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BOUNDS FOR THE BOXICITY OF MYCIELSKI GRAPHS

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Abstract. A *box* in Euclidean k-space is the Cartesian product $I_1 \times I_2 \times \cdots \times I_k$, where I_j is a closed interval on the real line. The *boxicity* of a graph G, denoted by $\mathrm{box}(G)$, is the minimum nonnegative integer k such that G can be isomorphic to the intersection graph of a family of boxes in Euclidean k-space.

Mycielski [11] introduced an interesting graph operation that extends a graph G to a new graph $M(G)$, called the *Mycielski graph* of G. In this paper, we observe the behavior of the boxicity of Mycielski graphs. The inequality $\mathrm{box}(M(G)) \geq \mathrm{box}(G)$ holds for a graph G, and hence we are interested in whether the boxicity of the Mycielski graph of G is more than that of G or not. Here we give bounds for the boxicity of Mycielski graphs: for a graph G with l universal vertices, the inequalities $\mathrm{box}(G) + \lceil l/2 \rceil \leq \mathrm{box}(M(G)) \leq \theta(\overline{G}) + \lceil l/2 \rceil + 1$ hold, where $\theta(\overline{G})$ is the *edge clique cover number* of the complement $\overline{G}$. Further observations determine the boxicity of the Mycielski graph $M(G)$ if G has no universal vertices or odd universal vertices and satisfies $\mathrm{box}(G) = \theta(\overline{G})$.

We also present relations between the Mycielski graph $M(G)$ and its generalizations $M_3(G)$ and $M_r(G)$ in the context of boxicity, which will encourage us to calculate the boxicity of $M(G)$ and $M_3(G)$.

1. Introduction

The notion of *boxicity* of graphs was introduced by Roberts [13]. It has applications in some research fields, like niche overlap in ecology (see [14, 15]) and fleet maintenance in operations research (see [12]). Roberts [13] proved that the maximum boxicity of graphs with n vertices is $\lfloor n/2 \rfloor$ (also see [7]), where $\lfloor x \rfloor$ denotes the largest integer at most x. Cozzens [6] proved that the task of computing boxicity of graphs is NP-hard. Some researchers have attempted to calculate or bound boxicity of graphs with special structure. Roberts [13] showed that the boxicity of a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is the number of n_i which is at least 2. Scheinerman [16] proved that the boxicity of outer planar graphs is at most 2. Thomassen [17] proved

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that the boxicity of planar graphs is at most 3. Cozzens and Roberts [7] investigated the boxicity of split graphs. As Chandran et al. [5] say, not much is known about boxicity of most of the well-known graph classes. They proved that the boxicity of a graph G is at most $\mathrm{tw}(G)+2$, where $\mathrm{tw}(G)$ is the treewidth of G, and presented upper bounds for chordal graphs, circular arc graphs, AT-free graphs, co-comparability graphs, and permutation graphs. Recently, Chandran et al. [1] found the following relation between boxicity and chromatic number.

Theorem 1.1 ([1], Theorem 6.1)**.** *Let G be a graph with n vertices. If $\mathrm{box}(G) = n/2 - s$ for $s \geq 0$, the inequality $\chi(G) \geq n/(2s+2)$ holds, where $\chi(G)$ is the chromatic number of G.*

Theorem 1.1 implies that, if the boxicity of a graph with n vertices is very close to the maximum boxicity $\lfloor n/2 \rfloor$, the chromatic number of the graph must be very large. The converse does not hold in general. The complete graph K_n is an example of a graph whose boxicity is small, even though its chromatic number is large. Also there are bipartite graphs with arbitrary large boxicity (see Section 5.1 in [1] and also see [2]). However, a graph operation increasing chromatic number may admit increasing boxicity. For example, the *join* of two graphs, taking the disjoint union of two graphs and adding all edges between them is desired one. The behavior of boxicity has been studied in the context of various graph operations (see [3, 4, 18] for example). This paper is another attempt in this direction that studies the behavior of boxicity in the context of Mycielski's graph operation.

One purpose of this paper is to consider whether the behavior of boxicity is similar to that of chromatic number under Mycielski's graph operation. Mycielski [11] found an interesting graph operation that extends a graph G to a new graph $M(G)$, called the *Mycielski graph* of G or the *Mycielskian* of G. It is well-known that the chromatic number of the Mycielski graph of G is more than that of G, in fact, $\chi(M(G)) = \chi(G) + 1$ holds. We can construct (triangle-free) graphs with arbitrarily large chromatic numbers by using the graph operation. Here we present the definition of the graph $M(G)$. Let $V(G)_i$ be a copy of the vertex set $V(G)$ of a graph G, where $i \in \{1, 2\}$. For each vertex $v \in V(G)$, the symbol v_i denotes the vertex in $V(G)_i$ corresponding to v. The vertex set of $M(G)$ is defined to be $\{z\} \cup V(G)_1 \cup V(G)_2$, the disjoint union of the set of a single new vertex z and copies $V(G)_1$ and $V(G)_2$. The edge set of $M(G)$ is defined to be the union $E_1 \cup E_2 \cup E_3$, where

$$E_1 = \{u_1 v_1 \,|\, uv \in E(G)\},\ E_2 = \{u_1 v_2, v_1 u_2 \,|\, uv \in E(G)\},\ \text{and}$$
$$E_3 = \{z u_2 \,|\, u \in V(G)\}$$

and $E(G)$ denotes the edge set of G (see Figure 1 for example). Note that the inequality $\mathrm{box}(M(G)) \geq \mathrm{box}(G)$ holds for a graph G since $M(G)$ contains the subgraph induced by $V(G)_1$, isomorphic to G. So, first we are interested in whether the boxicity of the Mycielski graph $M(G)$ is strictly greater than G, the same as the behavior of the chromatic number under the graph operation, as mentioned at the beginning of this paragraph. Many researchers have studied Mycielski graphs and have compared a graph G with $M(G)$ under various graph invariants (see [8, 10] for example).

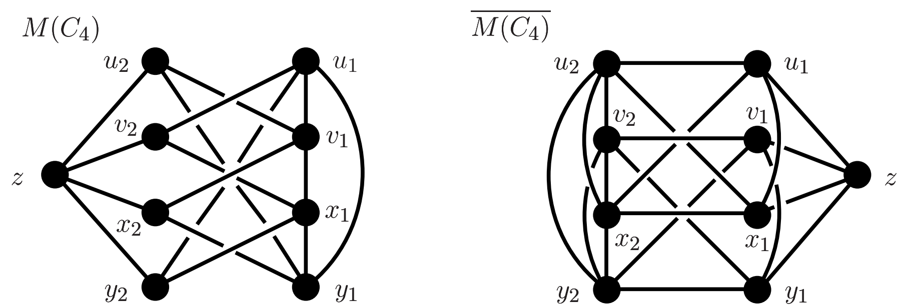

FIGURE 1. The Mycielski graph $M(C_4)$ of a cycle C_4 and its complement $\overline{M(C_4)}$.

In Section 3, we improve the trivial lower bound for the boxicity of the Mycielskian of a graph G in terms of the number of universal vertices of G. This implies that the boxicity of the Mycielski graph $M(G)$ is more than that of G if the graph G has universal vertices. Also note that there is a graph G without universal vertices such that the boxicity of the Mycielski graph $M(G)$ is more than that of G. While such examples of graphs appear, there is also a graph G such that $\text{box}(M(G)) = \text{box}(G)$. As a conclusion, the behavior of boxicity is not similar to that of chromatic number under Mycielski's graph operation in general. This leads to our next goal: Classify as many graphs as possible into $\text{box}(M(G)) > \text{box}(G)$ or $\text{box}(M(G)) = \text{box}(G)$.

In Section 4, we discuss upper bounds for the boxicity of Mycielski graphs. Chandran et al. [1] proved that the inequality $\text{box}(G) \leq \lfloor t(G)/2 \rfloor + 1$ holds for a graph G, where $t(G)$ is the minimum cardinality of a vertex cover of G. It is easy to see that $t(M(G)) \leq 2t(G) + 1$ for a graph G, and hence we have $\text{box}(M(G)) \leq \lfloor t(M(G))/2 \rfloor + 1 \leq t(G) + 1$. Here we present another upper bound for the boxicity of the Mycielskian of a graph G in terms of the edge clique cover number $\theta(\overline{G})$ of the complement $\overline{G}$. We also consider graphs that satisfy the equality $\text{box}(G) = \theta(\overline{G})$. The family of graphs satisfying $\text{box}(G) = \theta(\overline{G})$ contains complete multipartite graphs, for example. Other examples of such graphs appear at the end of Section 4. As a result, our observations determine the boxicity of their Mycielski graphs if the original graphs have no universal vertices or odd universal vertices.

In Section 5, we consider relations between the Mycielski graph and its generalization $M_r(G)$, called the *generalized Mycielski graph* of G, in the context of boxicity, where $r \geq 3$. We present upper bounds for the boxicity of the generalized Mycielski graph $M_r(G)$ in terms of $M(G)$ for a bipartite graph G or in terms of $M_3(G)$ for a graph G. These results provide motivation for calculating the boxicity of $M(G)$ and $M_3(G)$.

2. Preliminaries

In this paper, all graphs are finite, simple and undirected. We use $V(G)$ for the vertex set of a graph G. We use $E(G)$ for the edge set of a graph G. An edge of a graph with endpoints u and v is denoted by uv. A vertex v of G is said to be *universal* if v is adjacent to all vertices in $V(G) \setminus \{v\}$. A graph is said to be *trivial* if $E(G)$ is empty. For a subset V of $V(G)$, let $G - V$ be the subgraph induced by $V(G) \setminus V$. For a subset E of $E(G)$, let $G - E$ be the subgraph on $V(G)$ with $E(G) \setminus E$ as its edge set. A subset of $V(G)$ that induces a complete subgraph of G is called a *clique* of G. For a graph G, its complement is denoted by $\overline{G}$. The *intersection graph* of a nonempty family $\mathcal{F}$ of sets is the graph whose vertex set is $\mathcal{F}$ and F_1 is adjacent to F_2 if and only if $F_1 \cap F_2 \neq \emptyset$ for $F_1,\ F_2 \in \mathcal{F}$. The intersection graph of a family of closed intervals on the real line is called an *interval graph*. A graph G can be represented as the intersection graph of a family $\mathcal{F}$ if there is a bijection between $V(G)$ and $\mathcal{F}$ such that two vertices of G are adjacent if and only if the corresponding sets in $\mathcal{F}$ have nonempty intersection. A *box* in Euclidean k-space is the Cartesian product $I_1 \times I_2 \times \cdots \times I_k$, where I_j is a closed interval on the real line. The *boxicity* of a graph G, denoted by $\mathrm{box}(G)$, is the minimum nonnegative integer k such that G can be represented as (isomorphic to) the intersection graph of a family of boxes in Euclidean k-space. The boxicity of a complete graph is defined to be 0. If G is an interval graph, $\mathrm{box}(G) \leq 1$. If H is an induced subgraph of G, $\mathrm{box}(H) \leq \mathrm{box}(G)$ holds by the definition.

A graph is a *cointerval graph* if its complement is an interval graph. Lekkerkerker and Boland [9] presented the forbidden subgraph characterization of interval or cointerval graphs. Cointerval graphs do not contain the complement of a cycle of length at least 4 as an induced subgraph, for example. It is easy to see that the union of a cointerval graph and isolated vertices is also a cointerval graph. A *cointerval edge covering* of a graph G is a family $\mathcal{C}$ of cointerval spanning subgraphs of G such that each edge of G is in some graph of $\mathcal{C}$. For a set X, the cardinality of X is denoted by $|X|$.

The following theorem is useful to calculate of the boxicity of graphs.

Theorem 2.1 ([7], Theorem 3)**.** *Let G be a graph. Then,* $\mathrm{box}(G) \leq k$ *if and only if there is a cointerval edge covering $\mathcal{C}$ of $\overline{G}$ with $|\,\mathcal{C}\,| = k$.*

3. A lower bound for the boxicity of Mycielski graphs

For a complete graph K_n, it is easy to see that $\mathrm{box}(M(K_n)) \geq 1 > 0 = \mathrm{box}(K_n)$ since $M(K_n)$ is not complete by the definition of Mycielski graphs. We determine the boxicity of $M(K_n)$ in the next section (see Lemma 4.1).

First we consider if the boxicity of the Mycielski graph of a graph G is more than that of G in general.

Question 3.1. *For a graph G, does the inequality* $\mathrm{box}(M(G)) > \mathrm{box}(G)$ *hold?*

The following example shows that there exists a graph G such that the equality $\mathrm{box}(M(G)) = \mathrm{box}(G)$ holds. Here C_n denotes a cycle with n vertices.

Example 3.2. The boxicity of the Mycielski graph of a cycle C_4 is equal to 2. To check this, we give a cointerval edge covering of the complement $\overline{M(C_4)}$ (see Figure 1).

Let H_1 and H_2 be the graphs in Figure 2. Both graphs are cointerval spanning subgraphs of $\overline{M(C_4)}$. Note that the disjoint union of a cointerval

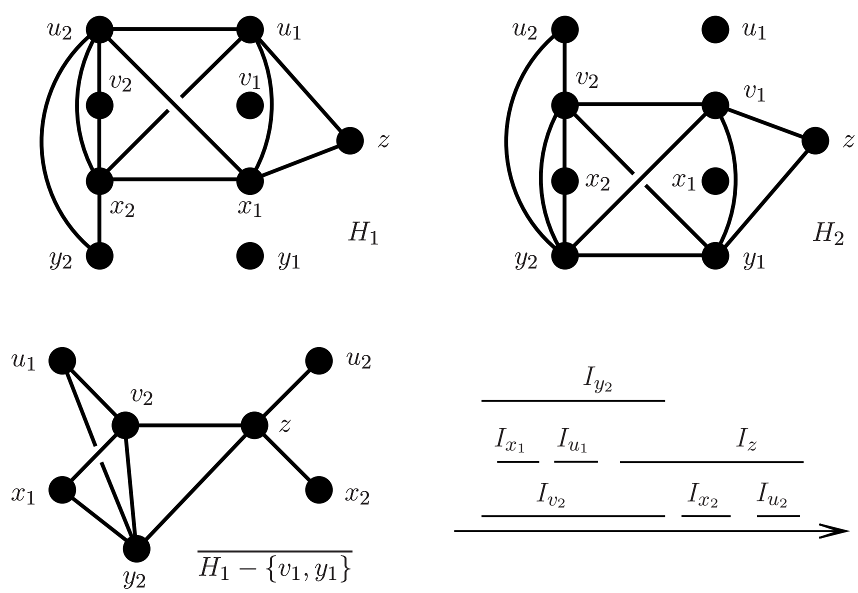

FIGURE 2. Cointerval spanning subgraphs H_1 and H_2 of $\overline{M(C_4)}$ and an interval representation of $\overline{H_1 - \{v_1, y_1\}}$.

graph and isolated vertices is also cointerval since these isolated vertices become pairwise adjacent universal vertices in the complement. Hence, it suffices to prove that $H_1 - \{v_1, y_1\}$ and $H_2 - \{u_1, x_1\}$ are cointerval, respectively. A family of intervals on the real line with intersection graph isomorphic to $\overline{H_1 - \{v_1, y_1\}}$ can be found as in the bottom of Figure 2. Similar arguments work for $H_2 - \{u_1, x_1\}$. Also see that H_1 and H_2 cover all edges of $\overline{M(C_4)}$. The family $\{H_1, H_2\}$ is a desired cointerval edge covering of $\overline{M(C_4)}$, and hence, $\mathrm{box}(M(C_4)) \leq 2$ by Theorem 2.1. Also note that $\mathrm{box}(M(C_4)) \geq \mathrm{box}(C_4) = 2$.

Question 3.3. *Is there a graph G such that the inequality* $\mathrm{box}(M(G)) > \mathrm{box}(G)$ *holds?*

The *distance between two vertices u and v* in a graph G is defined by length of the shortest path from u to v in G and is denoted by $d_G(u,v)$. If there exist no paths from u to v in G, define $d_G(u,v) = \infty$. Let H_1 and H_2 be subgraphs of G. The *distance between two subgraphs H_1 and H_2* in G, denoted by $d_G(H_1, H_2)$, is defined to be the minimum distance $\min\{d_G(v_1, v_2) \mid v_1 \in V(H_1),\, v_2 \in V(H_2)\}$. The following lemma is a generalization of Corollary 3.6 in [7].

Lemma 3.4. *Let G be a graph and H_1, H_2 induced subgraphs of the complement $\overline{G}$. If $d_{\overline{G}}(H_1, H_2) \geq 2$, the following inequality holds:*

$$\mathrm{box}(G) \geq \mathrm{box}(\overline{H_1}) + \mathrm{box}(\overline{H_2}).$$

Proof. If either H_1 or H_2 is trivial, say H_1, then $\overline{H_1}$ is complete. Hence, $\mathrm{box}(\overline{H_1}) = 0$. Since $\overline{H_2}$ is an induced subgraph of G, we see that

$$\mathrm{box}(G) \geq \mathrm{box}(\overline{H_2}) = \mathrm{box}(\overline{H_1}) + \mathrm{box}(\overline{H_2})$$

holds. In what follows, we may assume that H_1 and H_2 are nontrivial.

The assumption $d_{\overline{G}}(H_1, H_2) \geq 2$ means that $d_{\overline{G}}(v_1, v_2) \geq 2$ for any vertex v_1 of H_1 and v_2 of H_2. Hence, an edge of H_1 and an edge of H_2 form $2K_2$, the disjoint union of two edges, as an induced subgraph of $\overline{G}$. Moreover, we claim the following.

Claim 1. No cointerval spanning subgraphs of $\overline{G}$ contain an edge of H_1 and an edge of H_2.

Claim 2. We need at least $\mathrm{box}(\overline{H_i})$ cointerval spanning subgraphs of $\overline{G}$ to cover all edges of H_i, where $i \in \{1, 2\}$.

Claim 1 follows from the forbidden subgraph characterization of cointerval graphs. In fact, cointerval graphs do not contain $2K_2$ as an induced subgraph. Claim 2 follows from Theorem 2.1. A cointerval subgraph of $\overline{G}$ with edges of H_1 does not contain edges of H_2. Thus, the inequality $\mathrm{box}(G) \geq \mathrm{box}(\overline{H_1}) + \mathrm{box}(\overline{H_2})$ holds. □

We can derive a positive answer to Question 3.3 by using Lemma 3.4. The following lemma is useful to make our answer more precise. Here, $\lceil x \rceil$ denotes the smallest integer at least x.

Lemma 3.5 ([7], Lemma 3)**.** *Let G be a graph. Let $S_1 = \{a_1, a_2, \ldots, a_n\}$ and $S_2 = \{b_1, b_2, \ldots, b_n\}$ be disjoint subsets of $V(G)$ such that the only edges between S_1 and S_2 in $\overline{G}$ are the edges $a_i b_i$, where $i \in \{1, 2, \ldots, n\}$. Then,* $\mathrm{box}(G) \geq \lceil n/2 \rceil$.

Theorem 3.6. *For a graph G with l universal vertices, the following inequality holds:*

$$\mathrm{box}(M(G)) \geq \mathrm{box}(G) + \left\lceil \frac{l}{2} \right\rceil.$$

Proof. Let G be a graph and $x_1, x_2, \ldots, x_l$ universal vertices of G. Let H be the subgraph of G induced by $V(G) \setminus \{x_1, x_2, \ldots, x_l\}$. Note that $\mathrm{box}(H) = \mathrm{box}(G)$ holds. We consider the Mycielski graph $M(G)$ and its complement $\overline{M(G)}$. Let $X_j = \{(x_1)_j, (x_2)_j, \ldots, (x_l)_j\}$, the set of vertices in $V(G)_j$ corresponding to universal vertices of G. Let D_l be the subgraph of $\overline{M(G)}$ induced by the union of X_1 and X_2. Note that X_1 and X_2 are disjoint by their definition. It is not difficult to check that the only edges between X_1 and X_2 in $\overline{D_l}$ are the edges $(x_i)_1(x_i)_2$, where $i \in \{1, 2, \ldots, l\}$. In fact, the vertex $(x_i)_1 \in X_1$ is adjacent to all vertices in $V(G)_2 \setminus \{(x_i)_2\}$ in $M(G)$ and the vertex $(x_i)_2 \in X_2$ is adjacent to all vertices in $V(G)_1 \setminus \{(x_i)_1\}$ in $M(G)$ since x_i is a universal vertex of G. We see that $\mathrm{box}(D_l) \geq \lceil l/2 \rceil$ by Lemma 3.5.

We prove that $d_{\overline{M(G)}}(\overline{H}, \overline{D_l}) \geq 2$ holds. Let v be a vertex of $\overline{H}$ and x a vertex of $\overline{D_l}$. The vertex v is in $V(G)_1 \setminus X_1$ and the vertex x is in X_1 or X_2. We may represent x as $(x_i)_j$, where $j \in \{1, 2\}$. Since x_i is a universal vertex of G, the vertex $(x_i)_j$ is not adjacent to v in $\overline{M(G)}$. This implies that $d_{\overline{M(G)}}(v, x) \geq 2$ for a vertex v of $\overline{H}$ and a vertex x of $\overline{D_l}$, that is, $d_{\overline{M(G)}}(\overline{H}, \overline{D_l}) \geq 2$. Thus, the inequality

$$\mathrm{box}(M(G)) \geq \mathrm{box}(H) + \mathrm{box}(D_l) \geq \mathrm{box}(G) + \left\lceil \frac{l}{2} \right\rceil$$

holds by Lemma 3.4. □

Remark 3.7. *We note that the proof of Theorem 3.6 works on the generalized Mycielski graph* $M_r(G)$ *(see Section 5 for definition), that is, the inequality* $\mathrm{box}(M_r(G)) \geq \mathrm{box}(G) + \lceil l/2 \rceil$ *holds for a graph with* l *universal vertices. Further observations on* $\mathrm{box}(M_r(G))$ *appear in Section 5.*

In the proof of Theorem 3.6, we prove that $\mathrm{box}(D_l) \geq \lceil l/2 \rceil$ *by using Lemma 3.5. In fact, note that* $\mathrm{box}(D_l) = \lceil l/2 \rceil$*. Any two vertices in* X_1 *are not adjacent in* $\overline{M(G)}$ *since they are adjacent in* $M(G)$*. Hence,* X_1 *is independent in* $\overline{D_l}$*. Also note that* X_2 *is a clique in* $\overline{M(G)}$*, that is, in* $\overline{D_l}$ *by the definition of Mycielski graphs. See the argument behind the proof of Theorem 5 in* [7].

If we restrict our attention to the graph G *with only one universal vertex or only two universal vertices in the proof of Theorem 3.6, then Lemma 3.5 is superfluous. Note that* $\mathrm{box}(D_1) = \mathrm{box}(D_2) = 1$ *since* D_1 *is the trivial graph with two vertices and* D_2 *is the path with four vertices.*

Theorem 3.6 implies that $\mathrm{box}(M(G)) > \mathrm{box}(G)$ holds for a graph G with universal vertices. Also note that Mycielski's graph operation can be used to construct graphs with arbitrarily large boxicity (and chromatic number) the same as the join of graphs.

At the end of this section, we note that there is a graph G without universal vertices such that the boxicity of the Mycielski graph $M(G)$ is more than that of G. We give a simple example here. Also see Section 6.

Example 3.8. *Let P_n be a path with n vertices, where $n \geq 2$. We see that* $\mathrm{box}(M(P_n)) = 2$. *We can give a representation of $M(P_n)$ by a family of boxes in Euclidean 2-space. See Figure 3 below, where we write $V(P_n)_1 = \{1, 2, \ldots, n\}$ and $V(P_n)_2 = \{1', 2', \ldots, n'\}$ and for a vertex $v \in V(M(P_n)) = \{z\} \cup V(P_n)_1 \cup V(P_n)_2$, B_v denotes a box in Euclidean 2-space corresponding to the vertex v. Also note that $M(P_n)$ contains an induced cycle C_5.*

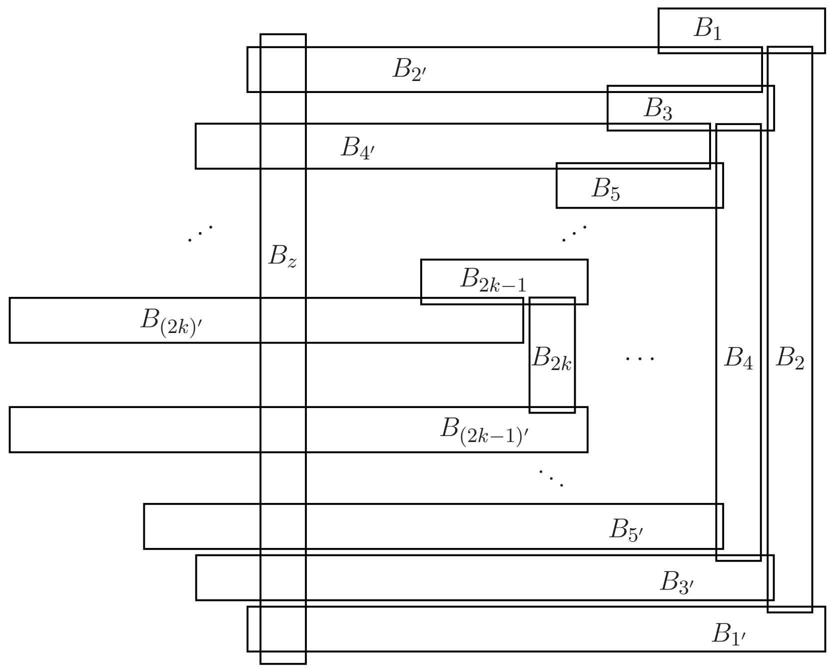

FIGURE 3. A representation of $M(P_{2k})$ by a family of boxes in Euclidean 2-space.

4. AN UPPER BOUND FOR THE BOXICITY OF MYCIELSKI GRAPHS

In this section, we give an upper bound for the boxicity of Mycielski graphs. Moreover we calculate the boxicity of Mycielski graphs of some of complete multipartite graphs. First we determine the boxicity of Mycielski graphs of complete graphs.

Lemma 4.1. *For a complete graph K_n, the following equalities hold:*

$$\mathrm{box}(M(K_n)) = \begin{cases} \lceil n/2 \rceil & \textit{if } n \textit{ is odd,} \\ \lceil n/2 \rceil + 1 & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Let H_0 be the subgraph of $M(K_n)$ induced by $V(M(K_n)) - \{z\}$. We have the inequality $\mathrm{box}(M(K_n)) \geq \mathrm{box}(H_0) \geq \lceil n/2 \rceil$ by Lemma 3.5.

Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. To see $\mathrm{box}(M(K_n)) \leq \lceil n/2 \rceil + 1$, we give cointerval subgraphs of $\overline{M(K_n)}$. Let G_0 be the subgraph of $\overline{M(K_n)}$ induced by $\{z, (v_n)_2\} \cup V(K_n)_1$. We define G_i to be the subgraph of $\overline{M(K_n)}$ induced by $\{(v_{2i-1})_1, (v_{2i})_1\} \cup V(K_n)_2$, where $i \in \{1, 2, \ldots, \lceil n/2 \rceil - 1\}$. Moreover, let $G_{\lceil n/2 \rceil}$ be the subgraph of $\overline{M(K_n)}$ induced by $\{(v_{n-1})_1, (v_n)_1\} \cup V(K_n)_2$. It is easy to see that the family $\{G_0, G_1, \ldots, G_{\lceil n/2 \rceil}\}$ is a cointerval edge covering of $\overline{M(K_n)}$, and hence $\mathrm{box}(M(K_n)) \leq \lceil n/2 \rceil + 1$ holds.

If n is odd, the family $\{G_0, G_1, \ldots, G_{\lceil n/2 \rceil - 1}\}$ is a cointerval edge covering of $\overline{M(K_n)}$, because the edge $(v_n)_1(v_n)_2$ is covered with the graph G_0. Hence we have the equality $\mathrm{box}(M(K_n)) = \lceil n/2 \rceil$.

If n is even, that is, $n = 2k$, we show that $\mathrm{box}(M(K_{2k})) > k$. Suppose to the contrary that $\overline{M(K_{2k})}$ can be covered with k cointerval (spanning) subgraphs $H_1, H_2, \ldots, H_k$ of $\overline{M(K_{2k})}$. Let $e_j = (v_j)_1(v_j)_2$ for $j \in \{1, 2, \ldots, 2k\}$. The graph H_i contains at most two edges in $\mathcal{E} = \{e_1, e_2, \ldots, e_{2k}\}$ since H_i is cointerval. In fact, the graph H_i must contain two edges in $\mathcal{E}$. Otherwise there is a graph H in $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ which contains only one edge in $\mathcal{E}$ or which contains no edges in $\mathcal{E}$. Hence the family $\mathcal{H} \setminus \{H\}$ of $k-1$ cointerval subgraphs of $\overline{M(K_{2k})}$ must cover at least $2k-1$ edges in $\mathcal{E}$, but this is impossible. On the other hand, there is a cointerval graph H_* in $\mathcal{H}$ which contains an edge $z(v)_1$, where the vertex v is in $V(K_{2k})$. We may assume that the graph H_* contains two edges e_s and e_t in $\mathcal{E}$. Hence we see

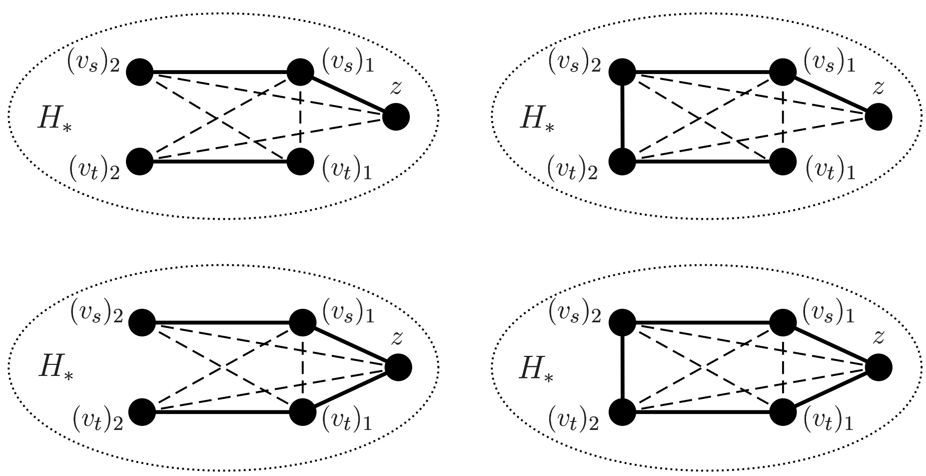

FIGURE 4. The subgraph H_* of $\overline{M(K_{2k})}$ containing edges e_s and e_t.

that $V(H_*) \supset \{(v_s)_1, (v_s)_2, (v_t)_1, (v_t)_2, z\}$. We note that

$$(v_s)_1(v_t)_1, (v_s)_1(v_t)_2, (v_t)_1(v_s)_2, z(v_s)_2, z(v_t)_2 \notin E(\overline{M(K_{2k})})$$

by the definition of Mycielski's construction. If $v \notin \{v_s, v_t\}$, it follows from Lemma 3.5 that $\mathrm{box}(\overline{H_*}) \geq 2$ since $(v)_1(v_s)_1, (v)_1(v_t)_1 \notin E(\overline{M(K_{2k})})$, a contradiction. Hence we may assume that $v = v_s$. We reach the four cases on the graph H_* indicated in Figure 4. These cases imply that $\mathrm{box}(\overline{H_*}) \geq$

2, which contradicts our assumption that H_* is cointerval. Thus we have $\mathrm{box}(M(K_{2k})) > k$. Hence we obtain the equality $\mathrm{box}(M(K_n)) = \lceil n/2 \rceil + 1$ if n is even. □

Remark 4.2. *We proved that the inequality* $\mathrm{box}(M(K_n)) \leq \lceil n/2 \rceil + 1$ *holds in the second paragraph of the proof of Lemma 4.1. We can also derive this inequality by using the minimum cardinality of a vertex cover of* $M(K_n)$*, that is, using the inequality* $\mathrm{box}(M(K_n)) \leq \lfloor t(M(K_n))/2 \rfloor + 1$*. A subset* U *of the vertex set of a graph* G *is a vertex cover of* G *if for each* $e \in E(G)$*, there is a vertex* $u \in U$ *such that* u *is in* e*. Note that* $t(M(K_n)) = n + 1$*.*

The *edge clique cover number* of a graph G, denoted by $\theta(G)$, is the minimum cardinality of a family of cliques that covers all edges of G. The following theorem gives us an upper bound for the boxicity of Mycielski graphs.

Theorem 4.3. *For a graph* G *with* l *universal vertices, the inequality*

$$\mathrm{box}(M(G)) \leq \theta(\overline{G}) + \left\lceil \frac{l}{2} \right\rceil + 1$$

holds. If l *is zero or odd, we have the inequality*

$$\mathrm{box}(M(G)) \leq \theta(\overline{G}) + \left\lceil \frac{l}{2} \right\rceil.$$

Proof. Let $\{A_1, A_2, \ldots, A_{\theta(\overline{G})}\}$ be a family of cliques in $\overline{G}$ that covers all edges of $\overline{G}$. Let $v_1, v_2, \ldots, v_l$ be all isolated vertices of $\overline{G}$ and write $J = \{v_1, v_2, \ldots, v_l\}$. Note that $V(G) = A_1 \cup A_2 \cup \cdots \cup A_{\theta(\overline{G})} \cup J$. We define

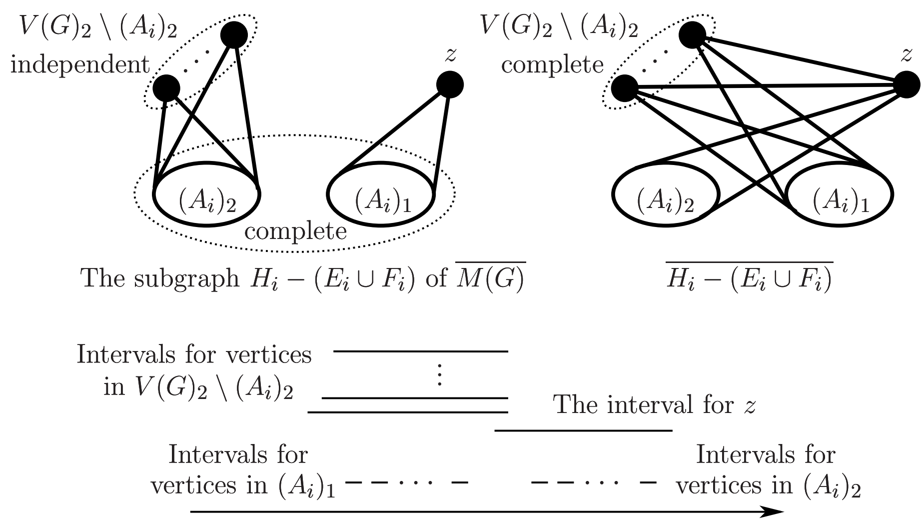

FIGURE 5. The subgraph $H_i - (E_i \cup F_i)$ and an interval representation of $\overline{H_i - (E_i \cup F_i)}$.

H_i to be the subgraph of $\overline{M(G)}$ induced by $(A_i)_1 \cup V(G)_2 \cup \{z\}$ and let $E_i = \{xy \,|\, x, y \in V(G)_2 \backslash (A_i)_2\}$ and $F_i = \{xy \,|\, x \in (A_i)_1, y \in V(G)_2 \backslash (A_i)_2\}$, where $i \in \{1, 2, \ldots, \theta(\overline{G})\}$. We can check that $H_i - (E_i \cup F_i)$ is a cointerval graph (see Figure 5). Note that the subgraph of $M(G)$ induced by $J_1 \cup J_2 \cup \{z\}$ is isomorphic to $M(K_l)$. Hence the edge set of the subgraph of $\overline{M(G)}$ isomorphic to $\overline{M(K_l)}$ can be covered with at most $\lceil l/2 \rceil + 1$ cointerval subgraphs as in the proof of Lemma 4.1. Let G_0 be the subgraph of $\overline{M(G)}$ induced by $\{z, (v_l)_2\} \cup J_1$ and G_i the subgraph of $\overline{M(G)}$ induced by $\{(v_{2i-1})_1, (v_{2i})_1\} \cup J_2$ for $i \in \{1, 2, \ldots, \lceil l/2 \rceil - 1\}$. Moreover, let $G_{\lceil l/2 \rceil}$ be the subgraph of $\overline{M(G)}$ induced by $\{(v_{l-1})_1, (v_l)_1\} \cup J_2$. We can check that $\theta(\overline{G}) + \lceil l/2 \rceil + 1$ cointerval subgraphs $H_1 - (E_1 \cup F_1), \ldots, H_{\theta(\overline{G})} - (E_{\theta(\overline{G})} \cup F_{\theta(\overline{G})}), G_0, G_1, \ldots, G_{\lceil l/2 \rceil}$ cover all edges of $\overline{M(G)}$.

Let e be an edge of $E(\overline{M(G)})$. If $e \cap \{z\} \neq \emptyset$, we see that $e \cap V(G)_1 \neq \emptyset$. Hence there is an $i \in \{1, 2, \ldots, \theta(\overline{G})\}$ such that $e \in E(H_i - (E_i \cup F_i))$ or $e \in E(G_0)$. If $e \cap \{z\} = \emptyset$, we have $e \subset V(G)_1 \cup V(G)_2$. Hence, if $e \subset V(G)_2$, in particular, $e \cap (A_i)_2 \neq \emptyset$, we see that $e \in E(H_i - (E_i \cup F_i))$. If $e \subset V(G)_2$ and $e \cap (A_i)_2 = \emptyset$ for any i, we see that $e \subset J_2$, and hence $e \in E(G_i)$ for $i \neq 0$. If $e \cap V(G)_1 \neq \emptyset$, we reach the following two cases:

$$\text{(i) } e \subset V(G)_1 \text{ or (ii) } e \cap V(G)_2 \neq \emptyset.$$

In case (i), the edge e is in some $(A_i)_1$ since the family $\{A_1, A_2, \ldots, A_{\theta(\overline{G})}\}$ of cliques covers all edges of $\overline{G}$, and hence we have $e \in E(H_i - (E_i \cup F_i))$.

Now we focus on case (ii). Let u be a vertex in $V(\overline{G})$ and C_u the union of cliques in $\{A_1, A_2, \ldots, A_{\theta(\overline{G})}\}$ containing the vertex u. If u is an isolated vertex in $\overline{G}$, let C_u be the set $\{u\}$. Then we note $u_1 \in V(G)_1$ is never adjacent to vertices in $V(G)_2 \setminus (C_u)_2$ on $\overline{M(G)}$ by the definition of Mycielski graphs. Hence the following two subcases occur:

(ii-1) the edge e connects a vertex of $(A_i)_1$ and a vertex of $(A_i)_2$ for some i or
(ii-2) the edge e connects a vertex $(v_i)_1$ and a vertex $(v_i)_2$, where $v_i \in J$.

Under subcase (ii-1), we notice that $e \in E(H_i - (E_i \cup F_i))$. Under subcase (ii-2), we see $e \in E(G_{\lceil i/2 \rceil})$. These arguments complete the proof of our first statement.

If $l = 0$, the graphs $H_1 - (E_1 \cup F_1), \ldots, H_{\theta(\overline{G})} - (E_{\theta(\overline{G})} \cup F_{\theta(\overline{G})})$ cover all edges of $\overline{M(G)}$. If l is odd, $H_1 - (E_1 \cup F_1), \ldots, H_{\theta(\overline{G})} - (E_{\theta(\overline{G})} \cup F_{\theta(\overline{G})})$, $G_0, G_1, \ldots, G_{\lceil l/2 \rceil - 1}$ cover all edges of $\overline{M(G)}$, because the edge $(v_l)_1(v_l)_2$ is covered with the graph G_0. Our second statement follows from similar arguments as above. □

Theorem 3.6 and Theorem 4.3 pretty much narrow the boxicity of Mycielskians of graphs that satisfy the equality $\mathrm{box}(G) = \theta(\overline{G})$. They also determine the boxicity of some Mycielski graphs.

Corollary 4.4. *For a graph G with l universal vertices that satisfies the equality* $\mathrm{box}(G) = \theta(\overline{G})$, *the inequalities*

$$\mathrm{box}(G) + \left\lceil \frac{l}{2} \right\rceil \leq \mathrm{box}(M(G)) \leq \mathrm{box}(G) + \left\lceil \frac{l}{2} \right\rceil + 1$$

hold. Moreover if l be zero or odd, the equality

$$\mathrm{box}(M(G)) = \mathrm{box}(G) + \left\lceil \frac{l}{2} \right\rceil$$

holds. □

We can give examples of graphs that satisfy $\mathrm{box}(G) = \theta(\overline{G})$. Recall that the boxicity of a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is the number of n_i which is at least 2. If $K_{n_1,n_2,\ldots,n_k}$ has l universal vertices, we obtain $\mathrm{box}(K_{n_1,n_2,\ldots,n_k}) = k - l = \theta(\overline{K_{n_1,n_2,\ldots,n_k}})$. Hence we have

$$\mathrm{box}(K_{n_1,n_2,\ldots,n_k}) + \left\lceil \frac{l}{2} \right\rceil = \left\lceil \frac{2k-l}{2} \right\rceil.$$

Corollary 4.5. *For a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ with l universal vertices, the inequalities*

$$\left\lceil \frac{2k-l}{2} \right\rceil \leq \mathrm{box}(M(K_{n_1,n_2,\ldots,n_k})) \leq \min \left\{ k, \left\lceil \frac{2k-l}{2} \right\rceil + 1 \right\}$$

hold. In particular, if l is zero or odd, the equality $\mathrm{box}(M(K_{n_1,n_2,\ldots,n_k})) = \lceil (2k-l)/2 \rceil$ *holds.* □

We present other examples of graphs that satisfy $\mathrm{box}(G) = \theta(\overline{G})$. The graph H whose complement is a chain of cliques is a desired one, where neighboring cliques share exactly one vertex and each clique has at least four vertices. Note that the graph H contains a complete multipartite graph $K_{2,2,\ldots,2}$ as an induced subgraph and the number of its partite sets is equal to that of maximal cliques of the complement $\overline{H}$.

Moreover if we consider a graph operation that extends a graph G to a new graph $\mathrm{Susp}_n(G)$, called the *n-suspension* of G, we can get more examples that we desire. The vertex set of $\mathrm{Susp}_n(G)$ is the union of $V(G)$ and the set of new vertices $\{x_1, x_2, \ldots, x_n\}$. The edge set of $\mathrm{Susp}_n(G)$ is the union of $E(G)$ and the set $\{x_i v \,|\, v \in V(G),\, i \in \{1, 2, \ldots, n\}\}$. Here we assume that n is an integer at least 2. We see that $\mathrm{box}(\mathrm{Susp}_n(G)) = \mathrm{box}(G) + 1$ and $\theta(\overline{\mathrm{Susp}_n(G)}) = \theta(\overline{G}) + 1$ for a graph G by Theorem 2.1 and Lemma 3.4. Hence if the graph G satisfies $\mathrm{box}(G) = \theta(\overline{G})$, the equality $\mathrm{box}(\mathrm{Susp}_n(G)) = \theta(\overline{\mathrm{Susp}_n(G)})$ holds. We note that the family of graphs satisfying $\mathrm{box}(G) = \theta(\overline{G})$ is not narrow at all.

5. Relations between the boxicity of Mycielski graphs and generalized Mycielski graphs

In this section, we consider relations between Mycielski graphs and their generalizations in the context of boxicity.

Let G be a graph and r an integer at least 2. Let $V(G)_i$ be a copy of $V(G)$, where $i \in \{1, 2, \ldots, r\}$. For each vertex $v \in V(G)$, the symbol v_i denotes the vertex in $V(G)_i$ corresponding to v. The *generalized Mycielski graph* of G, denoted by $M_r(G)$, is the graph whose vertex set is $\{z\} \cup (\cup_{i=1}^{r} V(G)_i)$, the disjoint union of the set of an additional new vertex z and copies $V(G)_1, \ldots, V(G)_r$ of $V(G)$, and whose edge set is $\cup_{i=1}^{r+1} E_i$, where

$$\begin{aligned} E_1 &= \{u_1 v_1 \mid uv \in E(G)\}, \\ E_i &= \{u_{i-1} v_i, v_{i-1} u_i \mid uv \in E(G)\} \quad \text{for } i \in \{2, 3, \ldots, r\}, \text{ and} \\ E_{r+1} &= \{z u_r \mid u \in V(G)\}. \end{aligned}$$

Note that the graph $M_2(G)$ is identical to $M(G)$. First, we present a relation between $\operatorname{box}(M_r(G))$ and $\operatorname{box}(M_2(G))$ for a bipartite graph G.

Theorem 5.1. *The inequality* $\operatorname{box}(M_r(G)) \leq \operatorname{box}(M_2(G)) + 2$ *holds for a bipartite graph G and $r \geq 2$.*

Proof. We partition $V(G)$ into two partite sets V_1 and V_2. Fix a family $\{B_x\}$ of boxes in the optimal dimensional space which represents $M_2(G)$. Note that $B_{u_1} \cap B_{u_2} = \emptyset$, $B_{v_1} \cap B_{v_2} = \emptyset$, and $B_{u_2} \cap B_{v_2} = \emptyset$ for distinct two vertices u and v of G by the definition of $M_2(G)$. Moreover we note that $uv \in E(G)$, $B_{u_1} \cap B_{v_1} \neq \emptyset$, $B_{u_1} \cap B_{v_2} \neq \emptyset$, and $B_{u_2} \cap B_{v_1} \neq \emptyset$ are equivalent each other. First we define the family $\{B'_{v_i}\}$ of boxes in $(\operatorname{box}(M_2(G)) + 1)$-dimensional space to give a box-representation of the graph $M_r(G) - \{z\}$ as follows: for each vertex $v \in V(G)$,

$$B'_{v_1} = B_{v_1} \times \{0\},$$

$$B'_{v_{2i}} = \begin{cases} B_{v_2} \times [i-1, i-1/2] & \text{if } v \in V_1, \\ B_{v_2} \times [-(i-1/2), -(i-1)] & \text{if } v \in V_2, \end{cases} \quad \text{for } i \in \{1, 2, \ldots, \lfloor r/2 \rfloor\},$$

and

$$B'_{v_{2i-1}} = \begin{cases} B_{v_1} \times [-(i-1), -(i-3/2)] & \text{if } v \in V_1, \\ B_{v_1} \times [i-3/2, i-1] & \text{if } v \in V_2, \end{cases} \quad \text{for } i \in \{2, 3, \ldots, \lceil r/2 \rceil\}.$$

Take a vertex $v \in V(G)$ and $k \in \{1, 2, \ldots, r\}$, then consider the adjacency of the vertex v_k of $M_r(G) - \{z\}$ from the above family $\{B'_{v_i}\}$ that we defined. It is easy to see that the box B'_{v_k} does not have intersection with boxes corresponding to vertices in $\{v_1, v_2, \ldots, v_r\} \setminus \{v_k\}$. We also see that B'_{v_k} does not have intersection with boxes that correspond to vertices in $V(G)_k \setminus \{v_k\}$ for $k \in \{2, 3, \ldots, r\}$. Clearly, the family $\{B'_{v_1}\}_{v \in V(G)}$ represents the subgraph of $M_r(G) - \{z\}$ induced by $V(G)_1$, so we may assume $k \geq 2$.

If the vertex v is adjacent to a vertex u in G, we can check that the box B'_{v_k} has nonempty intersection only with boxes $B'_{u_{k-1}}$ and $B'_{u_{k+1}}$ for $2 \leq k \leq r-1$, and only with the box $B'_{u_{r-1}}$ for $k = r$ in the family $\{B'_{u_1}, B'_{u_2}, \ldots, B'_{u_r}\}$. If the vertex v is not adjacent to a vertex u in G, no boxes in the family $\{B'_{u_1}, B'_{u_2}, \dots, B'_{u_r}\}$ have nonempty intersection with B'_{v_k} since $B_{u_1} \cap B_{v_1} = \emptyset$, $B_{u_1} \cap B_{v_2} = \emptyset$, and $B_{u_2} \cap B_{v_1} = \emptyset$ hold. Hence the family $\{B'_{v_i}\}$ represents the graph $M_r(G) - \{z\}$.

Now, we define the family $\{B''_x\}$ of boxes in $(\mathrm{box}(M_2(G))+2)$-dimensional space that represents $M_r(G)$ as follows:

$$\begin{aligned} B''_{v_i} &= B'_{v_i} \times \{0\} \quad \text{for } i \neq r, \\ B''_{v_r} &= B'_{v_r} \times [0,1], \\ B''_z &= B \times \{1\}, \end{aligned}$$

where B is a box in $(\mathrm{box}(M_2(G)) + 1)$-dimensional space that contains all boxes in $\{B'_{v_r}\}_{v \in V(G)}$. We can check easily that the family $\{B''_x\}$ represents $M_r(G)$, which completes the proof of our theorem. □

We believe that the inequality $\mathrm{box}(M_r(G)) \leq \mathrm{box}(M_2(G)) + c$ holds for a graph G and some small constant c in general. The next theorem shows a relation between $\mathrm{box}(M_r(G))$ and $\mathrm{box}(M_3(G))$ for a graph G. These results provide further motivation to investigate the boxicity of $M_2(G)$ and $M_3(G)$.

Theorem 5.2. *The inequality* $\mathrm{box}(M_r(G)) \leq \mathrm{box}(M_3(G)) + 1$ *holds for a graph* G *and* $r \geq 3$.

Proof. Let $\{B_x\}$ be a family of boxes in the optimal dimensional space which represents $M_3(G)$. We note that for distinct two vertices u and v of G,

$$\begin{aligned} &B_{u_i} \cap B_{u_j} = \emptyset,\ B_{v_i} \cap B_{v_j} = \emptyset, \text{ where } i, j \in \{1,2,3\} \text{ and } i \neq j, \\ &B_{u_i} \cap B_{v_i} = \emptyset, \text{ where } i \in \{2,3\}, \text{ and} \\ &B_{u_1} \cap B_{v_3} = \emptyset,\ B_{v_1} \cap B_{u_3} = \emptyset \end{aligned}$$

hold by the definition of $M_3(G)$. In addition we note that $uv \in E(G)$, $B_{u_1} \cap B_{v_1} \neq \emptyset$, $B_{u_1} \cap B_{v_2} \neq \emptyset$, $B_{u_2} \cap B_{v_1} \neq \emptyset$, $B_{u_2} \cap B_{v_3} \neq \emptyset$, and $B_{u_3} \cap B_{v_2} \neq \emptyset$ are equivalent each other. By using similar techniques from the previous theorem, we can present the family $\{B'_x\}$ of boxes in $(\mathrm{box}(M_3(G)) + 1)$-dimensional space that represents the graph $M_r(G)$ as follows: for each vertex $v \in V(G)$, define

$$\begin{aligned} B'_{v_i} &= B_{v_i} \times \{0\} & &\text{for } i \in \{1,2\}, \\ B'_{v_{2i-1}} &= B_{v_3} \times [i-2, i-3/2] & &\text{for } i \in \{2,3,\dots,\lceil r/2 \rceil\}, \\ B'_{v_{2i}} &= B_{v_2} \times [i-3/2, i-1] & &\text{for } i \in \{2,3,\dots,\lfloor r/2 \rfloor\}, \end{aligned}$$

and for the additional vertex z' of $M_r(G)$,

$$B'_{z'} = \begin{cases} B \times \{r/2 - 1\} & \text{if } r \text{ is even,} \\ B_z \times \{\lfloor r/2 \rfloor - 1/2\} & \text{if } r \text{ is odd,} \end{cases}$$

where B is a box in $(\mathrm{box}(M_3(G)))$-dimensional space that contains all boxes in $\{B_{v_2}\}_{v \in V(G)}$ and z is the additional vertex of $M_3(G)$. Note that any pair of distinct two boxes in $\{B'_{v_1}, B'_{v_2}, \dots, B'_{v_r}\}$ does not have intersection for a vertex v of G, and also note that any pair of distinct two boxes in $\{B'_{v_k}\}_{v\in V(G)}$ does not have intersection for $k \in \{2,3,\ldots,r\}$.

Fix a vertex $v \in V(G)$ and $k \in \{1,2,\ldots,r\}$. We consider the adjacency of the vertex v_k of $M_r(G)$. Clearly, the family $\{B'_{v_1}\}_{v\in V(G)}$ represents the subgraph of $M_r(G)$ induced by $V(G)_1$, and hence we may assume $k \geq 2$. If the vertex v is adjacent to a vertex u in G, we can verify that the box B'_{v_k} has nonempty intersection only with boxes $B'_{u_{k-1}}$ and $B'_{u_{k+1}}$ for $2 \leq k \leq r-1$, and only with $B'_{u_{r-1}}$ for $k = r$ in $\{B'_{u_1}, B'_{u_2}, \ldots, B'_{u_r}\}$. If the vertex v is not adjacent to a vertex u in G, no boxes in the family $\{B'_{u_1}, B'_{u_2}, \ldots, B'_{u_r}\}$ have nonempty intersection with B'_{v_k}. In addition, the box B'_{v_k} has nonempty intersection with $B'_{z'}$ if and only if $k = r$ for each $v \in V(G)$. Hence our arguments guarantee that the family $\{B'_x\}$ represents the graph $M_r(G)$. $\square$

6. Concluding Remarks: graphs with $\mathrm{box}(M(G)) > \mathrm{box}(G)$

We proved that the boxicity of the Mycielski graph of a graph G with universal vertices is more than that of G. As examples of complete multipartite graphs without universal vertices, one may expect that the equality $\mathrm{box}(M(G)) = \mathrm{box}(G)$ holds for a graph G without universal vertices. However, we note that there is a graph G without universal vertices such that $\mathrm{box}(M(G)) > \mathrm{box}(G)$ holds. For examples, nontrivial interval graphs without universal vertices satisfy this condition. The Mycielski graph of such an interval graph is not interval because it contains a cycle with 5 vertices as an induced subgraph. Another example of a graph without universal vertices that satisfy $\mathrm{box}(M(G)) > \mathrm{box}(G)$ is a cycle with at least 5 vertices. The author verified in a manuscript that the boxicity of Mycielski graph of a cycle with at least 5 vertices is equal to 3.

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