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ON THE TOTAL SIGNED DOMINATION NUMBER OF THE CARTESIAN PRODUCT OF PATHS

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ABSTRACT. Let G be a finite connected simple graph with a vertex set $V(G)$ and an edge set $E(G)$. A total signed dominating function of G is a function $f : V(G) \cup E(G) \to \{-1, 1\}$, such that $\sum_{y \in N_T[x]} f(y) \geq 1$ for all $x \in V(G) \cup E(G)$. The total signed domination number of G is the minimum weight of a total signed dominating function on G. In this paper, we prove lower and upper bounds on the total signed domination number of the Cartesian product of two paths, $P_m \Box P_n$.

1. Introduction

Let G be a finite connected simple graph with a vertex set $V(G)$ and an edge set $E(G)$. For $v \in V(G)$, the open neighborhood of v is $N(v) = \{u \mid (u, v) \in E(G)\}$, and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For $e \in E(G)$, the open neighborhood of e is $N(e) = \{g \mid g \in E(G) \text{ is adjacent to } e\}$, and the closed neighborhood of e is $N[e] = N(e) \cup \{e\}$. For an element $x \in V(G) \cup E(G)$, the total closed neighborhood of x is $N_T[x] = \{y | y \text{ is adjacent to } x \text{ or } y \text{ is incident with } x, y \in V(G) \cup E(G)\} \cup \{x\}$. We use [6] for terminology and notation which are not defined here.

The fundamental concept concerning domination, namely the domination number of a graph, was originally defined by means of a dominating set. This definition may be transferred into an equivalent definition done by means of a dominating function (the characteristic function of a dominating set). A function $f : V(G) \to \{0, 1\}$ is called a domination function on G, if $\sum_{x \in N[v]} f(x) \geq 1$ for each $v \in V(G)$. The weight of f is $w(f) = \sum_{v \in V(G)} f(v)$. The minimum of weights $w(f)$, taken over all dominating functions on G, is called the domination number $\gamma(G)$ of G.

The variations of the domination number may be obtained by replacing the set $\{0, 1\}$ by another set of numbers. If the closed interval $[0, 1]$ on the real line is taken instead of $\{0, 1\}$, then the fractional domination number is

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defined; by exchanging $\{0,1\}$ for $\{-1,1\}$, the signed domination number is obtained.

A signed dominating function is defined as $f : V(G) \to \{-1,1\}$ such that $\sum_{x\in N[v]} f(x) \geq 1$ for all $v \in V(G)$. The weight of f is $w(f) = \sum_{v\in V(G)} f(v)$. The signed domination number $\gamma_s(G)$ of G is the minimum weight of a signed dominating function on G.

A total signed dominating function is defined as $f : V(G) \cup E(G) \to \{-1,1\}$, such that $F(x) = \sum_{y\in N_T[x]} f(y) \geq 1$ for all $x \in V(G) \cup E(G)$. The weight of f is $w(f) = \sum_{x\in V(G)\cup E(G)} f(x)$. The total signed domination number $\gamma_s^*(G)$ of G is the minimum weight of a total signed dominating function G. In [3], Lu gave lower bounds for the total signed domination number of a graph G and computed the exact values of $\gamma_s^*(C_n)$ and $\gamma_s^*(P_n)$ $(n \geq 3)$. In [4], Yuan and his collaborators studied the total signed domination number of $n \cdot C_m$. Zou [7] gave the lower bounds on the total signed domination number of some graphs.

For two graphs G and H, the Cartesian product of G and H is the graph denoted $G\Box H$, where $v_{i,j} \in V(G\Box H)$ if and only if $v_i \in V(G)$ and $v_j \in V(H)$, and $(v_{i_1,j_1}, v_{i_2,j_2}) \in E(G\Box H)$ if and only if $i_1 = i_2$ and $(j_1, j_2) \in E(H)$ or $j_1 = j_2$ and $(i_1, i_2) \in E(G)$. The study of domination numbers of products of graphs was initiated by Vizing [5]. A survey and recent results on Vizing's conjecture can be found in [1].

In this paper, we study the total signed domination number of Cartesian products of two paths. We prove a lower bound on the total signed domination number of $P_m\Box P_n$ $(m, n \geq 2)$,

$$\gamma_s^*(P_m\Box P_n) \geq \left\lceil \frac{15mn - 3m - 3n - 40}{45} \right\rceil_{\mathcal{P}(3mn-m-n)}.$$

We then construct some total signed dominating functions and with them, present an upper bound of $\gamma_s^*(P_m\Box P_n)$,

$$\gamma_s^*(P_m\Box P_n) \leq \frac{mn + m + n + 2}{2}.$$

The following are some important results on the total signed domination number of P_n and the signed domination number of $P_m\Box P_n$.

Theorem 1.1. (Lu [3]) *For any graph G,*

$$\gamma_s^*(G) \geq \left\lceil \frac{\delta(G) - \Delta(G) + 1}{\delta(G) + \Delta(G) + 1}(|E(G)| + |V(G)|) \right\rceil_{\mathcal{P}(|E(G)|+|V(G)|)}$$

where $\mathcal{P}(s)$ is defined to be the parity of s, that is, $\mathcal{P}(s) = odd$ if s is odd and $\mathcal{P}(s) = even$ if s is even. Furthermore, this bound is sharp.

Based on the Theorem 1.1, we can easily obtain the lower bounds for the total signed domination number $\gamma_s^*(P_m\Box P_n)$.

Corollary 1.2. *For any positive integers $m, n \geq 2$,*

$$\gamma_s^*(P_m \Box P_n) \geq \left\lceil -\frac{3mn - m - n}{7} \right\rceil_{\mathcal{P}(3mn-m-n)}.$$

Theorem 1.3 (Lu [3])**.** *For $n \geq 3$,*

$$\gamma_s^*(P_n) = \begin{cases} \lceil \frac{2n-1}{5} \rceil + 1, & \text{if } n \pmod 5 \equiv 0 \text{ or } 4, \\ \lceil \frac{2n-1}{5} \rceil, & \text{if } n \pmod 5 \equiv 1 \text{ or } 3, \\ \lceil \frac{2n-1}{5} \rceil + 2, & \text{if } n \pmod 5 \equiv 2. \end{cases}$$

Theorem 1.4 (Haas [2])**.**

$$\gamma_s(P_2 \Box P_n) = \begin{cases} n, & n \text{ even}, \\ n+1, & n \text{ odd}. \end{cases}$$

For $n \geq 3$,

$$\frac{7n}{5} - \frac{8}{5} \leq \gamma_s(P_3 \Box P_n) \leq \frac{7n}{5} + 2 - \frac{2(n \pmod 5)}{5}.$$

For $m, n \geq 4$,

$$\frac{mn + 4m + 4n - 24}{5} \leq \gamma_s(P_m \Box P_n) \leq \frac{mn + 8n + 4m}{5}.$$

2. Lower bounds on the total signed domination number of graph $P_m \Box P_n$

In this section, we prove that lower bounds on the total signed domination number of $P_m \Box P_n$ $(m, n \geq 2)$ can be greater than zero (Corollary 1.2).

Let $G = P_m \Box P_n$ with $V(G) = \{v_{i,j} \mid 0 \leq i \leq m-1,\ 0 \leq j \leq n-1\}$ and $E(G) = \{e_{i,j} \mid e_{i,j} = (v_{i,j}, v_{i+1,j}),\ 0 \leq i \leq m-1,\ 0 \leq j \leq n-2\} \cup \{e'_{i,j} \mid e'_{i,j} = (v_{i,j}, v_{i+1,j}),\ 0 \leq i \leq m-2,\ 0 \leq j \leq n-1\}$ (see Figure 1).

Theorem 2.1. *For any integers $m, n \geq 2$,*

$$\gamma_s^*(P_m \Box P_n) \geq \left\lceil \frac{15mn - 3m - 3n - 40}{45} \right\rceil_{\mathcal{P}(3mn-m-n)}.$$

Proof. Let f be an arbitrary total signed dominating function of graph $G = P_m \Box P_n$. Then we have

$$\sum_{y \in V(G) \cup E(G)} \sum_{x \in N_T[y]} f(x) \geq 3mn - m - n. \tag{2.1}$$

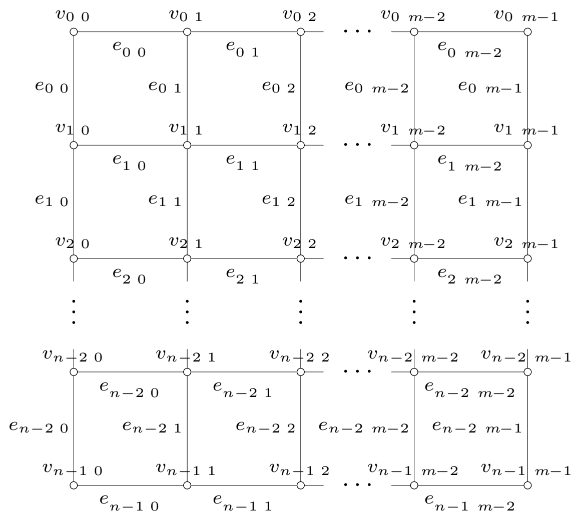


FIGURE 1. Graph $P_m \Box P_n$.

Clearly for $0 \leq i \leq n-1$, $F(v_{i,0}) - f(v_{i,1}) \geq 0$ and $F(v_{i,m-1}) - f(v_{i,m-2}) \geq 0$. Similarly for $0 \leq i \leq m-1$, $F(v_{0,i}) - f(v_{1,i}) \geq 0$ and $F(v_{n-1,i}) - f(v_{n-2,i}) \geq 0$. Therefore the sum

$$\begin{array}{l}\sum_{i=0}^{n-1}(F(v_{i,0}) - f(v_{i,1})) + \sum_{i=0}^{n-1}(F(v_{i,m-1}) - f(v_{i,m-2})) + \\ \sum_{i=0}^{m-1}(F(v_{0,i}) - f(v_{1,i})) + \sum_{i=0}^{m-1}(F(v_{n-1,i}) - f(v_{n-2,i})) \geq 0.\end{array}$$

Using the fact that $f(x) \geq -1$ for all $x \in V(G)$, we conclude

$$\begin{array}{l}3\sum_{i=0}^{n-1} f(v_{i,0}) + 2\sum_{i=0}^{n-2} f(e_{i,0}) + \sum_{i=0}^{n-1} f(e'_{i,0}) \\ 3\sum_{i=0}^{n-1} f(v_{i,m-1}) + 2\sum_{i=0}^{n-2} f(e_{i,m-1}) + \sum_{i=0}^{n-1} f(e'_{i,m-2}) + \\ 3\sum_{i=0}^{m-1} f(v_{0,i}) + 2\sum_{i=0}^{m-2} f(e'_{0,i}) + \sum_{i=0}^{m-1} f(e_{0,i}) + \\ 3\sum_{i=0}^{m-1} f(v_{n-1,i}) + 2\sum_{i=0}^{m-2} f(e'_{n-1,i}) + \sum_{i=0}^{m-1} f(e_{n-1,i}) \\ \geq -8.\end{array} \tag{2.2}$$

Note that since $F(v_{0,0}) \geq 1$, $f(v_{0,0}) + f(e'_{0,0}) + f(e_{0,0}) \geq -1$. Analogously, we have $f(v_{n-1,0}) + f(e'_{n-1,0}) + f(e_{n-2,0}) \geq -1$, $f(v_{0,m-1}) + f(e'_{0,m-1}) + f(e_{0,m-1}) \geq -1$ and $f(v_{n-1,m-1}) + f(e'_{n-1,m-1}) + f(e_{n-2,m-1}) \geq -1$. Since $F(x) \geq 1$ for all x,

$$\sum_{i=0}^{n-1} F(e_{i,0}) + \sum_{i=0}^{n-1} F(e_{i,m-1}) + \sum_{i=0}^{m-1} F(e'_{0,i}) + \sum_{i=0}^{m-1} F(e'_{n-1,i}) \geq 2m + 2n - 4.$$

It follows that

$$\begin{array}{l}2\sum_{i=0}^{n-1} f(v_{i,0}) + 3\sum_{i=0}^{n-2} f(e_{i,0}) + 2\sum_{i=0}^{n-1} f(e'_{i,0}) \\ +2\sum_{i=0}^{n-1} f(v_{i,m-1}) + 3\sum_{i=0}^{n-2} f(e_{i,m-1}) + 2\sum_{i=0}^{n-1} f(e'_{i,m-2}) \\ +2\sum_{i=0}^{m-1} f(v_{0,i}) + 3\sum_{i=0}^{m-2} f(e'_{0,i}) + 2\sum_{i=0}^{m-1} f(e_{0,i}) \\ +2\sum_{i=0}^{m-1} f(v_{n-1,i}) + 3\sum_{i=0}^{m-2} f(e'_{n-1,i}) + 2\sum_{i=0}^{m-1} f(e_{n-2,i}) \\ \geq 2m + 2n - 12.\end{array} \tag{2.3}$$

Adding equation (2.2) and (2.3),

$$\begin{aligned}
&5\sum_{i=0}^{n-1} f(v_{i,0}) + 5\sum_{i=0}^{n-2} f(e_{i,0}) + 3\sum_{i=0}^{n-1} f(e'_{i,0})\\
&+5\sum_{i=0}^{n-1} f(v_{i,m-1}) + 5\sum_{i=0}^{n-2} f(e_{i,m-1}) + 3\sum_{i=0}^{n-1} f(e'_{i,m-2})\\
&+5\sum_{i=0}^{m-1} f(v_{0,i}) + 5\sum_{i=0}^{m-2} f(e'_{0,i}) + 3\sum_{i=0}^{m-1} f(e_{0,i})\\
&+5\sum_{i=0}^{m-1} f(v_{n-1,i}) + 5\sum_{i=0}^{m-2} f(e'_{n-1,i}) + 3\sum_{i=0}^{m-1} f(e_{n-2,i})\\
&\geq 2n + 2m - 20.
\end{aligned}$$

This implies

$$\begin{aligned}
&10\sum_{i=0}^{n-1} f(v_{i,0}) + 10\sum_{i=0}^{n-2} f(e_{i,0}) + 5\sum_{i=0}^{n-1} f(e'_{i,0})\\
&+10\sum_{i=0}^{n-1} f(v_{i,m-1}) + 10\sum_{i=0}^{n-2} f(e_{i,m-1}) + 5\sum_{i=0}^{n-1} f(e'_{i,m-2})\\
&+10\sum_{i=0}^{m-1} f(v_{0,i}) + 10\sum_{i=0}^{m-2} f(e'_{0,i}) + 5\sum_{i=0}^{m-1} f(e_{0,i})\\
&+10\sum_{i=0}^{m-1} f(v_{n-1,i}) + 10\sum_{i=0}^{m-2} f(e'_{n-1,i}) + 5\sum_{i=0}^{m-1} f(e_{n-2,i})\\
&\geq 4m + 4n - 40 - \sum_{i=0}^{n-1} f(e'_{i,0}) - \sum_{i=0}^{n-1} f(e'_{i,m-2}) - \sum_{i=0}^{m-1} f(e_{0,i})\\
&\quad - \sum_{i=0}^{m-1} f(e_{n-2,i})\\
&\geq 2m + 2n - 40.
\end{aligned} \tag{2.4}$$

Finally observe,

$$\begin{aligned}
&\sum_{y\in V(G)\cup E(G)} \sum_{x\in N_T[y]} f(x)\\
= \;& 9\sum_{y\in V(G)\cup E(G)} f(y) - 2\sum_{i=0}^{n-1} f(v_{i,0}) - 2\sum_{i=0}^{n-2} f(e_{i,0})\\
&- \sum_{i=0}^{n-1} f(e'_{i,0}) - 2\sum_{i=0}^{n-1} f(v_{i,m-1}) - 2\sum_{i=0}^{n-2} f(e_{i,m-1})\\
&- \sum_{i=0}^{n-1} f(e'_{i,m-2}) - 2\sum_{i=0}^{m-1} f(v_{0,i}) - 2\sum_{i=0}^{m-2} f(e'_{0,i})\\
&- \sum_{i=0}^{m-1} f(e_{0,i}) - 2\sum_{i=0}^{m-1} f(v_{n-1,i}) - 2\sum_{i=0}^{m-2} f(e'_{n-1,i})\\
&- \sum_{i=0}^{m-1} f(e_{n-2,i}).
\end{aligned} \tag{2.5}$$

From equations (2.1),(2.4), and (2.5) it follows that

$$\begin{aligned}
& 45\sum_{y\in V(G)\cup E(G)} f(y)\\
= \;& 5\sum_{y\in V(G)\cup E(G)} \sum_{x\in N_T[y]} f(x)\\
&+10\sum_{i=0}^{n-1} f(v_{i,0}) + 10\sum_{i=0}^{n-2} f(e_{i,0}) + 5\sum_{i=0}^{n-1} f(e'_{i,0})\\
&+10\sum_{i=0}^{n-1} f(v_{i,m-1}) + 10\sum_{i=0}^{n-2} f(e_{i,m-1}) + 5\sum_{i=0}^{n-1} f(e'_{i,m-2})\\
&+10\sum_{i=0}^{m-1} f(v_{0,i}) + 10\sum_{i=0}^{m-2} f(e'_{0,i}) + 5\sum_{i=0}^{m-1} f(e_{0,i})\\
&+10\sum_{i=0}^{m-1} f(v_{n-1,i}) + 10\sum_{i=0}^{m-2} f(e'_{n-1,i}) + 5\sum_{i=0}^{m-1} f(e_{n-2,i})\\
\geq \;& 5(3mn - m - n) + 2m + 2n - 40 = 15mn - 3m - 3n - 40.
\end{aligned}$$

Hence

$$\gamma_s^*(P_m \Box P_n) \geq \left\lceil \frac{15mn-3m-3n-40}{45} \right\rceil_{\mathcal{P}(3mn-m-n)}.$$

□

3. Upper bounds on the total signed domination number of graph $P_m \Box P_n$

In this section, we present upper bounds on the total signed domination number of $P_m \Box P_n$ for $m, n \geq 2$. We introduce the following notation to define a total signed dominating function of $P_m \Box P_n$,

$$f = \begin{pmatrix} f(v_{0,0}) & f(e'_{0,0}) & f(v_{0,1}) & f(e'_{0,1}) & f(v_{0,2}) & \cdots & f(v_{0,m-2}) & f(e'_{0,m-2}) & f(v_{0,m-1}) \\ f(e_{0,0}) & & f(e_{0,1}) & & f(e_{0,2}) & \cdots & f(e_{0,m-2}) & & f(e_{0,m-1}) \\ f(v_{1,0}) & f(e'_{1,0}) & f(v_{1,1}) & f(e'_{1,1}) & f(v_{1,2}) & \cdots & f(v_{1,m-2}) & f(e'_{1,m-2}) & f(v_{1,m-1}) \\ f(e_{1,0}) & & f(e_{1,1}) & & f(e_{1,2}) & \cdots & f(e_{1,m-2}) & & f(e_{1,m-1}) \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ f(v_{n-2,0}) & f(e'_{n-2,0}) & f(v_{n-2,1}) & f(e'_{n-2,1}) & f(v_{n-2,2}) & \cdots & f(v_{n-2,m-2}) & f(e'_{n-2,m-2}) & f(v_{n-2,m-1}) \\ f(e_{n-2,0}) & & f(e_{n-2,1}) & & f(e_{n-2,2}) & \cdots & f(e_{n-2,m-2}) & & f(e_{n-2,m-1}) \\ f(v_{n-1,0}) & f(e'_{n-1,0}) & f(v_{n-1,1}) & f(e'_{n-1,1}) & f(v_{n-1,2}) & \cdots & f(v_{n-1,m-2}) & f(e'_{n-1,m-2}) & f(v_{n-1,m-1}) \end{pmatrix}.$$

Lemma 3.1. *For $m \geq 2$ and $n = 2$,*

$$\gamma_s^*(P_m \Box P_n) \leq m.$$

Proof. It is sufficient to define a function f with $w(f) = m$. Let

$$f = \begin{pmatrix} -1 & 1 & \cdots & -1 & 1 & -1 \\ 1 & & \cdots & 1 & & 1 \\ 1 & -1 & \cdots & 1 & -1 & 1 \end{pmatrix}.$$

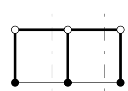


FIGURE 2. Graphs $P_3 \Box P_2$ corresponding to f.

By adding each column, one can see $w(f) = m$. For $m = 3$ and $n = 2$, see Figure 2, where black vertices (thick edges) stand for $f(x) = 1$, and white vertices (thin edges) stand for $f(x) = -1$. $\Box$

Lemma 3.2. *For $m \geq 2$ and $n \geq 3$,*

$$\gamma_s^*(P_m \Box P_n) \leq \begin{cases} \frac{mn+m+4}{2}, & m \pmod 4 \equiv 0 \text{ and } n \pmod 4 \equiv 0, \\ \frac{mn+m+2}{2}, & m \pmod 4 \equiv 2 \text{ or } n \pmod 4 \equiv 2, \\ \frac{mn+m-2}{2}, & m \pmod 2 \equiv 0 \text{ and } n \pmod 2 \equiv 1, \\ \frac{mn+m+n-5}{2}, & m \pmod 2 \equiv 1 \text{ and } n \pmod 2 \equiv 1. \end{cases}$$

Proof.
Case 1: $m \pmod 2 \equiv 0$ and $n \pmod 2 \equiv 0$.

Subcase 1.1. $m \pmod 4 \equiv 0$ and $n \pmod 4 \equiv 0$.

It is sufficient to define a function f with $w(f) = (mn+m+4)/2$. Let

$$f = \left(\begin{array}{cccccccccccccccccccccccc}
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & \cdots & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\
1 & & 1 & & -1 & & 1 & & \cdots & 1 & & 1 & & -1 & & 1 & & 1 & & 1 & & -1 & & 1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & \cdots & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & & -1 & & 1 & & 1 & & \cdots & 1 & & -1 & & 1 & & 1 & & 1 & & -1 & & 1 & & -1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & \cdots & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & & 1 & & -1 & & 1 & & \cdots & 1 & & 1 & & -1 & & 1 & & 1 & & 1 & & -1 & & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & \cdots & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
-1 & & -1 & & 1 & & -1 & & \cdots & -1 & & -1 & & 1 & & -1 & & -1 & & -1 & & 1 & & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & \cdots & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & & 1 & & -1 & & 1 & & \cdots & 1 & & 1 & & -1 & & 1 & & 1 & & 1 & & -1 & & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & \cdots & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & & -1 & & 1 & & 1 & & \cdots & 1 & & -1 & & 1 & & 1 & & 1 & & -1 & & 1 & & -1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & \cdots & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & & 1 & & -1 & & 1 & & \cdots & 1 & & 1 & & -1 & & 1 & & 1 & & 1 & & -1 & & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & \cdots & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
-1 & & -1 & & 1 & & -1 & & \cdots & -1 & & -1 & & 1 & & -1 & & -1 & & -1 & & 1 & & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & \cdots & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & & 1 & & -1 & & 1 & & \cdots & 1 & & 1 & & -1 & & 1 & & 1 & & 1 & & -1 & & 1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & \cdots & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & & -1 & & 1 & & 1 & & \cdots & 1 & & -1 & & 1 & & 1 & & 1 & & -1 & & 1 & & -1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & \cdots & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & & 1 & & -1 & & 1 & & \cdots & 1 & & 1 & & -1 & & 1 & & 1 & & 1 & & -1 & & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & \cdots & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1
\end{array}\right).$$

By adding each column, one can see $w(f) = (mn + m + 4)/2$ (see Figure 3 for $m = 12$ and $n = 12$).

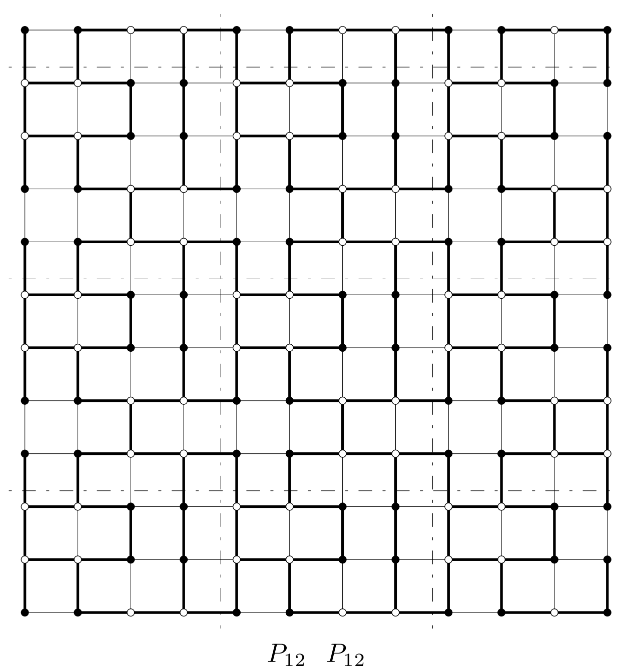


FIGURE 3. Graph $P_{12}\square P_{12}$ corresponding to f.

Subcase 1.2. $m \pmod 4 \equiv 2$ or $n \pmod 4 \equiv 2$.

It is sufficient to define a function f with $w(f) = (mn+m+2)/2$. Let

$$f = \left(\begin{array}{cccccccccccccccccccc}
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & & 1 & & 1 & & 1 & & \cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & & 1 & & -1 & & -1 & & \cdots & 1 & & 1 & & -1 & & -1 & & 1 & & 1 \\
-1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & & 1 & & 1 & & 1 & & \cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & & 1 & & -1 & & -1 & & \cdots & 1 & & 1 & & -1 & & -1 & & 1 & & 1 \\
-1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & & 1 & & 1 & & 1 & & \cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & & 1 & & -1 & & -1 & & \cdots & 1 & & 1 & & -1 & & -1 & & 1 & & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & \cdots & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & & 1 & & 1 & & -1 & & \cdots & -1 & & 1 & & 1 & & -1 & & -1 & & 1 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & \cdots & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1
\end{array}\right).$$

By adding each column, one can see $w(f) = (mn + m + 2)/2$ (see Figure 4 for $m = 10$ and $n = 8$).

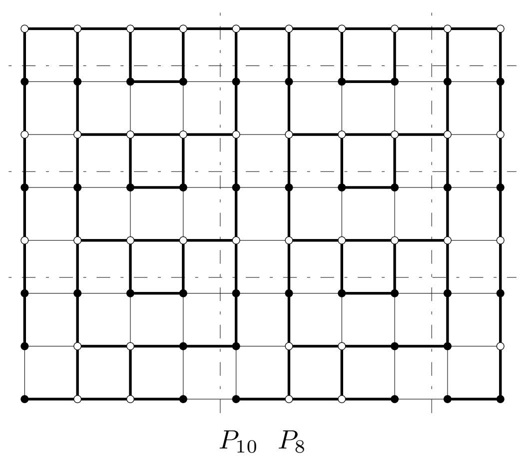


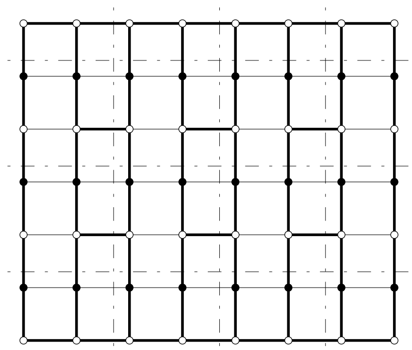
FIGURE 4. Graphs $P_{10} \Box P_8$ corresponding to f.

Case 2: $m \pmod 2 \equiv 0$ and $n \pmod 2 \equiv 1$.

It is sufficient to define a function f with $w(f) = (mn+m-2)/2$. Let

$$f = \left(\begin{array}{cccccccccccc}
-1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & & 1 & & \cdots & 1 & & 1 & & 1 & & 1 \\
1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & & 1 & & \cdots & 1 & & 1 & & 1 & & 1 \\
-1 & -1 & -1 & 1 & \cdots & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & & 1 & & \cdots & 1 & & 1 & & 1 & & 1 \\
\vdots & & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots \\
1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & & 1 & & \cdots & 1 & & 1 & & 1 & & 1 \\
-1 & -1 & -1 & 1 & \cdots & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & & 1 & & \cdots & 1 & & 1 & & 1 & & 1 \\
1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & & 1 & & \cdots & 1 & & 1 & & 1 & & 1 \\
-1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{array}\right).$$

By adding each column, one can see $w(f) = (mn+m-2)/2$ (see Figure 5 for $m = 8$ and $n = 7$).



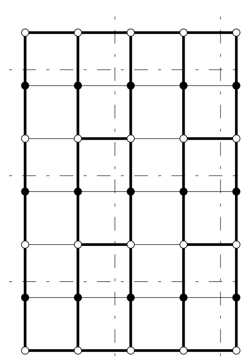
P_8 P_7

FIGURE 5. Graphs $P_8 \square P_7$ corresponding to f.

Case 3: $m \pmod 2 \equiv 1$ and $n \pmod 2 \equiv 1$.

It is sufficient to define a function f with $w(f) = (mn+m-5)/2$. Let

$$f = \begin{pmatrix}
-1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & \cdots & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & \cdots & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & \cdots & 1 & 1 & 1 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & \cdots & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & \cdots & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & \cdots & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & \cdots & -1 & 1 & -1 & 1 & -1
\end{pmatrix}.$$



P_5 P_7

FIGURE 6. Graphs $P_5 \square P_7$ corresponding to f.

By adding each column, one can see $w(f) = (mn + m - 5)/2$ (see Figure 6 for $m = 5$ and $n = 7$).

□

By Lemma 3.1 and Lemma 3.2, we have

Theorem 3.3. *For any integers $m, n \geq 2$,*

$$\gamma_s^*(P_m \Box P_n) \leq \begin{cases} m, & n = 2, \\ \frac{mn+m+4}{2}, & n \geq 3 \text{ and } m \pmod 4 \equiv 0 \text{ and } n \pmod 4 \equiv 0, \\ \frac{mn+m+2}{2}, & n \geq 3 \text{ and } m \pmod 4 \equiv 2 \text{ or } n \pmod 4 \equiv 2, \\ \frac{mn+m-2}{2}, & n \geq 3 \text{ and } m \pmod 2 \equiv 0 \text{ and } n \pmod 2 \equiv 1, \\ \frac{mn+m+n-5}{2}, & n \geq 3 \text{ and } m \pmod 2 \equiv 1 \text{ and } n \pmod 2 \equiv 1. \end{cases}$$

that is,

$$\gamma_s^*(P_m \Box P_n) \leq \frac{mn + m + n + 2}{2}.$$

By Theorem 2.1 and Theorem 3.3, we have

Theorem 3.4. *For any integers $m, n \geq 2$,*

$$\left\lceil \frac{15mn - 3m - 3n - 40}{45} \right\rceil_{\mathcal{P}(3mn-m-n)} \leq \gamma_s^*(P_m \Box P_n) \leq \frac{mn + m + n + 2}{2}.$$

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