions, and we invoke a Möbius inversion argument. Further, when $\mathcal{J} = \{s_1, \dots, s_\ell\}$ is finite, we can express this as follows:
(3.3)

$$\sum_{0<n} f_{\mathcal{J},n} = \exp\left(\sum_{0<k} \frac{(-1)^{k+1}}{k} \sum_{i_1+i_2+\cdots+i_\ell=k} \binom{k}{i_1\ i_2\ \dots\ i_\ell} p_{s_1 i_1 + s_2 i_2 + \cdots + s_\ell i_\ell}\right).$$

Theorem 3.4. *Let $\mathcal{J} = \{j_1, \dots, j_\ell\}$ be a finite set of ℓ distinct positive integers. Then $\mathbf{G}_{\mathcal{J}}$, the X-generating function for the symmetric species of labelled simple graphs with edge weights from $\mathcal{J}$ satisfies:*

$$\begin{aligned}
\mathbf{G}_{\mathcal{J}} &= \left(\sum_n f_{\mathcal{J},n}[e_2]\right) = \mathbf{F}_{\mathcal{J}}[e_2] \\
&= \exp\left(\sum_{0<n} \frac{a_n}{2}(p_n^2 - p_{2n})\right),
\end{aligned}$$

with a_n as defined in Proposition 3.3.

Proof. This follows from Propositions 3.2 and 3.3, and the fact that $e_2 = p_1^2 - p_2$. □

Returning to our example:

$$\mathbf{G}_{\{2,3\}} = \exp\left(\sum_{0<n} \frac{(-1)^{n+1}}{2n} \sum_{i=0}^{n} \binom{n}{i} p_{3n-i}^2 - p_{6n-2i}\right).$$

4. D-finite symmetric series

Recall that a series $S \in K[[x_1, \dots, x_n]]$ is *D-finite* in $x_1, \dots, x_n$ when the set of all partial derivatives and their iterates, $\partial^{i_1+\dots+i_n} F/\partial x_1^{i_1} \cdots \partial x_n^{i_n}$, spans a finite-dimensional vector space over the field $K(x_1, \dots, x_n)$. This was generalized to an infinite number of variables by Gessel [9], who had symmetric functions in mind. A series $S \in K[[x_1, x_2, \dots]]$ is *D-finite* in the x_i if the specialization to 0 of all but a finite (arbitrary) set of variables results in a D-finite function (in the finite sense). In this case, many of the properties of the finite multivariate case hold true. One notable exception is closure under algebraic substitution, which requires additional hypotheses.

The definition is then applied to symmetric series by considering the algebra of symmetric series as generated over $\mathbb{Q}$ by the set of power sum symmetric functions $\{p_1, p_2, \dots\}$. A symmetric series is called D-*finite* when it is D-finite as a function of the p_i's. The applicability of this definition will be apparent in a moment.

The two prototypical examples, $\mathbf{H}$ and $\mathbf{E}$ are easily seen to be D-finite, as any such specialization of variables results in an exponential of a polynomial, which is clearly D-finite. Similarly, from the expression in Proposition 3.3 we see that the same argument will hold for any $\mathbf{F}_\mathcal{J}$.

Theorem 4.1. *For any set of positive integers $\mathcal{J}$, the series $\mathbf{F}_\mathcal{J}$ and $\mathbf{G}_\mathcal{J}$ are both D-finite symmetric series with respect to the p-basis.*

Proof. For both of these symmetric series any specialization of the p variables so that only a finite number are nonzero leaves an exponential of a polynomial, which is easily shown to be D-finite in the remaining variables. We immediately conclude the D-finiteness of both $\mathbf{F}_\mathcal{J}$ and $\mathbf{G}_\mathcal{J}$, given the two previous results □

4.1. Extracting the generating functions.

In the power notation for integer partitions, $\lambda = 1^{n_1} \dots k^{n_k}$ indicates that i occurs n_i times in λ, for $i = 1, 2, \dots, k$. The normalization constant

$$z_\lambda := 1^{n_1} n_1! \cdots k^{n_k} n_k!$$

plays the role of the square of a norm of p_λ in the following important formula:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda, \tag{4.1}$$

where $\delta_{\lambda,\mu}$ is 1 if $\lambda = \mu$ and 0 otherwise.

The scalar product is useful for coefficient extraction because $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$. If we write S in the form $\sum_\lambda c_\lambda m_\lambda$, then the coefficient of $x_1^{\lambda_1} \dots x_k^{\lambda_k}$ in S is $c_\lambda = \langle S, h_\lambda \rangle$.

The closure under the Hadamard product of D-finite series [15] yields the consequence:

Theorem 4.2 (Gessel [9])**.** *Let $\mathbf{S}$ and $\mathbf{T}$ be elements of $\mathbb{Q}[z][[p_1, p_2, \dots]]$, D-finite in the p_i's and z, and further suppose that $\mathbf{T}$ involves only finitely* *many of the p_i's. Then $\langle \mathbf{S}, \mathbf{T} \rangle$ is D-finite as a function of z, provided it is well-defined as a power series.*

From this and Lemma 4.1, our main theorem follows almost immediately.

Theorem 4.3. *Let $\mathcal{J}$ and $\mathcal{D}$ be sets of positive integers, and suppose additionally that $\mathcal{D}$ is finite. Let $G_{\mathcal{J},\mathcal{D}}(z)$ be the generating function for the class $\mathcal{G}_{\mathcal{J},\mathcal{D}}$ of well-labelled graphs where edge weights are from $\mathcal{J} \cup \{0\}$ and the sum of weights of the edges incident to any given vertex is an element of $\mathcal{D}$. Then,*

$$G_{\mathcal{J},\mathcal{D}}(z) = \sum_n |\mathcal{G}_{\mathcal{J},\mathcal{D},n}|\, z^n$$

is D-finite. Furthermore, there exist algorithms to compute the differential equation satisfied by $G_{\mathcal{J},\mathcal{D}}(z)$.

We remark that the present implementations algorithms to compute the differential equation satisfied by $G_{\mathcal{J},\mathcal{D}}(z)$ have significant space requirements, which limit their utility in practice.

Proof. The series are combinatorial generating functions, and so they exist. We remark that

$$G_{\mathcal{J},\mathcal{D}}(z) = \left\langle \mathbf{G}_{\mathcal{J}}, \frac{1}{1 - \sum_{d \in \mathcal{D}} h_d z^d} \right\rangle. \tag{4.2}$$

The first argument to the scalar product is D-finite by Lemma 4.1. Furthermore, $\sum_{d\in\mathcal{D}} h_d z^d$ is a polynomial in the power sum basis and z when $\mathcal{D}$ is finite. Hence the second argument is rational, and D-finite. The result follows by Theorem 4.2. We address the computability in the next section. ☐

Example 4.4. *We continue the running example. The series can be expanded:*

$$\begin{aligned}\mathbf{G}_{\{3,2\}} = \mathbf{F}_{\{3,2\}}[e_2] &= (m_2 + m_3 + m_{2,2} + m_{3,2} + m_{2,2,2} + m_{3,3} + \dots)\,[e_2] \\ &= m_{2,2} + m_{3,3} + m_{4,2,2} + 3m_{2,2,2,2} + m_{5,3,2} + m_{3,3,2,2} \\ &\quad + m_{6,2,2,2} + 2\,m_{4,4,2,2} + 6\,m_{4,2,2,2,2} + m_{4,4,4} + m_{6,3,3} \\ &\quad + 15\,m_{2,2,2,2,2,2} + 3\,m_{3,3,3,3} + \dots\end{aligned}$$

To extract the generating function of 2-regular graphs examine the coefficients of $m_{(2,2,\dots,2)}$:

$$G_{\{3,2\},\{2\}}(z) = \left\langle \mathbf{G}_{\{3,2\}}, \sum_{0<n} h_2^n z^n \right\rangle = z^2 + 3z^4 + 15z^6 + 105z^8 + \dots.$$

In this case, it is corresponds the number of matchings. The generating function for the graphs where each vertex is either of degree 2 or 3 is given by

$$G_{\{3,2\},\{2,3\}}(z) = \left\langle \mathbf{G}_{\{3,2\}}, \sum_{0<n} \sum_{k=0}^{n} h_3^k h_2^{n-k} z^n \right\rangle = 2z^2 + 7z^4 + 36z^6 + 429z^8 + \dots.$$

An extraction for any degree sequence is possible. The D-finiteness result can be generalized to handle infinite $\mathcal{D}$ provided $\sum_n \sum_{\lambda\vdash n:\lambda_i\in\mathcal{D}} h_\lambda t^n$ is a D-finite symmetric series. In general, to determine the generating function of graphs with a fixed set of degree sequences D, it suffices to consider the series $\sum_{\lambda\in D} h_\lambda z^{|\lambda|}$.

We could also mark something other than number of vertices. As we noted, the symmetric series $\sum h_n z^n$ is D-finite, and in fact, for any finite k, the series $\sum h_n^k z^n$ is D-finite. This would extract the generating function for the subclass of all regular graphs on k vertices from a given graph class with n marking the regularity. The resulting generating function is also D-finite.

4.2. **Comments on effective computation.** There are two computational tools at hand to compute $G_{\mathcal{J},\mathcal{D}}(z)$. One could iteratively expand $f_{\mathcal{J},n}[e_2]$ in the monomial basis as we did in the previous example. It might be slightly more efficient to expand the exponential expression. In practice, we were able to get some small results with this strategy.

Alternatively, we can make use of the fact that Gessel's result is effective: given the system of differential equations satisfied by symmetric series $\mathbf{F}$ and $\mathbf{G}$, there are algorithms [4] to compute the differential equation satisfied by the scalar product. (Of course, at least one of $\mathbf{F}$ and $\mathbf{G}$ must contain other variables). It is straightforward to define the system satisfied by $\mathbf{G}_{\mathcal{J}}$, since it is expressed as an exponential of a polynomial. Consequently in theory we can compute the differential equation satisfied by $G_{\mathcal{J},\mathcal{D}}(z)$. In practice, using current algorithms, the computations are too resource intensive to deliver results when the number of variables are greater than five. We were able to confirm the correctness in small cases.

5. Other generalizations

By playing with the inner series in the plethysm, we can enumeration other families of objects, such as hypergraphs, or cyclic coverings of sets. The details are essentially given in [18]. As mentioned above, we could be interested in other kinds of functions for the growth of $\mathbf{d}$. We can extract other subseries using similar methods. The resulting generating function will be D-finite provided the series used to do the extraction is also D-finite. A different future direction would be to try to adapt the approach of de Panafieu and Ramos [5] for multigraphs to the weighted edge versions.

References

1. F. Bergeron, *Une combinatoire du pléthysme*, J. Combin. Theory Ser. A **46** (1987), no. 2, 291–305. MR 914661 (89b:05017)
2. F. Bergeron, G. Labelle, and P. Leroux, *Combinatorial species and tree-like structures*, Encyclopedia of Mathematics and its Applications, vol. 67, Cambridge University Press, Cambridge, 1998, Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota. MR 1629341 (2000a:05008)

3. M. Bousquet-Mélou, *Rational and algebraic series in combinatorial enumeration*, International Congress of Mathematicians. Vol. III, Eur. Math. Soc., Zürich, 2006, pp. 789–826. MR 2275707 (2007k:05008)
4. F. Chyzak, M. Mishna, and B. Salvy, *Effective scalar products of D-finite symmetric functions*, J. Combin. Theory Ser. A **112** (2005), no. 1, 1–43. MR 2167474 (2006d:05181)
5. É. de Panafieu and L. Ramon, *Graphs with degree constraints*, arXiv:1506.03061v1, 2015.
6. P. Flajolet, S. Gerhold, and B. Salvy, *On the non-holonomic character of logarithms, powers, and the nth prime function*, Electron. J. Combin. **11** (2004/06), no. 2, Article 2, 16. MR 2195433
7. ———, *On the non-holonomic character of logarithms, powers, and the nth prime function*, Electron. J. Combin. **11** (2004/06), no. 2, Article 2, 16. MR 2195433 (2006h:05013)
8. A. Gainer-Dewar and I. M. Gessel, *Enumeration of bipartite graphs and bipartite blocks*, Electron. J. Combin. **21** (2014), no. 2, Paper 2.40, 20. MR 3244806
9. I. M. Gessel, *Symmetric functions and P-recursiveness*, J. Combin. Theory Ser. A **53** (1990), no. 2, 257–285. MR 1041448 (91c:05190)
10. I. P. Goulden, D. M. Jackson, and J. W. Reilly, *The Hammond series of a symmetric function and its application to P-recursiveness*, SIAM J. Algebraic Discrete Methods **4** (1983), no. 2, 179–193. MR 699770 (84g:05011)
11. C. Greenhill and B. D. McKay, *Asymptotic enumeration of sparse nonnegative integer matrices with specified row and column sums*, Adv. in Appl. Math. **41** (2008), no. 4, 459–481. MR 2459445 (2009i:05021)
12. A. Henderson, *Species over a finite field*, J. Algebraic Combin. **21** (2005), no. 2, 147–161. MR 2142405 (2006c:05017)
13. A. Joyal, *Une théorie combinatoire des séries formelles*, Adv. in Math. **42** (1981), no. 1, 1–82. MR 633783 (84d:05025)
14. G. Labelle, *Sur la symétrie et l'asymétrie des structures combinatoires*, Theoret. Comput. Sci. **117** (1993), no. 1-2, 3–22, Conference on Formal Power Series and Algebraic Combinatorics (Bordeaux, 1991). MR 1235165 (94j:05006)
15. L. Lipshitz, *The diagonal of a D-finite power series is D-finite*, J. Algebra **113** (1988), no. 2, 373–378. MR 929767 (89c:13027)
16. B. D. McKay, *Enumeration of sparse symmetric matrices without 1.*, Mathematisches Forschungsinstitut Oberwolfach **Report No. 12/2014** (2014), 367–377, DOI: 10.4171/OWR/2014/12.
17. M. Méndez, *Multisets and the combinatorics of symmetric functions*, Adv. Math. **102** (1993), no. 1, 95–125. MR 1250467 (95g:05103)
18. M. Mishna, *Automatic enumeration of regular objects*, J. Integer Seq. **10** (2007), no. 5, Article 07.5.5, 18. MR 2304413 (2008a:05016)
19. R. P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR 1676282 (2000k:05026)

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